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PREFACE

The volume includes theoretical and applicational papers from the field of the multicriteria decision making. The authors are faculty members of the Karol Adamiecki University of Economics in Katowice, Department of the Operations Research, and researchers from Poland and abroad, collaborating with the Department.

In the paper ***Multiple criteria decision making in frozen decision processes*** **M. Chmielewski** and **I. Kaliszewski** consider a decision process in which an algorithm has to be chosen, and not changed afterwards, before all feasible variants are known, as it happens e.g. in public tenders.

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In the paper ***On stability of educational rankings*** **G. Koloch, T. Kuszewski**, and **T. Szapiro** investigate the evaluations of educational institutions. A general, mathematically based concept of stability of ranking is used.

In the paper ***Computer-based support of multicriteria cooperative decisions – some problems and ideas*** **L. Kruś** extends the bargaining game model to the multicriteria case to describe cooperation problems.

In the paper ***Affinity set and its applications*** **M. Larbani** and **Y.W. Chen** propose a new forecasting method without historical memory, based on game theory and affinity set, and compare it with the simple regression model for their performances on decision of buying in or selling out stocks in a dynamic market.

In the paper ***An application of interactive multiple criteria technic in labor planning*** **M. Nowak** proposes a new decision aiding procedure and illustrates its application.

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In the paper ***Representing partial information on preferences with the help of linear transformation of objective space*** **D. Podkopaev** considers the case in which information about preferences in a multiobjective optimization problem is represented in the form of upper bounds on trade-off coefficients, defined for any pair of objective functions.

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The volume editor would like to thank the authorities of the Karol Adamiecki University of Economics for the support in editing the current volume
in the series ***Multiple Criteria Decision Making***.

Tadeusz Trzaskalik

Marek Chmielewski

Ignacy Kaliszewski

MULTIPLE CRITERIA DECISION MAKING IN FROZEN DECISION PROCESSES

Abstract

We consider a decision process of choosing an algorithm for selecting the most preferred variant. We consider the case when an algorithm has to be chosen and not changed afterwards before all feasible variants are known, as it happens e.g. in public tenders.

The fact that the chosen algorithm cannot be changed is the cause of potential regret the decision maker can resent when confronted with the selected variant.

We show how some formal tools of interactive Multiple Criteria Decision Making can be employed to confine decision maker's regret.

Keywords

Interactive multiple criteria decision making, nonautonomous processes.

INTRODUCTION

Decision processes for selecting the most preferred variant can be differentiated with respect to rights hold by the involved parties. In *autonomous processes* the sole actor of the decision process is the *decision maker* (DM). This means that the DM can carry out a decision process in a fully sovereign

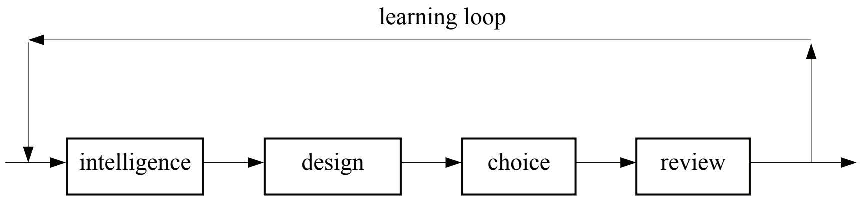

Fig. 1 Four phases of decision making process

manner. In contrast to autonomous processes, in *nonautonomous processes* manners of carrying out decision processes are restricted by rights to the process other parties may hold.

A widely adopted consensus is that every decision process consists of four phases usually closed in a feedback learning loop, namely the phase of *intelligence*, *design* (modeling), *choice*, and *review* [5] (Figure 1). With regard to the definition given above, a decision process is autonomous if in each phase the DM is sovereign in his decisions. The DM enjoys the maximal sovereignty in *interactive decision making*, i.e. when in the third phase the DM selects the most preferred variant interacting with a model, directing himself only by his own preferences.

A decision process ceases to be autonomous when the DM's sovereignty is restricted. A form of restriction can be, for example, the necessity to negotiate with the involved parties the manner in which the process is carried out or just to explain and give grounds for the manner adopted.

Below we focus on nonautonomous processes in which the only restriction is that the DM is bound to chose an unequivocal *selection algorithm* for selecting the most preferred variant and to made this algorithm public without any possibility to modify it in the future. We call such a decision process a *frozen process*.

In frozen processes consequences of impertinent choice of selection algorithm are irreversible, where pertinent choice is understood as follows: *a selection algorithm is chosen pertinently if the most preferred variant selected by this algorithm is that, which would be selected as the most preferred also in the case of an autonomous decision process*. Impertinent choice of a selection algorithm is a source of DM's (posterior) *regret*[1].

In general, chances to choose a pertinent algorithm are small. This is the case of e.g. public tenders, where parties involved are a tender calling entity, bidders, supervisory bodies, and to some extent, the whole society with its monitoring institutions (state agencies, media). Though in such cases consequences of impertinent choice of selection algorithm are mainly borne by a tender calling entity, they clearly also impact, explicitly or implicitly, other parties involved.

Further examples of frozen decision processes are public valuation procedures of individuals or institutions such as open competitions or rankings of universities.

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[1] "Regret is a negative, cognitively based emotion that we experience when realizing or imagining that our present situation would have been better, had we decided differently" [6].

The issue whether a decision process is frozen or not is context dependent. For example, banks are autonomous in credibility assessments of their clients. However, a credit officer cannot alter a credibility assessment procedure at his discretion, hence from his standpoint the assessment process is frozen.

Our aim is to provide the DM with means (methodologies and tools) to assess consequences of decisions, similar to means available for him in the case of autonomous processes. But as in the case of autonomous processes the main objective is to select the most preferred variant, in the case of frozen processes the main objective is to choose an algorithm which reflects DM's implicit and/or explicit, and usually partial preferences with respect to that variant. In other words, the objective is to minimize DM's posterior regret understood here as DM's *emotion resulting from comparing his implicit preferences for the most preferred variant and the most preferred variant selected by the chosen algorithm*.

In frozen decision processes the range of information about variants can vary between two extremes, from full information (all variants are known before a selection algorithm is chosen) to lack of any information (no variant is known before a selection algorithm is chosen). In general, DM's regret should be minimal in the former case. In all other cases the DM can make use of *hypothetical variants*. Hypothetical variants convey DM' his expectations about and, at the same time, his expertise on possible variants. Creating and evaluating hypothetical variants is the main (and practically only) tool to confine DM's regret.

In this paper we analyze frozen decision processes. Our aim is to identify the extent of freedom the DM possesses when choosing selection algorithm, as well as to identify his ability to minimize his regret. We also aim at establishing a methodology for supporting the DM in choosing selection algorithms.

We adopt a simplifying assumption that the only party which in a frozen decision process can resent regret is the DM.

The outline of the paper is as follows. In Section 1 we introduce necessary definitions and notation. In Section 2 we present those elements of Multiple Criteria Decision Making [4, 1. 3], which we use in Section 3 to propose how to support choice of selection algorithms in the case of frozen decision processes. Next, we present conclusions and we point to further possible directions of research.

1. DEFINITIONS AND NOTATION

Let x denote a decision variant, X – a space of decision variants, X_0 – a finite subset of feasible variants, and $X_0 \subseteq X$. Then the Multiple Criteria Decision Making (MCDM) problem is formulated as:

$$\begin{aligned} &\text{“}max\text{”}\, f(x) \\ &x \in X_0 \subseteq X \end{aligned} \tag{1}$$

where $f : X \to R^k,\ f = f(f_1, \ldots, f_k);\ f_i : X \to R,\ i = 1, \ldots, k,\ k \geq 2$, are objective (criteria) functions; “max” denotes the operator of deriving all efficient variants in X_0 according to the definition of efficiency given below.

In MCDM to compare variants x one makes use of their *outcomes* $f(x)$. Relations between outcomes in space R^k induce relations between variants in space X.

Below we make use of the following notation: $y = f(x)$ and $Z = f(X_0)$. Outcome $\overline{y} \in Z$ is called efficient if $y_i \geq \overline{y}_i,\ i = 1, \ldots, k,\ y \in Z$ implies $y = \overline{y}$. Variant $\bar{x} \in X_0$ is called efficient if $\bar{y} = f(\bar{x})$ is efficient.

Observe that in frozen processes the DM deals with X_0 and Z, which may contain hypothetical variants and hypothetical outcomes.

2. THE PROPOSED APPROACH

As a vehicle for supporting the DM in choosing selection algorithm in the case of frozen decision processes we employ interactive MCDM methods, which on the base of model (1) enable the DM to select the most preferred outcome (variant).

In interactive MCDM at each iteration the DM evaluates at least one pair of outcomes and establishes preference $\succ$ between them. The selection of the most preferred outcome is made possible by assuming that DM's preferences are consistent with his *implicit value function*.

Since outcomes are vectors of numbers, it is reasonable to assume that DM's preference relation $\succ$ is a partial order. Since there is no reason to exclude the case that two or more variants have the same outcome, the relation induced in the set of feasible variants X_0 by relation $\succ$ is in general a quasi-partial order.

Consistency between the implicit value function and preference relation $\succ$ (we assume, by analogy to interactive MCDM, that consistency holds), entails implication:

$$y \succ y' \ \Rightarrow \ v(y) > v(y') \tag{2}$$

which establishes a set of conditions on the DM's implicit value function.

We employ here *weighted linear scalarizing functions*, widely used in MCDM, namely:

$$\sum_i \lambda_i y_i \tag{3}$$

where $\lambda_i \geq 0, \ i = 1, ..., k$. For these functions condition (2) reduces to:

$$y \succ y' \ \Rightarrow \ \sum_i \lambda_i y_i > \sum_i \lambda_i y'_i \tag{4}$$

Evaluating h pairs of outcomes results in a system of inequalities:

$$\sum_i \lambda_i y_i^h > \sum_i \lambda_i y_i'^h, \ h = 1, ... \tag{5}$$

where y^h and y'^h are elements of pair h.

It is easy to show that if $y' \succ y''$ and $y''' \succ y'', \ y' \neq y'' \neq y'''$ and $y'_i \geq y'''_i, \ \ i = 1, ..., k$ (i.e. y' *dominates* y'''), then inequality $\sum_i \lambda_i y'''_i > \sum_i \lambda_i y''_i$ is redundant. Hence, it is suffcient to evaluate only pairs of efficient outcomes. This entails the following observation.

Lemma 2.1

The most preferred outcome selected consistently to implication (2), where $v(\cdot)$ are weighted linear scalarizing functions, is efficient.

Derivation of efficient outcomes is carried with the use of *scalarizing functions*, which attain their extremal values at eficient outcomes. In particular, one can make use of weighted linear scalarizing functions. For each $\lambda_i, \ i = 1, ..., k$, maximization of the corresponding linear scalarizing function yields an efficient outcome [4, 1, 3].

Each vector λ satisfying (5) and $\lambda_i \geq 0, \ i = 1, ..., k,$ defines function $\sum_i \lambda_i y_i$ preserving (in the sense of (4)) all *h* relations $y^h \succ y'^h$

The set of all such vectors (we denote this set as $\bar{\Lambda}$) defines the extent of exibility the DM has when choosing a selection algorithm consisting in maximizing a linear scalarizing function.

Clearly, $\bar{\Lambda} \subseteq \Lambda = \{\lambda \mid \lambda_i \geq 0,\ i = 1, ..., k\}$. Without loss of generality we assume that elements of Λ satisfy additional condition $\sum_i \lambda_i = 1$.

3. SUPPORTING CHOICE OF SELECTION ALGORITHMS

It is rational to assume that at the start of choosing a selection algorithm the DM has only some vague preferences with respect to the most preferred outcome. The aim of the algorithm selection (frozen) process is to specify those preferences and evoke more partial preferences. Recall that set X_0 can, and in some cases should, include hypothetical variants.

If selection algorithm is chosen when all variants are known, one can expect that the DM chooses always an algorithm which selects the most preferred variant and hence there is no cause for regret. However, this is not always the case, as shown in the following example.

Example 3.1

Let three variants be given which outcomes are y^1 = (2, 6), y^2 = (3, 3), y^3 = (6, 2). *Let outcome* y_2 *be the most preferred outcome. Then, by* (2):

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 > \lambda_1 y_1^1 + \lambda_2 y_2^1 \quad (hence\ \lambda_1 > 3\lambda_2)$$

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 > \lambda_1 y_1^3 + \lambda_2 y_2^3 \quad (hence\ 3\lambda_1 < \lambda_2)$$

But it is easy to check that this system is inconsistent, i.e. $\bar{\Lambda} = \emptyset$. *This entails that there is no function* $\sum_i \lambda_i y_i$ *such that it attains its largest value for* y^2.

Weighted linear functions allow deriving the most preferred outcomes only if they are located on the convex hull (i.e. the smallest polyhedral set containing *Z*) of *Z*. Below, for the sake of clarity of presentation, we assume that this is the case. We admit that this is an oversimplification of reality, but we do this because weighted linear functions are predominantly used in interactive MCDM (a good example of such a „standard" are methods to select a winner in public tenders). Relaxation of this assumption would require using e.g. weighted Tchebycheff functions [4, 1, 3] as scalarizing functions. This, however, falls outside of the scope of this paper.

The idea of supporting choice of selection algorithm consists in analyzing set $\bar{\Lambda}$. Recall that elements of this sets are vectors λ satisfying system of conditions (5), where y^h and y'^h are pairs of outcomes for which the DM expressed preference (in the sense of relation $\succ$).

In particular, if all outcomes are known and the DM points to the most preferred (and hence efficient) outcome y^j, then the set of conditions:

$$y^j \succ y, \text{ for each } \; y \in Z \setminus \{y^j\} \tag{7}$$

is to hold. Then set $\bar{\Lambda}$ contains all vectors λ, for which:

$$\sum_i \lambda_i y_i^j > \sum_i \lambda_i y_i \;\text{ for each }\; y \in Z \setminus \{y^j\} \tag{8}$$

In this case we distinguish set $\bar{\Lambda}$ as Λ^j. Hence, for each $\lambda \in \Lambda^j$ the algorithm (Algorithm Δ) defined as follows:

$$\text{“select } \arg(\max_{y \in Z} \sum_i \lambda_i y_i)\text{”}$$

selects y^j. In other words, set Λ^j is *the stability set* of outcome y^j with respect to perturbations of λ.

Example 3.2

Given are three variants with outcomes as in Example 3.1. Set $\Lambda^1 \subseteq \bar{\Lambda}$ *such that for all Set* $\lambda \in \Lambda^1$ *Algorithm* Δ *selects outcome* y^1 *is given by the system of inequalities*:

$$\lambda_1 y_1^1 + \lambda_2 y_2^1 > \lambda_1 y_1^2 + \lambda_2 y_2^2 \quad (\textit{hence } 3\lambda_2 > \lambda_1)$$

$$\lambda_1 y_1^1 + \lambda_2 y_2^1 > \lambda_1 y_1^3 + \lambda_2 y_2^3 \quad (\textit{hence } \lambda_2 > \lambda_1)$$

On the other hand, set $\Lambda^3 \subseteq \bar{\Lambda}$ *such that for all* $\lambda \in \Lambda^3$ *Algorithm* Δ *selects outcome* y^3 *is given by the system of inequalities:*

$$\lambda_1 y_1^3 + \lambda_2 y_2^3 > \lambda_1 y_1^1 + \lambda_2 y_2^1 \; (\textit{hence } \lambda_1 > \lambda_2)$$

$$\lambda_1 y_1^3 + \lambda_2 y_2^3 > \lambda_1 y_1^2 + \lambda_2 y_2^2 \; (\textit{hence } 3\lambda_2 > \lambda_2)$$

Sets Λ^1, Λ^3 *are obviously disjoint, as illustrated in Figure 2.*

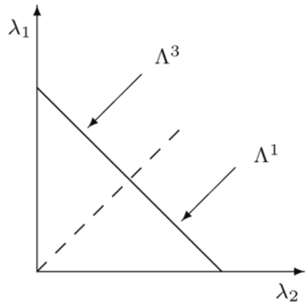

Fig. 2. Stability regions of y^1 and y^3 (Example 3.2)

If (8) does not hold, set $\bar{\Lambda}$ contains vectors λ for which Algorithm Δ can select different outcomes (and therefore to different variants) and DM's regret varies accordingly.

Example 3.3

Given are three variants with outcomes as in Example 3.1. Assume that set $\bar{\Lambda}$ *defined by the following condition*:

$$\lambda_1 y_1^1 + \lambda_2 y_2^1 > \lambda_1 y_1^2 + \lambda_2 y_2^2 \quad (hence\ 3\lambda_2 > \lambda_1)$$

Then for some $\lambda \in \bar{\Lambda}$ *Algorithm* Δ *can select the following outcomes:* y^1 *and* y^3.

In general, $cl(\bar{\Lambda}) = \bigcup_j cl(\Lambda^j),\ \Lambda_j \cap \Lambda_i = \emptyset,\ i \neq j$, where Λ^j is the set of vectors λ such that for any $\lambda \in \Lambda^j$ Algorithm Δ selects only y^j, $j = 1,...,L$, $L \leq |Z|$, and $cl(\cdot)$ denotes closure of a set. Set Λ^i determine stability regions of outcomes y^i, $i = 1, ..., L$.

Lemma 3.1

The most preferred outcome is a vertex of the convex hull of Z.

Proof

By the adopted assumption, the most preferred outcome, say outcome y^m, is located on the convex hull of *Z*. Suppose it is not a vertex. Then there exists at least one outcome y such that:

$$\sum_i \lambda_i y_i^m = \sum_i \lambda_i y_i \ \text{ for some } \lambda \in \Lambda^j$$

Hence, by (2), $y^m \not> y$, which by (8) is a contradiction.

■

From Lemma 3.1 we infer the following obvious observation.

Lemma 3.2

Partition of set $cl(\bar{\Lambda})$ into subsets $cl(\Lambda^i)$ depends only on effcient vertices of the convex hull of Z.

By the above lemma system of inequalities (5) can be confined exclusively to inequalities generated by efficient vertices of the convex hull of *Z*.

The DM can control the level of his regret imposing conditions that certain outcomes (and therefore the corresponding variants) are not selected by Algorithm Δ. This is equivalent to imposing conditions on admissible vectors λ. Namely, if he wants that y^j is not selected by Algorithm Δ he has to set weights $\lambda \in \bar{\Lambda}$ such that $\lambda \notin \Lambda^j$.

On the other hand, the DM can control the level of his regret imposing the condition that a certain outcome (and the corresponding variant) is selected by Algorithm Δ as the most preferred. Here again, this is equivalent to imposing a condition on admissible vectors λ. Namely, if he wants that y^j is selected by Algorithm Δ he has to set weights $\lambda \in \bar{\Lambda}$ such that $\lambda \in \Lambda^j$.

Identification of sets of weights for which Algorithm Δ selects given outcome y^j leads to the following question: which vector λ from set Λ^j ensures the maximum stability of the most preferred outcome y^j with respect to perturbations of decision problem parameters?

Let us consider first stability of the most preferred outcome y^j with respect to perturbations of vectors λ (in the sense of value $||\lambda - \lambda'||_2$, where λ' is perturbed vector). Outcome y^j is the most stable (robust) with respect to perturbations of vector λ if such a vector is the most distant from all constraints which define set Λ^j. Such a vector can be defined in the following way.

Vector λ most distant from a constraint, which defines Λ^j as a consequence of relation $y^j \succ y^l$, can be found by solving the following optimization problem:

$$\max_{\lambda \in cl(\Lambda^j)} \sum_i \lambda_i (y_i^j - y_i^l)$$

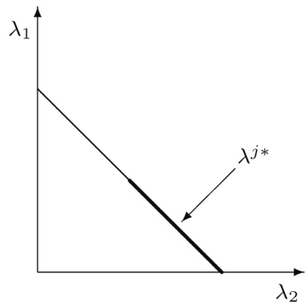

Fig. 3. Stability of an outcome in the weight space

For all constraints which define Λ^j vector λ maximizing all $\sum_i \lambda_i(y_i^j - y_i^l)$, $l = 1, ..., L$ simultaneously, can be found by solving the following optimization problem (Figure 3):

$$\begin{gathered} \max t \\ \sum_i \lambda_i y_i^j - \sum_i \lambda_i y_i^l \geq t, \ l = 1, ..., L \\ \lambda \in cl(\Lambda^j) \end{gathered} \tag{9}$$

Consider now stability of the most preferred outcome y^j with respect to variations of the remaining outcomes (in the sense of value $||y - y'||_2$ where y' is perturbed outcome). Recall that in the considered problem some variants can be hypothetical and therefore their outcomes can vary. Outcome y^j is the most stable with respect to variations of outcomes y^l, $l = 1, ..., L$, $y^l \neq y^j$, if the minimal of differences:

$$\sum_j \lambda_j y_j^i - \sum_j \lambda_j y_j^l, \ l = 1, ..., L$$

is maximal (Figure 4). Hence, vector λ satisfying this requirement can be again found by solving problem (9).

Therefore vector λ for which outcome y^j is the most stable with respect to variations of weights at the same time ensures the highest stability of y^j with respect to variations of other outcomes. As the former observation pertains to the weight space, the latter observation pertains to the outcome space.

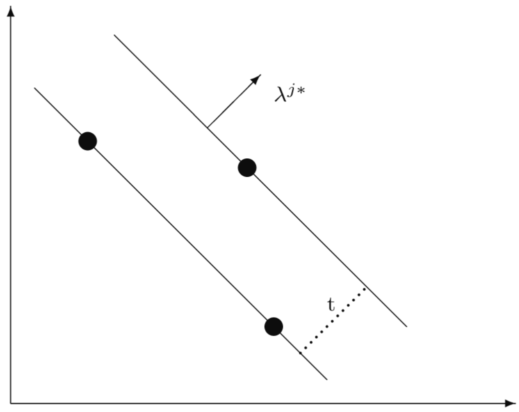

Fig. 4. Stability of an outcome in the outcome space

The interpretation in the weight space is rather straightforward. Vector λ^{j*} which solves (9) corresponds to the point in set Λ^j located in the same distance to all constraints which define this set.

More interesting is interpretation of vector λ^{j*} in outcome space. This vector in normal to the hyperplane tangent to y^j and at the same time maximizing the minimal distance to all efficient outcomes adjacent to y^j.

CONCLUDING REMARKS AND DIRECTIONS FOR FURTHER RESEARCH

Probably the most spectacular area of the above considerations can be public tenders, where public money is involved. In many occasions the variant selected is not the most preferred (in the earlier defined sense), but by the rules of public tenders (where decision processes are frozen) it is the winner. This causes regret, often formulated verbally: If I (we) had known that *such* variants were proposed, I (we) would have chosen a different selection algorithm.

Careful analysis of frozen decision process problems with the help of the technics discussed above could in many cases reduce regret. This is of particular importance when tender is organized for the first time in a field new to the DM, where a significant regret can materialize.

It is rather obvious that the process of choosing an algorithm for selection of the most preferred variant, besides considering hypothetical variants, has to also address selection of criteria. One should not regard model (1) as given and fixed, but model specification should be also an element of the process of algorithm selection. Appropriate model specification in frozen decision processes will be a subject of further research.

Another issue for further research will be also the possible use of other scalarizing functions than linear weighted functions.

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Petr Fiala

MULTIOBJECTIVE COMBINATORIAL AUCTIONS*

Abstract

Auctions are important market mechanisms for the allocation of goods and services. Combinatorial auctions are those auctions in which bidders can place bids on combinations of items. The advantage of combinatorial auctions is that the bidder can more fully express his preferences. This is particular important when items are complements. Allowing bidders more fully to express preferences often leads to improved economic efficiency and greater auction revenues. A typical combinatorial auction problem is the so called winner determination problem. The problem illustrates the possibility to formulate combinatorial auctions as mathematical programming problems as well as the complexity of combinatorial auctions. Auctions with complex bid structures are called multiobjective auctions, since they address multiple objectives in the negotiation space. Multiobjective optimization can be helpful for detailed analysis of combinatorial auctions. Buyers can specify weights and aspiration levels that express their desired values on the attributes of the items to be purchased. Interactive methods for multiobjective optimization are proposed for analysis of combinatorial auctions and for negotiation process.

Keywords

Combinatorial auctions, preference elicitation, multiobjective optimization, negotiation, interactive methods, Dynamic Network Process.

INTRODUCTION

Auctions are important market mechanisms for the allocation of goods and services. They are preferred often to other common processes because they are open, quite fair, easy to understand by participants, and lead to economically efficient outcomes. Many modern markets are organized as auctions. Design of auctions is a multidisciplinary effort made of contributions from economics,

* The research project was supported by Grant No. 402/07/0166 „Combinatorial auctions – modeling and analysis" from the Grant Agency of Czech Republic.

operations research, informatics, and other disciplines. Combinatorial auctions are those auctions in which bidders can place bids on combinations of items called bundles. The advantage of combinatorial auctions is that the bidder can more fully express his preferences. This is particularly important when items are complements. The auction designer also derives value from combinatorial auctions. Allowing bidders more fully to express preferences often leads to improved economic efficiency and greater auction revenues. However, alongside their advantages, combinatorial auctions raise a host of questions and challenges [see 5; 6].

Auction theory has attracted tremendous interest from both the economic side as well as the Internet industry. An auction is a competitive mechanism to allocate resources to buyers based on predefined rules. These rules define the bidding process, how the winner is determined, and the final agreement. In electronic commerce transactions, software agents that negotiate on behalf of buyers and sellers conduct auctions. The popularity of auctions and the requirements of e-business have led to growing interest in the development of complex trading models [1; 2; 9].

Classification of auctions is based on some specific characteristics as:

1. The numbers of sellers and buyers.
2. The number of items.
3. Traded items (indivisible, divisible, pure commodities, structured commodities).
4. Participants' roles in auctions (one-sided, multilateral auctions).
5. Preferences of the participants.
6. The form of the private information participants have about preference.
7. Objectives of auctions (optimization, allocation rules, pricing rules).
8. Evaluating criteria.
9. Complexity of bids (simply, related bids).
10. Organization of auctions (single-round, multi-round, sequential, parallel, price schemes).

The problem, called the winner determination problem, has received considerable attention in the literature. The problem is formulated as: Given a set of bids in a combinatorial auction, find an allocation of items to bidders that maximizes the seller's revenue. It introduced many important ideas, such as the mathematical programming formulation of the winner determination problem, the connection between the winner determination problem and the set packing problem as well as the issue of complexity.

Iterative combinatorial auctions with multiple objectives are proposed in the paper as complex trading models. A solution procedure is presented.

1. WINNER DETERMINATION PROBLEM

Many types of combinatorial auctions can be formulated as mathematical programming problems. From among different types of combinatorial auctions we present an auction of indivisible items with one seller and several buyers. Let us suppose that one seller offers a set G of m items, $j = 1, 2, \ldots, m$, to n potential buyers. Items are available in single units. A bid made by buyer i, $i = 1, 2, \ldots, n$, is defined as:

$$B_i = \{S, v_i(S)\}$$

where:

$S \subseteq M$ is a combination of items,
$v_i(S)$ is the valuation or offered price by buyer i for the combination of items S.

The objective is to maximize the revenue of the seller given the bids made by buyers. Constraints are imposed such that no single item is allocated to more than one buyer and that no buyer obtains more than one combination.

1.1. Problem formulation

Let $x_i(S)$ be a bivalent variable specifying if the combination S is assigned to buyer i ($x_i(S) = 1$). The winner determination problem can be formulated as follows

$$\sum_{i=1}^{n} \sum_{S \subseteq M} v_i(S)\, x_i(S) \rightarrow \quad \max$$

subject to:

$$\sum_{S \subseteq M} x_i(S) \leqq 1, \quad \forall i,\ i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} \sum_{S \subseteq M} x_i(S) \leqq 1, \quad \forall j \in M$$

$$x_i(S) \in \{0, 1\}, \ \forall S \subseteq M, \ \forall i,\ i = 1, 2, \ldots, n$$

The objective function expresses the revenue. The first constraint ensures that no bidder receives more than one combination of items. The second constraint ensures that overlapping sets of items are never assigned.

The winner determination problem, i.e. determination of the items that each bidder wins, is not difficult in the case of non-combinatorial auctions. It would take $O(nm)$ time where n is the number of bidders and m is the number of items. But in the case of combinatorial auctions, the winner determination problem is much more complex.

1.2. Complexity of the problem

Complexity is a fundamental question in combinatorial auction design. There are some types of complexity:
– computational complexity,
– valuation complexity,
– strategic complexity,
– communication complexity.

Computational Complexity covers such questions as: How much computation is required to compute an outcome given the bid information of the bidders. This is an extremely important question because winner determination problem is an NP-complete optimization problem. The winner determination problem turns out to be an instance of a weighted set packing problem. The weighted set packing problem is a problem of finding a disjoint collection of weighted subsets of a larger set with maximal total weight. Weighted set packing is a classical NP-complete problem.

Valuation complexity deals with such questions as: How much computation is required to provide preference information within a mechanism? Estimating every possible bundle of items requires exponential space and hence exponential time. Bidders need to determine valuations for 2^m -1 possible bundles.

Strategic complexity concerns such questions as: Which of the 2^m -1 bundles to bid on? What is the best strategy for bidding? Must bidders model the behavior of other bidders and solve problems to compute an optimal strategy? For instance, in a sealed bid combinatorial procurement scenario, sellers will need to take not only their valuation of the bundles into consideration, but also the bidding behavior of their competitors. This requires sophisticated bidding logic.

Communication complexity concerns such questions as: How much communication should be exchanged between bidders and auctioneer until an equilibrium price is reached and the mechanism computes an outcome. The amount of communication between the bidders and the auctioneer can become quite high. For instance, in an iterative combinatorial auction, where individual valuations are revealed progressively in an iterative manner, the communication costs could be high if the auction were conducted in a distributed manner over space and/or time. The problem of communication complexity can be addressed through the design of careful bidding languages that provide expressive, but concise bids.

2. MULTIDIMENSIONAL AUCTIONS

Multidimensional auctions are examples of generalization of auctions. These auctions can be classified as:

– multiunit auction,
– multiitem auction,
– multiobjective auction,
– multiround auction.

Multiunit auctions contain multiple units of items and makes possible volume discount auctions. In multiitem auctions one can place bids on combinations of items; such auctions are called combinatorial auctions. In combinatorial auctions multiple objectives can be defined, for instance, as:

– revenue maximization – the seller should extract the highest possible price,
– efficiency – the buyers with the highest valuation get the goods,
– collusion possibility.

Auctions with complex bid structures are also called multiobjective auctions, since they address multiple attributes of the items (quality, quantity, price) in the negotiation space. Multiobjective optimization can be helpful for detailed analysis of combinatorial auctions.

In the iterative approach, there are multiple rounds of bidding and allocation and the problem is solved in an iterative and incremental way. Iterative combinatorial auctions are attractive to bidders because they learn about their rivals' valuations through the bidding process, which could help them to adjust their own bids.

Combinations of multidimensional characteristics are possible. We suggest to use an iterative process for multiobjective combinatorial auctions. Multiobjective combinatorial auctions require several key components to solve the process:

– preference elicitation model,
– multiobjective optimization model,
– negotiation model.

The preference elicitation model is used to let the buyer express his preferences. The preferences are modeled by a combination of the dynamic version of Analytic Network Process and aspiration levels. The multiobjective optimization model selects the best offer for the buyer. For analysis of iterative combinatorial auctions we propose to use interactive methods for multiobjective optimization. The negotiation model helps to find a consensus by auctions. Auctions have emerged as a particularly interesting tool for negotiations. Combinatorial auctions provide a mechanism for negotiation between buyers and sellers. Various concepts of negotiation models can be used for modeling combinatorial auctions.

3. PREFERENCE ELICITATION

The key feature that makes combinatorial auctions most appealing is the ability for bidders to express complex preferences over bundles of items, involving complementarity and substitutability. Items are complements when a set of items has greater utility than the sum of the utilities for the individual items. Items are substitutes when a set of items has less utility than the sum of the utilities for the individual items.

Two items A and B are complementary, if the following holds:

$$v(\{A, B\}) > v(\{A\}) + v(\{B\})$$

Two items A and B are substitute, if the following holds:

$$v(\{A, B\}) < v(\{A\}) + v(\{B\})$$

Different elicitation algorithms may require different means of representing the information obtained by bidders. Sandholm and Boutilier [15] describe a general method for representing an incompletely specified valuation functions. A constraint network is a labeled directed graph consisting of one node for each bundle b representing the elicitor's knowledge of the preferences of a bidder. A directed edge (a, b) indicates that bundle a is preferred to bundle b. Figure 1 represents an example of a constraint network for bundles of three items (A,B,C).

The constraint network representation is conceptually useful and can be represented explicitly for use in various elicitation algorithms. But its explicit representation is generally tractable only for small problems, since it contains 2^m nodes. For preference elicitation of bundles in a constraint network Analytic Network Process can be used.

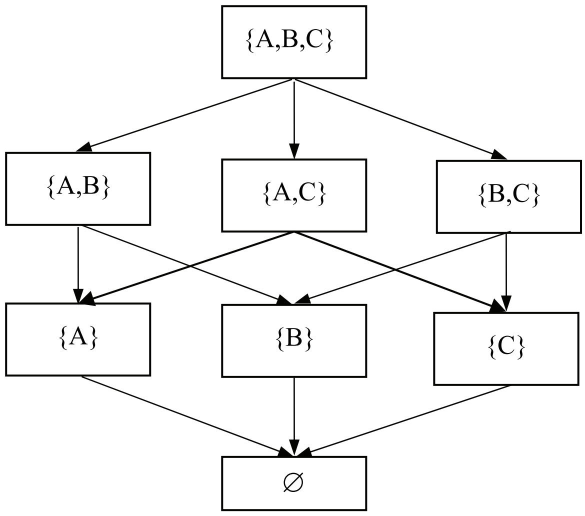

Fig. 1. Constraint network

The Analytic Hierarchy Process (AHP) is the method for setting priorities [11]. A priority scale based on reference is the AHP way to standardize nonunique scales in order to combine multiple performance measures. The AHP derives ratio scale priorities by making paired comparisons of elements on a common hierarchy level by using a 1 to 9 scale of absolute numbers. The absolute number from the scale is an approximation to the ratio w_j/w_k; it is then possible to derive values of w_j and w_k. The AHP method uses the general model for synthesis of the performance measures in the hierarchical structure:

$$u_i = \sum_{j=1}^{n} v_j w_{jk}$$

The Analytic Network Process (ANP) is the method [12] that makes it possible to deal systematically with all kinds of dependence and feedback in the performance system. The well-known AHP theory is a special case of the Analytic Network Process that can be very useful for incorporating connections in the system.

The structure of the ANP model is described by clusters of elements connected by their dependence on one another. A cluster groups elements share a set of attributes. At least one element in each of the clusters is connected to some element in another cluster. The connections indicate the flow of influence between the elements (see Figure 2).

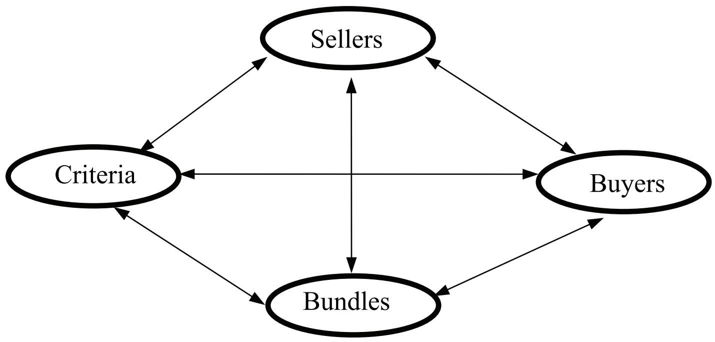

Fig. 2. Clusters and connections in multiobjective combinatorial auctions

The clusters in multiobjective combinatorial auctions can be sellers, buyers, bundles of items, and also evaluating criteria. Paired comparisons are inputs for preference elicitation in combinatorial auctions. A supermatrix is a matrix of all elements by all elements. The weights from the paired comparisons are placed in the appropriate column of the supermatrix. The sum of each column corresponds to the number of comparison sets. The weights in the column corresponding to the cluster are multiplied by the weight of the cluster. Each column of the weighted supermatrix sums to one and the matrix is column stochastic. Its powers can stabilize after some iterations to limited supermatrix. The columns of each block of the matrix are identical in many cases, though not always, and we can read the global priority of units.

Recent work has focused on the question of how to limit the amount of valuation information provided by bidders by adaptively limiting the precision of the bids that are specified.

Combinatorial auctions can be divided into auctioneer-side allocation auctions and bidder-side allocation auctions. The bidder-side allocation auctions were developed for small problems where bidders can cooperate in order to find a better allocation in each iteration without external help. In the auctioneer-side allocation auctions the auctioneer solves the winner determination problem after the bids are collected. The auctioneer provides then some kind of feedback to support the bidders in improving their bids in the next iteration. Usually the bidder's current winning bids and item prices are used as the feedback. The key challenge in the iterative combinatorial auction design is to provide information feedback to the bidders after each iteration. Assigning prices to items and/or item bundles was adopted as the most intuitive mechanism of providing feedback.

The AHP and ANP are static, but in decision analysis in the modern world it is very important to take time into account. The DHP/DNP (Dynamic Hierarchy Process/Dynamic Network Process) methods have been introduced [12]. There are two ways to study dynamic decisions: structural, by including scenarios, and functional by explicitly involving time in the judgment process. For the functional dynamics there are analytic or numerical solutions. The basic idea of the numerical approach is to obtain the time dependent principal eigenvector by simulation.

The Dynamic Network Process seems to be the appropriate instrument for analyzing dynamic network effects [7]. The method is appropriate also for the specific features of multiobjective combinatorial auctions. The method computes time dependent weights for bundles of items of weights of bidders (Figure 3).

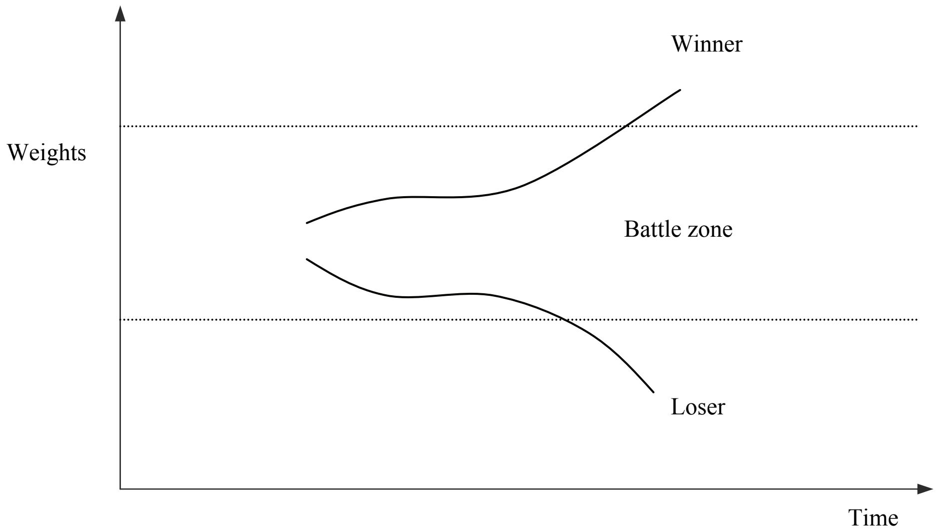

Fig. 3. Time dependent weights

In the multiobjective combinatorial auction model we take into account the auctioneer, bidders, criteria, and packages as clusters and different types of connections in the system. There are also some dependencies and feedback among elements and clusters. The dynamic version of the model is tested.

We used the alpha version of the ANP software Super Decisions developed by Creative Decisions Foundation (CDF) for some experiments for testing the possibilities of the expression and evaluation of the multi-objective combinatorial auction models (Figure 4).

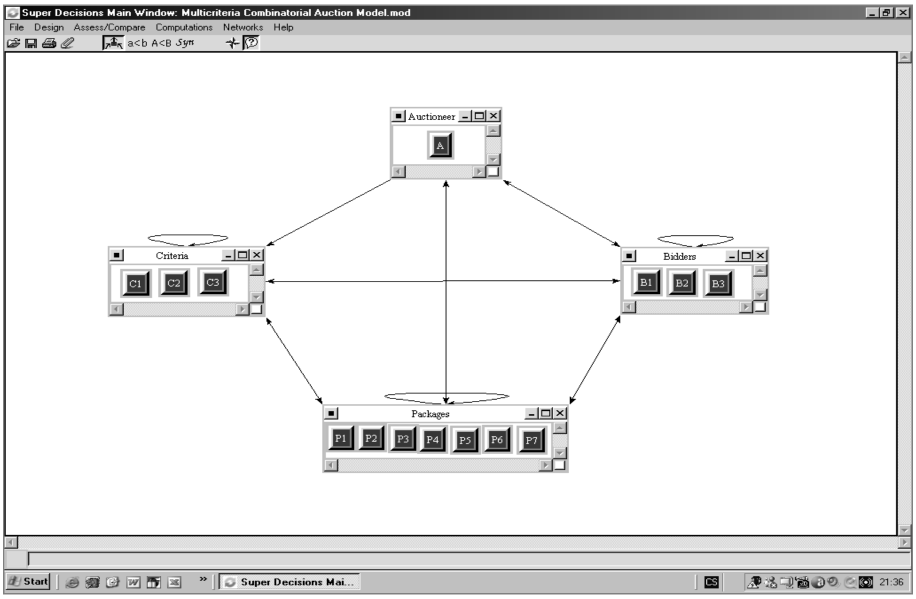

Fig. 4. Multiobjective Combinatorial Auction Model

We propose to combine the weight model of preference elicitation with multiobjective model based on aspiration levels ALOP [8]. The general formulation of an individual multiobjective decision problem is expressed as follows:

$$z(x) = (z_1(x), z_2(x), ..., z_k(x)) \rightarrow \text{“max”}$$
$$x \in X \tag{1}$$

where X is a decision space, x is a decision alternative and $z_1, z_2, ..., z_k$ are the objectives. The decision space is defined by objective restrictions and by mutual goals of all the agents in the aspiration level formulation. The decision alternative x is transformed by the objectives to objective values $z \in Z$, where Z is an objective space. Every agent has his own objectives.

It appears that people tend to satisfy given conditions rather than attempt to optimize them. That means substituting the goals of reaching specified aspiration levels for the goals of maximizing. We denote by $y^{(t)}$ aspiration levels of the objectives and by $\Delta y^{(t)}$ changes of aspiration levels in the step t. We search for alternatives such that:

$$z(x) \geq y^{(t)}$$
$$x \in X \tag{2}$$

According to heuristic information from the results of the condition (2) the agent changes the aspiration levels of objectives for step t + 1:

$$y^{(t+1)} = y^{(t)} + \Delta y^{(t)} \tag{3}$$

We can formulate the multiobjective decision problem as a state space representation. The state space corresponds to the objective space Z, where the states are the aspiration levels of the objectives $y^{(t)}$ and the operators are changes of the aspiration levels $\Delta y^{(t)}$. The start state is a vector of the initial aspiration levels and the goal state is a vector of the objective levels for the best alternative.

4. MULTIOBJECTIVE OPTIMIZATION MODEL

To find the ideal alternative we use the depth-first search method with backtracking procedure. The heuristic information is the distance between an arbitrary state and the goal state.

We propose an interactive procedure ALOP (Aspiration Levels Oriented Procedure) for multiobjective linear programming problems, where the decision space X is determined by the linear constraints:

$$X = \{x \in R^n; Ax \le b, x \ge 0\} \tag{4}$$

and $z_i = c_i x$, $i = 1, 2, \dots, k$, are linear objective functions. Then $z(x) = Cx$, where C is a coefficient matrix of objectives.

The decision alternative $x = (x_1, x_2, \dots, x_n)$ is a vector of n variables. The agent states the aspiration levels $y^{(t)}$ for the objectives values. There are three possibilities for the aspiration levels $y^{(t)}$. The problem (2) can be feasible, infeasible, or else; has a unique nondominated solution. We verify the three possibilities by solving the problem:

$$v = \sum_{i=1}^{k} w_i^+ \, d_i^+ \to \min \tag{5}$$

$$Cx - d^+ = y^{(t)}$$

$$x \in X \, , d^+ \ge 0$$

The value of the objective function in the problem (5) can be interpreted as an increase of utility.

If the following holds:

– $v > 0$, then the problem is feasible and d_i^+ are proposed changes $\Delta y^{(t)}$ of aspiration levels which achieve a nondominated solution in the next step,

– $v = 0$, then we obtained a nondominated solution,
– the problem is infeasible, then we search for the nearest solution to the aspiration levels by solving the goal programing problem:

$$v = \sum_{i=1}^{k} \frac{1}{\bar{z}_i} (d_i^+ + d_i^-) \to \min \tag{6}$$

$$Cx - d^+ + d^- = y^{(t)}$$

$$x \in X, d^+ \geq 0, d^- \geq 0$$

The solution of the problem (6) is feasible with changes of the aspiration levels $\Delta y(t) = d^+ - d^-$. For small changes of nondominated solutions the duality theory is applied. Dual variables to the objective constraints in the problem (6) are denoted by u_i, $i = 1, 2, ..., k$.

If the following holds:

$$\sum_{i=1}^{k} u_i \Delta y_i^{(t)} = 0 , \tag{7}$$

then for some changes $\Delta y^{(t)}$ the value $v = 0$ is not changed and we obtained another nondominated solution. The agent can state k-1 small changes of the aspiration levels $\Delta y_i^{(t)}$, $i = 1, 2, ... , k$, $i \neq r$, then the change of the aspiration level for criterion r is calculated from (7). The agent chooses a forward direction or backtracking. Results of the procedure ALOP are the path of tentative aspiration levels and the ideal solution.

5. NEGOTIATION MODEL

We propose a two-phase interactive approach for solving multiobjective negotiation problems:
1. Finding the ideal alternative for individual agents.
2. Finding a consensus for all the agents.

In the first phase each agent searches for the ideal alternative by the ALOP procedure. In the second phase a consensus can be obtained by the search process and the principle of cooperativeness is applied. The heuristic information for the agent is the distance between his proposal and the opponent's proposal. We assume that all the agents found their ideal alternatives. We propose an interactive procedure GROUP-ALOP for searching for a consensus.

For simplicity we assume the model with two agents:

$$\begin{aligned} &z^1(x) \rightarrow \text{“max”} \\ &z^2(x) \rightarrow \text{“max”} \\ &x \in X \end{aligned} \tag{8}$$

The agents search for a consensus on a common decision space X and change the aspiration levels of the objectives y^1, y^2. The sets of feasible alternatives for the aspiration levels y^1 and y^2 are X^1 and X^2.

$$\begin{aligned} &z^1(x) \geq y^1 \qquad && z^2(x) \geq y^2 \\ &x \in X && x \in X \end{aligned} \tag{9}$$

The consensus set S of the negotiations is the intersection of feasible sets X^1 and X^2:

$$S = X^1 \cap X^2 \tag{10}$$

When the aspiration levels change, the consensus set S is also changed. The agents search for one element consensus set S by alternating the consensus proposals. The image of partner's proposal can be taken as the aspiration levels in one’s own objectives space. In searching for a consensus the distance between the proposals is heuristic information. The paths of the tentative aspiration levels can be used for the backtracking procedure. The forward directions can be directed by the proposed new aspiration levels in step $t + 1$:

$$y^1(t+1) = (1-\alpha)y^1(t) + \alpha z^1(x^2)$$
$$y^2(t+1) = (1-\beta)y^2(t) + \beta z^2(x^1)$$

where $\alpha, \beta \in \langle 0,1 \rangle$ are the coefficients of cooperativeness.

CONCLUSIONS

Combinatorial auction is an important subject of intensive economic research which promises to increase efficiency and reduces exposure to risk in an economic environment where synergy is significant. The winner determination problem is by far the most researched issue in combinatorial auctions. The problem illustrates the possibility to formulate combinatorial auctions as mathematical programming problems and also the complexity of them.

We propose to use multiobjective iterative combinatorial auctions. Multiobjective optimization can be helpful for detailed analysis of combinatorial auctions. Iterative process helps the bidders express their preferences. A possible flexible approach is presented. The approach is based on the Dynamic Network Process and Aspiration Level Oriented Procedure. The combination of such approaches can give more complex views on auctions. The iterative method is used for multiobjective optimization and also negotiation which model helps to find a consensus by auctions.

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Dorota Górecka

MULTICRITERIA DECISION AIDING IN ORDERING PROJECTS CO-FINANCED BY THE EUROPEAN UNION STRUCTURAL FUNDS[*]

Abstract

This paper describes briefly the procedure of evaluation and selection of applications for project co-financing by the European Regional Development Fund that was used in Poland in the period 2004-2006. The paper brings up the issue of improving this procedure by using multicriteria approach based on outranking relations. As an example the problem of ordering projects submitted to Measure 1.2 Environmental Protection Infrastructure by means of ELECTRE methods is presented.

Keywords

Multicriteria decision aiding, European Union Structural Funds, ELECTRE methods.

INTRODUCTION

After entering the European Union on May 1, 2004 Poland has become eligible for support from the EU Structural Funds and the Cohesion Fund. One of the conditions of taking advantage of the opportunity to benefit from transfers from the EU is rational allocation of the financial means depending, among other things, on proper choice of projects that are going to be co-financed. This issue is even more important in the present programming period in which Poland is to receive around 67 billion EUR[1] in regional subsidies from

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[1] As a result of the indexing the total allocation granted to Poland for 2007-2013, i.e. 59,5 billion EUR, by CPI, additional 7,7 billion EUR was gained.

the EU. In order to help the decision-makers in this challenging and difficult task multiple criteria analysis technics can be applied, especially ones based on the outranking relation.

1. EUROPEAN UNION REGIONAL POLICY

The primary objective of the European Union Structural (Regional) Policy[2] is to strengthen the social and economic cohesion of the European Union by means of reducing the disparities in the EU and speeding up the economic convergence of the less-developed regions.

Among the instruments of the European Union Structural Policy there are e.g. Structural Funds and Cohesion Fund. The purpose of the funds is to support the restructuring and modernization of the economies of the member states. They are directed towards those countries and regions that without financial aid cannot achieve the average level of EU economic development [3].

In the programming period 2000-2006 were four structural funds:

– The European Regional Development Fund (ERDF),
– The European Social Fund (ESF),
– The European Agricultural Guidance and Guarantee Fund (EAGGF),
– The Financial Instrument for Fisheries Guidance (FIFG).

The EU Structural Policy is based on four basic principles, namely concentration, programming, additionality, and partnership [3].

The programming concept aims at focussing the member states' efforts on the stable multiannual development programmes which strive for the sustainable solution of the problems of the given region in line with the objectives defined by the EU in each programming period. The previous programming period covered the years 2000-2006, and the current one – the years 2007-2013.

In the programming process each member state has to submit the National Development Plan (NDP) that constitutes the basis for negotiations with the European Commission on the other document – Community Support Framework (CSF). The CSF corresponds to Operational Programmes and contains both the member state's and the Funds' strategy and priorities for action, their specific objectives, as well as the contribution from the Funds and other financial resources [7].

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[2] In the new programming period 2007-2013 it is called Cohesion Policy.

The Community Support Framework for Poland in the years 2004-2006 was implemented with help of seven Operational Programmes:
– Sectoral Operational Programme (SOP) Improvement of the Competitiveness of Enterprises,
– SOP Human Resources Development,
– SOP Restructuring and Modernising Food Sector and Rural Areas Development,
– SOP Fisheries and Fish Processing,
– SOP Transport,
– The Integrated Regional Operational Programme,
– Operational Programme Technical Assistance.

2. INTEGRATED REGIONAL OPERATIONAL PROGRAMME

The Integrated Regional Operational Programme was established to create conditions for the increase of competitiveness of the regions in terms of economic performance, labor productivity, education, innovation, individual income, quality, and quantity of technical infrastructure, as well as to prevent the exclusion of certain specified areas. It aimed at improving the long-term economic development of the country, its economic, social, and territorial cohesion, as well as its integration with the European Union [9].

The following priorities were implemented within the IROP:
– Priority 1 – Development and modernization of the infrastructure to enhance the competitiveness of regions (co-financed by the ERDF),
– Priority 2 – Strengthening the human resources development in the regions (co-financed by the ESF),
– Priority 3 – Local development (co-financed by the ERDF),
– Priority 4 – Technical assistance (co-financed by the ERDF).

An allocation of 4 083,9 million EUR was provided for the implementation of the IROP in the years 2004-2006, out of which 2 968,5 million was from the resources of the structural funds (85,2% from the ERDF and 14,8% from the ESF). The beneficiaries eligible for the support were, first of all, territorial self-government units, their unions, alliances, and associations, but also entities performing public services on the basis of an agreement with territorial self-government units, as well as higher education institutions, schools, health care institutions, and non-profit organizations [10].

The beneficiaries received the support for the realization of concrete projects compatible with the objectives of the appropriate measure of the IROP. The proposals for the projects had to be prepared in the form of an application for co-financing and submitted with required annexes to the proper institution.

3. APPRAISAL AND SELECTION PROCEDURE

The appraisal and selection procedure for the applications applying for co-financing under the ERDF comprised five stages:

1. The formal appraisal of the application made by the competent department of the Marshal Office. The criteria of the formal evaluation involved e.g.: completeness of the application, completeness of the annexes, compliance with the objectives of the measure and with the list of projects provided in the IROP Programme Complement, appropriateness of the sources of finance, eligibility of expenditures.
2. The appraisal by the Panel of Experts as regards all content-related criteria.
3. The recommendation by the Regional Steering Committee to the Voivodship Board.
4. The resolution of the Voivodship Board on project selection.
5. Signing the Agreement Granting Structural Funding with final beneficiary [4].

The Panel of Experts consisted of three persons[3] appraising the projects independently. During the evaluation process the projects were scored according to the criteria approved by the IROP Monitoring Committee and enclosed in the IROP Programme Complement.

The criteria were weighted with the maximum weight 4, the same as the maximum number of points for a given criterion. The score 1 meant that the project was very weakly compatible with the criterion and the score 4, that it was strongly consistent with the criterion. The appraisal of each expert was defined by the weighted sum of the partial scores for specific criteria and the final score of the project was calculated as the arithmetic mean of the scores of all experts participating in the Panel. If the difference between experts' scores for any given criterion amounted to 3 points (before considering its weight) another Panel of Experts appraised the project and its evaluation was binding. As the result of the assessment a ranking list of the projects was drawn up.

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[3] When project's value is in excess of 2 million EUR the fourth expert from the so-called State List can participate in the appraisal process.

The ranking of the projects that were appraised positively, i.e. received at least 60% of points, was submitted to the Regional Steering Committee. The RSC could change the ranking of the projects taking into account their coherence with and the significance for the regional development strategy, as well as their impact on the realization of the region's needs. Afterwards the RSC forwarded the final version of the ranking of projects selected to the Voivodship Board.

On the basis of recommendation of the RSC the Voivodship Board made the final decision on project selection taking into consideration, e.g. the amount of allocation for the voivodship for a given measure, and based on this, could possibly change the RCS' ranking [4].

4. ENVIRONMENTAL PROTECTION INFRASTRUCTURE

Under the Priority 1 six Measures were implemented. One of them was the Measure 1.2 Environmental Protection Infrastructure. The main aim of this Measure was to improve the quality of the natural environment through: reduction of the amount of pollution emitted into the air, water, and soil; improvement of the flood control conditions; boosting the use of energy from renewable sources; and improvement of environmental management. The additional goals which resulted from the main one were: to raise the standard of living of inhabitants, to increase the investment and tourist attractiveness of the territorial units, and to achieve the environmental standards included in the environmental Directives implemented in Polish law [10].

Within this measure support was provided for infrastructure projects concerned:

- water supply, water intake, and wastewater treatment,
- waste management,
- improvement of the quality of air,
- flood control,
- use of renewable sources of energy

with total value between 4 and 40 million PLN (projects with total cost over 40 million were co-financed from the Cohesion Fund, while these with total value below 4 million were implemented within the IROP Priority 3 Local Development), as well as for projects related to environmental protection management with minimum total value of 1,2 million PLN [9].

Final Beneficiaries of the Measure were: territorial self-government units, their unions, alliances and associations, entities rendering public services in which majority of shares or stock is owned by territorial self-government unit, entities chosen in a public tender procedure conducting public services on the basis of an agreement with territorial self-government unit on providing services in the field of environmental protection, administrative government units in voivodship, and units of the public finance sector [10].

Maximum share of ERDF funds in eligible costs of the project amounted to 75%. In the case of investments generating substantial net profit, supplementary financing from the European Union could not exceed 50% while in the case of projects to which the rules of granting the State Aid were applied – 35%.

The content-related appraisal of the projects was made according to the following requirements [9].

Table 1

Preliminary evaluation

No.	Criteria	Evaluation	
1	Properly prepared economic analysis of the project	YES	NO
2	Properly prepared financial analysis of the project	YES	NO
3	Legitimacy of the technical solutions	YES	NO
4	Legitimacy and suitable amount of eligible costs	YES	NO
5	Cohesion of the information included in the application and in the annexes	YES	NO

Table 2

Substantial and technical criteria

No.	Criteria	Weight	Maximum result
1	Impact on realization of accession commitments in the area of environmental protection	4	16
2	Complementarity with other projects (especially with projects funded by the IROP or Cohesion Fund)	1	4
3	Long-term sustainability and institutional feasibility of the project (guarantees the financial stability of the project and sustainability of and institutional structure)	2	8
4	Cost effectiveness of the project	4	16
5	Correctness of indicators	1	4
6	Technical feasibility	1	4
7	Comprehensive projects including projects implemented jointly by more than one self-government unit	2	8
TOTAL			60

5. MULTICRITERIA AGGREGATION PROCEDURE

The possibility of ranking the projects with help of arithmetic mean of the weighted scores given by the members of the Panel of Experts seems somewhat illusory, especially in view of uncertainty, inaccuracy, instability, and indefiniteness characteristic for decision-making problems.

An interesting alternative is the approach based on the outranking relations and on the fundamental partial comparability axiom in which incomparability plays a key role. It introduces indifference thresholds and preference thresholds in order to build outranking relations that represent decision--makers' preferences and constitute partial relations of the global preferences. In this kind of approach there is place for incomparability, explained e.g. by the lack of sufficient information to define preferential situation [8]. The procedures exploited according to this approach are usually less demanding for their users at the informational level and result in more balanced recommendations than those belonging to the first approach of a single criterion synthesis [1]. They can definitely improve the procedure of appraising and selecting projects applying for co-financing from the European Union.

6. APPLICATION OF THE ELECTRE METHODS TO THE PROJECT SELECTION

Among the procedures based on outranking relations the ELECTRE methods originated by B. Roy and his co-workers stand out[4]. Their usefulness for decision aiding process connected with selection of the projects applying for the support from the European Union Structural Funds will be illustrated by a real-life example of the applications reported in the Measure 1.2 in one of the voivodships in the programming period 2004-2006.

Seven infrastructure projects are considered. All of them deal with the surface water protection and include construction and modernization of wastewater and rainwater collection networks and wastewater treatment plants. Table 3 provides the performance matrix for these seven projects and nine criteria used to evaluate them.

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[4] The precise description of these methods can be found in Roy and Bouyssou [6]. The methodology of multicriteria decision aiding is presented in [5].

Table 3

Values of the criteria for projects

Criteria	Projects						
	A	**B**	**C**	**D**	**E**	**F**	**G**
Capital input [million PLN]	8,42	9,24	9,25	5,93	20,0	26,01	31,55
Net Present Value Ratio	0,0012	0,0003	-0,4621	-0,1825	-0,7935	-0,1208	0,1871
Average annual result measure (direct ecological effect) [g per year]	82 601	373 194	143 036	48 229	183 300	220 424	205 874
Cost efficiency indicator	0,3942	0,5758	0,8828	0,2355	0,2640	0,3206	0,1427
Additional employment [PLN per year]	2 052	2 253	11 484	30 100	129 642	13 853	3 562
Health benefits [PLN per year]	4 564	0	203 577	6 500	0	0	0
Influence on investment attractiveness [PLN per year]	6 735	6 580	179 225	2 000	856 019	0	0
Influence on tourist attractiveness [PLN per year]	20 205	248 312	0	15 000	0	0	0
Number of people using the project	2 000	9 128	1 550	582	3 900	2782	4784

The thresholds and weights are defined in Table 4.

Table 4

Decision-maker's model of preferences

Criteria	Preference's direction	Coefficients of importance	Indifference threshold	Preference threshold	Veto threshold
Capital input	min	0,125	1	3	20
Net Present Value Ratio	max	0,125	0,03	0,15	0,8
Direct ecological effect	max	0,225	10 000	30000	300 000
Cost efficiency indicator	max	0,175	0,03	0,1	0,7
Additional employment	max	0,0625	500	2000	250 000
Health benefits	max	0,0625	500	2000	250 000
Influence on investment attractiveness	max	0,0625	1 000	5000	900 000
Influence on tourist attractiveness	max	0,0625	1 000	5000	900 000
Number of persons using the project	max	0,1	100	500	10000

At the beginning, the ELECTRE I method was used for selecting environmental infrastructure projects[5].

Tables 5 and 6 present the complete concordance matrix and the discordance set.

Table 5

Matrix of concordance indexes

	A	B	C	D	E	F	G
A	1	0,375	0,4125	0,75	0,55	0,6125	0,4875
B	0,625	1	0,6375	0,75	0,875	0,9375	0,8125
C	0,5875	0,3625	1	0,625	0,55	0,4875	0,55
D	0,25	0,25	0,375	1	0,375	0,375	0,55
E	0,45	0,1875	0,5125	0,625	1	0,475	0,55
F	0,3875	0,125	0,575	0,625	0,65	1	0,775
G	0,5125	0,25	0,5125	0,45	0,575	0,4125	1

Table 6

Discordance set

	A	B	C	D	E	F	G
A	0	0	0	0	0	0	0
B	0	0	0	0	0	0	0
C	0	0	0	0	0	0	0
D	0	1	0	0	0	0	0
E	0	0	0	0	0	0	1
F	0	0		1	0	0	0
G	1	1	1	1	0	0	0

An outranking relation exists if the concordance and non-discordance conditions are fulfilled simultaneously. The table below presents the outranking relation for the concordance index not less than 0,6.

[5] The procedure is described in [2].

Table 7

Outranking relation for concordance index not less than s = 0,6

	A	B	C	D	E	F	G
A	1	0	0	1	0	1	0
B	1	1	1	1	1	1	1
C	0	0	1	1	0	0	0
D	0	0	0	1	0	0	0
E	0	0	0	1	1	0	0
F	0	0	0	0	1	1	1
G	0	0	0	0	0	0	1

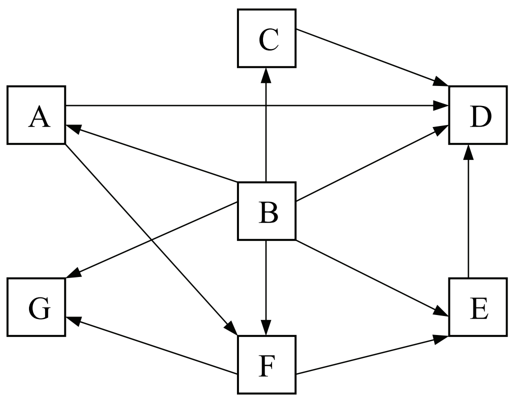

Fig. 1. Outranking relation for s = 0,6

Tables 8 and 9 show the situation after the increase of the concordance index to 0,625 and 0,65 respectively.

Table 8

Outranking relation for concordance index not less than s = 0,625

	A	B	C	D	E	F	G
A	1	0	0	1	0	0	0
B	1	1	1	1	1	1	1
C	0	0	1	1	0	0	0
D	0	0	0	1	0	0	0
E	0	0	0	1	1	0	0
F	0	0	0	0	1	1	1
G	0	0	0	0	0	0	1

Table 9

Outranking relation for concordance index not less than s = 0,65

	A	B	C	D	E	F	G
A	1	0	0	1	0	0	0
B	0	1	0	1	1	1	1
C	0	0	1	0	0	0	0
D	0	0	0	1	0	0	0
E	0	0	0	0	1	0	0
F	0	0	0	0	1	1	1
G	0	0	0	0	0	0	1

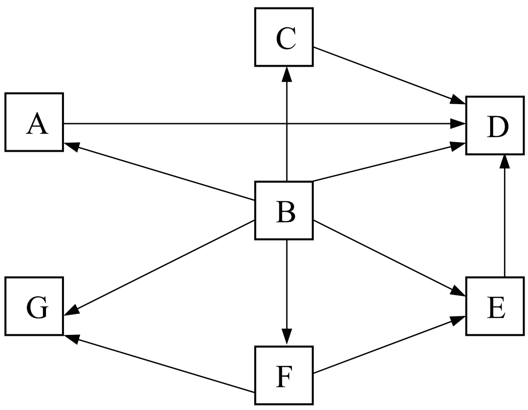

Fig. 2. Outranking relation for s = 0,625

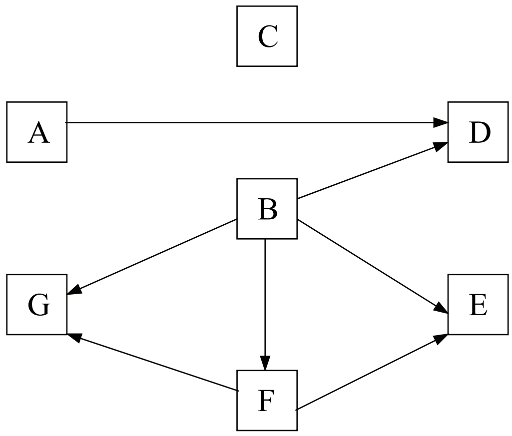

Fig. 3. Outranking relation for s = 0,65

Table 10

The results of graphs' analysis

Level	s = 0,6		s = 0,625		s = 0,65	
	From the best variant to the weakest one	From the weakest variant to the best one	From the best variant to the weakest one	From the weakest variant to the best one	From the best variant to the weakest one	From the weakest variant to the best one
1	B	B	B	B	A, B	B
2	A, C	A	A, C, F	F	D, F	A, F
3	F	F	E, G	A, C, E	E, G	D, E, G
4	E, G	C, E	D	D, G		
5	D	D, G				
					C – isolated	C – isolated

In all cases, project B turned out to be the best and should be recommended for co-financing. According to the results of the graph analysis, project A can also be worth considering as it entered the highest level when the concordance index amounted to 0,65. On the other hand, projects D and G were placed on the lowest level in almost all cases which leads to the conclusion that these are the weakest solutions. Furthermore, taking into account the case of s = 0,65, project C can be regarded as weakly comparable.

Subsequently, ELECTRE III was applied to order the projects[6]. Table 11 presents the credibility matrix produced for the case study.

Table 11

Credibility matrix

	A	B	C	D	E	F	G
A	1,00	0,01	0,09	0,78	0,07	0,61	0,51
B	0,94	1,00	0,33	0,75	0,40	0,94	0,84
C	0,73	0,12	1,00	0,63	0,30	0,49	0,28
D	0,25	0,00	0,02	1,00	0,06	0,48	0,55
E	0,01	0,00	0,05	0,45	1,00	0,27	0,00
F	0,13	0,01	0,06	0,00	0,09	1,00	0,78
G	0,00	0,00	0,00	0,00	0,07	0,59	1,00

In the next step, two preorders using the descending and ascending distillation were constructed on the basis of the credibility matrix. They are shown in Table 12 together with the final order.

Table 12

Preorders and the final ranking

Descending distillation		Ascending distillation		Final ranking	
class	projects	class	projects	class	projects
1	**B**	6	**B, C**	1	**B**
2	**C**	5	**E**	2	**C**
3	**A**	4	**D**	3	**A, E**
4	**F**	3	**A**	4	**F, D**
5	**E**	2	**F**	5	**G**
6	**D**	1	**G**		
7	**G**				

[6] The description of the procedure can be found in [2].

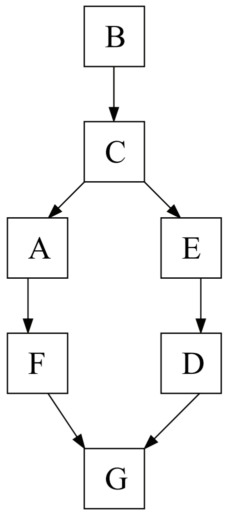

Fig. 4. Final ranking

According to the analysis, project B turned out again to be the strongest and project C was second-strongest. Project G turned out to be the worst variant and could be excluded from the further analysis. Additionally, it is worth mentioning that projects A and F are incomparable with projects E and D.

CONCLUSIONS

In reality, projects A and C were selected for co-financing as project B did not fulfil one of the formal requirements. Under these circumstances the analysis conducted in the paper proved that the ELECTRE methods can be used for solving the problem of selecting or ordering projects applying for the support from EU Structural Funds. Moreover, their application can enhance the appraisal procedure and improve the decision-making process.

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Josef Jablonský

A SPREADSHEET-BASED SYSTEM FOR DEA MODELS*

Abstract

DEA models are tools for evaluation of efficiency and performance of decision making units. They are based on the definition of efficiency as the ratio of the sum of outputs produced by the unit divided by the sum of inputs spent in the production process. A standard LP solver is the only requirement for solving DEA models. Unfortunately specialized optimization packages are not available to typical users. In this case it is possible to use the built-in MS Excel solver. This solver has many limitations, but it is usually a sufficient tool for DEA models. The paper describes an original MS Excel add-on application that offers a simple tool for solving several standard DEA models. This application includes basic envelopment models with constant and variable returns to scale, SBM models, models with undesirable inputs and outputs and models with uncontrollable input and outputs. The application allows to calculate super-efficiency measures for most of the models mentioned. The functionality and main features of the system are illustrated by a simple case study – an evaluation of performance of pension funds in the Czech Republic.

Keywords

Data envelopment analysis, efficiency, Excel, spreadsheets.

INTRODUCTION

Data envelopment analysis (DEA) models are widely used as a tool for evaluation of efficiency, performance or productivity of homogenous decision making units, i.e. units that produce several identical or equivalent effects. These effects can be denoted as the outputs of the decision making units. We consider positive outputs of the unit, i.e. such that their higher values lead (assuming that other characteristics are unchanged) to higher performance

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of the unit. To obtain the outputs the decision making units require several inputs that are usually minimized, i.e. their lower values lead to higher performance of the unit. Assuming the simplest case – one input and one output – the performance of the units can be simply expressed as the ratio:

$$\frac{\text{output}}{\text{input}}$$

In such a case we can receive many different financial characteristics with data that can be taken from financial statements of the evaluated unit. These simple ratio characteristics do not correspond to each other. That is why for the evaluation of the overall efficiency of the decision making unit it is necessary to take into account several inputs and outputs simultaneously.

Let us consider the set of homogenous units $U_1, U_2, \ldots, U_n$ described by r outputs and m inputs. Let us denote by $\mathbf{X} = \{x_{ij}, i = 1, 2, \ldots, m, \; j = 1, 2, ..., n\}$ the matrix of inputs and $\mathbf{Y} = \{y_{kj}, k = 1, 2, \ldots, r, \; j = 1, 2, ..., n\}$ the matrix of outputs. In general, the measure of efficiency of the unit U_q can be expressed as:

$$\frac{\text{weighted sum of outputs}}{\text{weighted sum of inputs}} = \frac{\sum_k u_k y_{kq}}{\sum_j v_j x_{jq}}$$

where v_j, $j = 1, 2, ..., m$ is the weight assigned to the j-th input and u_k, $k = 1, 2, ..., r$ is the weight of the k-th output. The evaluation of the efficiency of the unit U_q by a DEA model consists in maximization of its efficiency score under the constraints that the efficiency scores of all other units cannot be greater than 1 (100%). The weights of all inputs and outputs have to be greater than zero in order for the model to include all the characteristics. Such a model can be formulated as follows:

$$\text{maximize} \quad \frac{\sum_k u_k y_{kq}}{\sum_j v_j x_{jq}}$$

$$\begin{aligned} \text{subject to} \quad & \frac{\sum_k u_k y_{kp}}{\sum_j v_j x_{jp}} \le 1, && p = 1,2,\ldots,n \\ & u_i \ge \varepsilon, && i = 1,2,\ldots, r \\ & v_j \ge \varepsilon, && j = 1,2,\ldots, m \end{aligned} \tag{1}$$

The model (1) is known as a primal CCR (Charnes, Cooper, Rhodes) model. From the computational point of view it can be more efficient to work with the dual formulation:

$$\text{minimize} \qquad z = \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{k=1}^{r} s_k^+\right)$$

$$\text{subject to} \qquad \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{iq}, \qquad i = 1, 2, ..., m \tag{2}$$

$$\sum_{j=1}^{n} \lambda_j y_{kj} - s_k^+ = y_{kq}, \qquad k = 1, 2, ..., r$$

$$\lambda_j \geq 0,\ s_k^+ \geq 0,\ s_i^- \geq 0,$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2,\ldots, \lambda_n)$, $\boldsymbol{\lambda}\geq 0$, is the vector of weights assigned to the evaluated units, $\mathbf{s}^+$ and $\mathbf{s}^-$ are vectors of positive and negative slacks in input and output constraints, ε is an infinitesimal constant and θ is a scalar variable expressing the reduction rate of inputs in order to reach the efficient frontier. The unit U_q is efficient if the following two conditions hold:

1. The optimum value of the variable θ^* is equal to 1.
2. The optimum values of all slacks $\mathbf{s}^+$ and $\mathbf{s}^-$ is equal to zero.

Apart from the information about the level of efficiency – the efficiency score θ * – the DEA models compute inputs and outputs of the so-called virtual units. This unit lies always on the efficient frontier and expresses how to improve inputs/outputs of the evaluated unit in order to reach the efficient frontier. The virtual units corresponding to the units identified as efficient by a DEA model are identical because the efficient units lie on the frontier. The virtual units corresponding to non-efficient units can be expressed in the case of the model (2) as follows:

$$x'_{iq} = \theta^* x_{iq} - s_i^-, \qquad i = 1, 2, ..., m$$

$$y'_{kq} = y_{kq} + s_k^+, \qquad k = 1, 2, ..., r$$

The variables $\mathbf{s}^+$, $\mathbf{s}^-$ are exactly the slack variables expressing the difference between virtual inputs/outputs and the appropriate inputs/outputs of the unit U_q.

The CCR model (2) assumes constant returns to scale (CRS). The modification of the CCR model taking into account variable returns to scale (VRS) can be derived from the model (2) by adding the convexity constraint $\mathbf{e}^T\boldsymbol{\lambda}=1$. Moreover, non-decreasing (NDRS) or non-increasing returns to scale (NIRS) can be considered by adding $\mathbf{e}^T\boldsymbol{\lambda} < 1$ or $\mathbf{e}^T\boldsymbol{\lambda} > 1$ respectively.

The model (2) is an input oriented DEA model, i.e. the aim of this model is to find how to reduce the inputs of non-efficient units in order to reach the efficient frontier. Similarly, it is possible to formulate an output oriented model. The basic modifications of the model (2) are given in the following list:

Input oriented models **Output oriented models** (3)

	Input oriented models		Output oriented models
min	$z = \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{k=1}^{r} s_k^+\right)$	max	$g = \phi + \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{k=1}^{r} s_k^+\right)$
st	$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{iq}$	st	$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{iq}$
	$\sum_{j=1}^{n} \lambda_j y_{kj} - s_k^+ = y_{kq}$		$\sum_{j=1}^{n} \lambda_j y_{kj} - s_k^+ = \phi y_{kq}$
	$\lambda_j \geq 0,\ s_k^+ \geq 0,\ s_i^- \geq 0$		$\lambda_j \geq 0,\ s_k^+ \geq 0,\ s_i^- \geq 0$
CRS	$\sum_{j=1}^{n} \lambda_j - \text{free}$	CRS	$\sum_{j=1}^{n} \lambda_j - \text{free}$
VRS	$\sum_{j=1}^{n} \lambda_j = 1$	VRS	$\sum_{j=1}^{n} \lambda_j = 1$
NDRS	$\sum_{j=1}^{n} \lambda_j < 1$	NDRS	$\sum_{j=1}^{n} \lambda_j < 1$
NIRS	$\sum_{j=1}^{n} \lambda_j > 1$	NIRS	$\sum_{j=1}^{n} \lambda_j > 1$

The DEA models can be classified from many points of view. The aim of this paper is not to describe in detail the modifications of the above formulated model (2), but to discuss the possibility of solving DEA models in spreadsheets and describe our original application for their solving.

1. SOLVING DEA MODELS IN SPREADSHEETS

The mathematical formulation of the DEA models shows that they can be solved as standard linear programming problems. The efficiency score for any of the decision making unit of the set of units is computed by solving one linear programming problem with (n+m+r+1) variables and (m+r) constraints. Even for a higher number of units (n) this is a low-sized LP problem that can be solved without difficulties by any of the professional optimization systems. In order to obtain the efficiency score for all the units the optimization problem of the mentioned size has to be solved n times. Problems with approximately one hundred units can be solved by means of any professional optimization systems in several seconds.

The built-in optimization solver in MS Excel is limited to problems with approx. 250 variables. This limit allows to solve DEA models (3) with n = 200 units and m = r = 20 inputs/outputs. The problem here is the necessity to repeat the optimization run n times in order to receive the appropriate results for all the units of the given set. That is why we decided to build an add-on application in the MS Excel environment that works with the internal MS Excel solver. In this way the system can be used on any computer with the MS Excels spreadsheet, i.e. on almost all computers. In this section we formulate the DEA models covered by the system and then describe how to work with this system.

The DEA Excel solver appears in the main Excel menu after its activation. The list of the available DEA models in the system is shown in Figure 1.

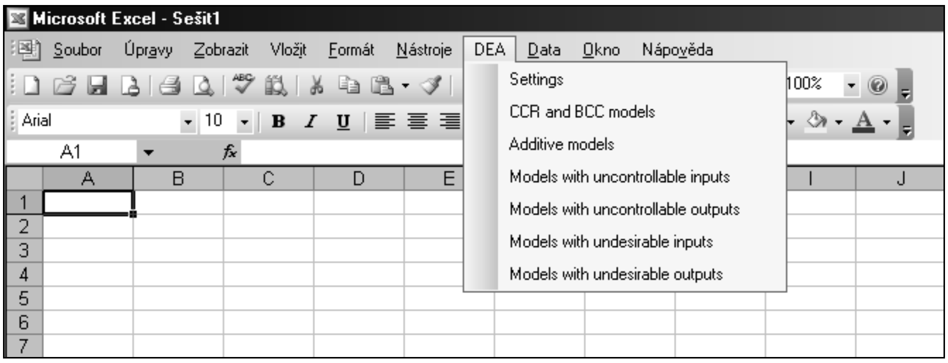


Fig. 1. The list of available DEA models

We will formulate them in detail in the following survey:

1. CCR and BCC models.
   They are often called envelopment models. These models were formulated in the introductory part of the paper – models (3).
2. Additive models.
   The additive models are often called SBM (slack based measure). This group of models measures the efficiency by means of slack variables only. In the application the following family of models in incorporated:

$$\max \quad z = \sum_{i=1}^{m} s_i^- + \sum_{k=1}^{r} s_k^+ \tag{4}$$

$$\text{st} \quad \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{iq}$$

$$\sum_{j=1}^{n} \lambda_j y_{kj} - s_k^+ = y_{kq}$$

$$\lambda_j \geq 0,\ s_k^+ \geq 0,\ s_i^- \geq 0$$

$$\text{CRS} \quad \sum_{j=1}^{n} \lambda_j - \text{free}$$

$$\text{VRS} \quad \sum_{j=1}^{n} \lambda_j = 1$$

$$\text{NDRS} \quad \sum_{j=1}^{n} \lambda_j < 1$$

$$\text{NIRS} \quad \sum_{j=1}^{n} \lambda_j > 1$$

   The objective function maximizes the sum of all slacks. Of course it is necessary to ensure the comparability of the inputs and outputs in this case. This can be simply done by normalization of all input and output values.
3. Models with uncontrollable inputs/outputs.
   In many applications some of the inputs or outputs cannot be directly controlled by the decision maker. In this case the uncontrollable characteristics have to be introduced into the model. The radial models (3) are modified as follows:

Input oriented models **Output oriented models** (5)

Input oriented models:

$$\min \quad z = \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{k=1}^{r} s_k^+\right)$$

st

$$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{iq}, \; i \in \mathrm{CI}$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{iq}, \quad i \notin \mathrm{CI}$$

$$\sum_{j=1}^{n} \lambda_j y_{kj} - s_k^+ = y_{kq}$$

$$\lambda_j \geq 0,\, s_k^+ \geq 0,\, s_i^- \geq 0$$

CRS $\sum_{j=1}^{n} \lambda_j$ – free

VRS $\sum_{j=1}^{n} \lambda_j = 1$

NDRS $\sum_{j=1}^{n} \lambda_j < 1$

NIRS $\sum_{j=1}^{n} \lambda_j > 1$

Output oriented models:

$$\max \quad g = \phi + \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{k=1}^{r} s_k^+\right)$$

st

$$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{iq}$$

$$\sum_{j=1}^{n} \lambda_j y_{kj} - s_k^+ = \phi y_{kq}, \; k \in \mathrm{CO}$$

$$\sum_{j=1}^{n} \lambda_j y_{kj} - s_k^+ = y_{kq}, \quad k \notin \mathrm{CO}$$

$$\lambda_j \geq 0,\, s_k^+ \geq 0,\, s_i^- \geq 0$$

CRS $\sum_{j=1}^{n} \lambda_j$ – free

VRS $\sum_{j=1}^{n} \lambda_j = 1$

NDRS $\sum_{j=1}^{n} \lambda_j < 1$

NIRS $\sum_{j=1}^{n} \lambda_j > 1$

where CI and CO is the set of indices of controllable inputs and outputs respectively. The family of models (5) formulated above is included in the DEA Excel solver.

4. Models with undesirable inputs/outputs.

   In typical cases inputs are to be minimized and outputs are to be maximized in DEA models, i.e. the lower value of inputs and the higher value of outputs lead to a higher efficiency score. It is not difficult to formulate a problem where some of the inputs and outputs will be of reverse nature. Such characteristics are denoted as undesirable inputs or outputs. Models with undesirable characteristics are included in the DEA Excel solver too. They are formulated as follows:

Input oriented (undesirable inputs)

$$
\begin{array}{ll}
\min & z = \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{k=1}^{r} s_k^+\right) \\
\text{st} & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{iq},\ i \in \text{DI} \\
 & \sum_{j=1}^{n} \lambda_j x'_{ij} + s_i^- = \theta x'_{iq},\ i \notin \text{DI} \\
 & \sum_{j=1}^{n} \lambda_j y_{kj} - s_k^+ = y_{kq} \\
 & \lambda_j \geq 0,\ s_k^+ \geq 0,\ s_i^- \geq 0 \\
\text{CRS} & \sum_{j=1}^{n} \lambda_j - \text{free} \\
\text{VRS} & \sum_{j=1}^{n} \lambda_j = 1 \\
\text{NDRS} & \sum_{j=1}^{n} \lambda_j < 1 \\
\text{NIRS} & \sum_{j=1}^{n} \lambda_j > 1
\end{array}
$$

Output oriented (undesirable outputs) (6)

$$
\begin{array}{ll}
\max & g = \phi + \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{k=1}^{r} s_k^+\right) \\
\text{st} & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{iq} \\
 & \sum_{j=1}^{n} \lambda_j y_{kj} - s_k^+ = \phi y_{kq},\ k \in \text{DO} \\
 & \sum_{j=1}^{n} \lambda_j y'_{kj} - s_k^+ = \phi y'_{kq},\ k \notin \text{DO} \\
 & \lambda_j \geq 0,\ s_k^+ \geq 0,\ s_i^- \geq 0 \\
\text{CRS} & \sum_{j=1}^{n} \lambda_j - \text{free} \\
\text{VRS} & \sum_{j=1}^{n} \lambda_j = 1 \\
\text{NDRS} & \sum_{j=1}^{n} \lambda_j < 1 \\
\text{NIRS} & \sum_{j=1}^{n} \lambda_j > 1
\end{array}
$$

where DI and DO are the set of indices of desirable inputs and outputs respectively and $x'_{ij} = x^{max}_{ij} - x_{ij}$, $i \notin$ DI, and $y'_{kj} = y^{max}_{kj} - y_{kj}$, $k \notin$ DO.

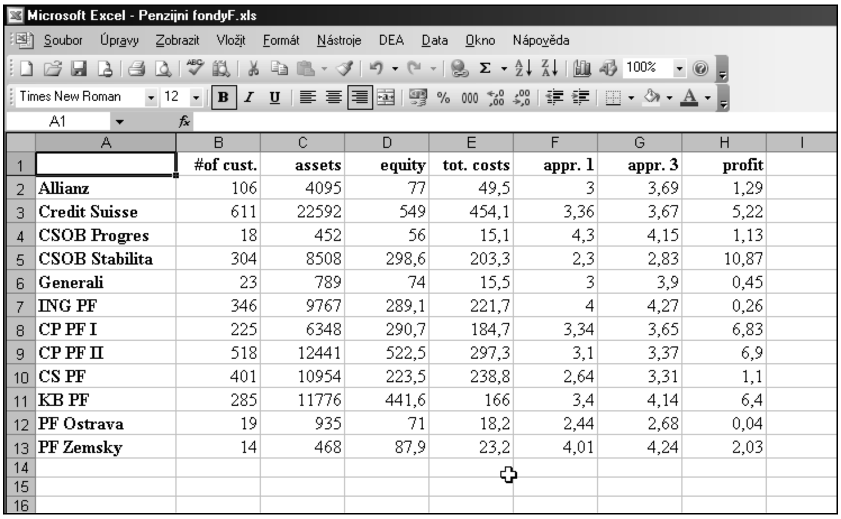
The efficiency score in the abovementioned DEA models (except additive models) is limited to 1 (100%). Depending on the selection of the DEA model and on the relation between the number of units on the one hand and the number of inputs and outputs on the other hand, the number of efficient units can be relatively high. That is why several definitions of super-efficiency were formulated in order to classify the efficient units. In super-efficiency models the efficiency score of inefficient units remains unchanged (lower than 1 in input oriented models) and the efficiency score of efficient units is higher than 1. In this way the model makes it possible to classify the efficient units – this can be one of the very important results of the analysis. The original super-efficiency model is the Andersen and Petersen model [1]. This model adds to the models (2), (3), (5), and (6) a new constraint $\lambda_q = 0$ (the weight of the evaluated unit is constant and equal to zero). This super-efficiency option can be used in the DEA Excel solver by checking the appropriate box. Nevertheless, it is necessary to mention that a feasible (and optimum) solution of the super--efficiency models exists always exactly under the assumption of constant returns to scale.

2. USING DEA EXCEL SOLVER – A SIMPLE CASE STUDY

The work with the system will be demonstrated on a small numerical example with a real economic background. It is the problem of evaluating the efficiency of the available pension funds in the Czech Republic. We have worked with the data set for 12 pension funds, each of them characterized by the following seven criteria (the data are from the year 2003):

1. INP 1 – the number of customers [thousands],
2. INP 2 – total assets [mil. CZK],
3. INP 3 – equity capital [mil. CZK],
4. INP 4 – total costs [mil. CZK],
5. OUT 1 – appreciation of the customer deposits for the last year (2003) [%],
6. OUT 2 – average appreciation of the customer deposits for the last three years (2001-2003) [%],
7. OUT 3 – net profit [mil. CZK].

For the DEA analysis, the first four criteria were taken as inputs and the remaining ones as outputs of the model. The data set for evaluation is given in the spreadsheet in Figure 2.



	#of cust.	assets	equity	tot. costs	appr. 1	appr. 3	profit
Allianz	106	4095	77	49,5	3	3,69	1,29
Credit Suisse	611	22592	549	454,1	3,36	3,67	5,22
CSOB Progres	18	452	56	15,1	4,3	4,15	1,13
CSOB Stabilita	304	8508	298,6	203,3	2,3	2,83	10,87
Generali	23	789	74	15,5	3	3,9	0,45
ING PF	346	9767	289,1	221,7	4	4,27	0,26
CP PF I	225	6348	290,7	184,7	3,34	3,65	6,83
CP PF II	518	12441	522,5	297,3	3,1	3,37	6,9
CS PF	401	10954	223,5	238,8	2,64	3,31	1,1
KB PF	285	11776	441,6	166	3,4	4,14	6,4
PF Ostrava	19	935	71	18,2	2,44	2,68	0,04
PF Zemsky	14	468	87,9	23,2	4,01	4,24	2,03

Fig. 2. Evaluation of pension funds – the data set

The new DEA menu item in MS Excel contains just the *Settings* option and then the selection of the DEA model that will be used for analysis (Figure 1). The *Settings* item contains the possibility to specify several parameters of the system, but they need not be changed as they are set to their defaults:

- *Language* – one of the two available language versions of the system (English, Czech),
- *Tolerance* – a constant with the initial value 10^{-6} which is used for testing of zero variables values (MS Excel solver often returns values very close to zero instead of zeroes),
- *Title* – the text displayed in the header part of the output of results,
- *Epsilon* – a constant of infinitesimal value – an initial value is 10^{-8},
- *Normalization of input data* – a switch (on/off) with the initial value "on" which specifies whether the normalization of input data should be conducted (a transformation to a comparable scale).

Let us suppose that the decision maker wants to apply the radial model with variable returns to scale (BCC model). After the appropriate family of models is selected from the main menu – *CCR and BCC models* in this case – the dialog box appears and the user can insert all the necessary information. The dialog box contains the following items:

- *DMU's labels* – a range with labels of the evaluated units (not obligatory – when it is not specified the system works with the default labels DMU1, DMU2,...),
- *Input/output labels* – ranges with the labels of the inputs and outputs (not obligatory – when it is not specified the system works with the default labels INP1, INP2,... and OUT1, OUT2...),
- *Matrix of inputs and outputs* – two continuous ranges containing the matrix of inputs and the matrix of outputs – in our example it is the range B2:E13 for the inputs and F2:H13 for the outputs,
- *Model orientation* – one of the two choices: input- or output-oriented model,
- *Returns to scale* – one of the four choices: CRS, VRS, NIRS, NDRS,
- *Super-efficiency* – the switch that sets up the selection of the super--efficiency model,
- *Optimization in two steps* – the switch that specifies whether the optimization is realized in one or two steps (the first step is the optimization of the reduction variable θ or the expansion variable ϕ and the second is the maximization of the slack variables $\mathbf{s}^{+}$ and $\mathbf{s}^{-}$,

– *Detailed/brief output of results* – two choices that switch on/off a brief and/or detailed output of the results – for the results the system creates single sheets with output information.

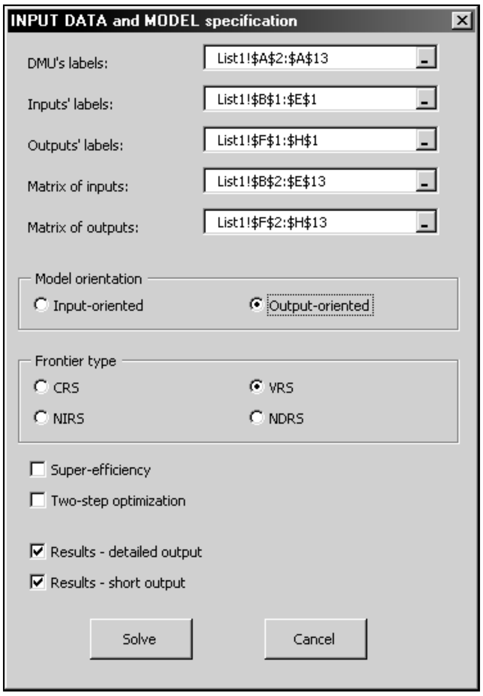


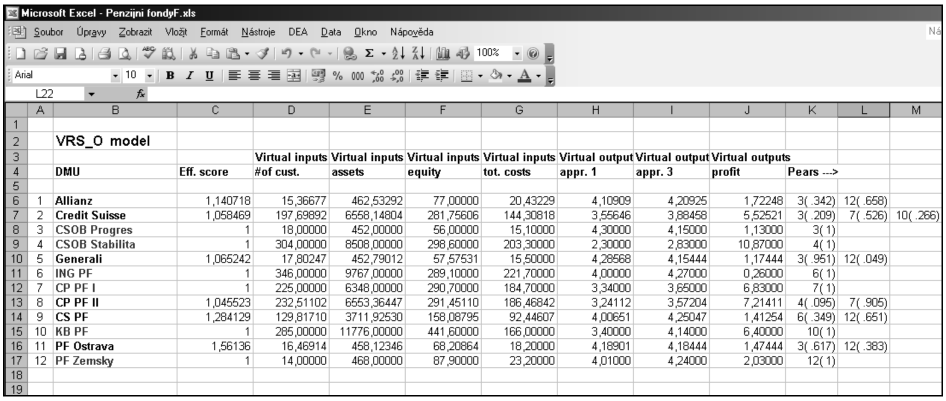
Fig. 3. Input data and model selection

The brief output information sheet for our example is presented in Figure 4. The sheet contains the following information:

1. The specification of the DEA model used in the analysis (VRS_O is an output oriented model with variable returns to scale).
2. The DMU labels and the efficiency scores computed by the model (the efficient units are marked by red colour).
3. The values of virtual inputs and outputs (target values for reaching the efficient frontier).
4. Non-zero weights of the units (the linear combination of units using these weights gives the virtual inputs and outputs).

Except this information the detailed output sheet contains the optimum values of slack variables and the original values of inputs and outputs.

The results of the BCC model for our example are given in Figure 4. It is clear that among twelve pension funds six were identified as efficient by the selected DEA model. The “worst” fund is PF Ostrava that has to improve its outputs by more than 56% in order to reach the efficient frontier. The brief output information sheet contains target values, i.e. the values of the input and output characteristics for improving the efficiency and reaching the efficient frontier.



VRS_O model

	DMU	Eff. score	Virtual inputs #of cust.	Virtual inputs assets	Virtual inputs equity	Virtual inputs tot. costs	Virtual output appr. 1	Virtual output appr. 3	Virtual outputs profit	Pears --->		
1	Allianz	1,140718	15,36677	462,53292	77,00000	20,43229	4,10909	4,20925	1,72248	3(.342)	12(.658)	
2	Credit Suisse	1,058469	197,69892	6558,14804	281,75606	144,30818	3,55646	3,88458	5,52521	3(.209)	7(.526)	10(.266)
3	CSOB Progres	1	18,00000	452,00000	56,00000	15,10000	4,30000	4,15000	1,13000	3(1)		
4	CSOB Stabilita	1	304,00000	8508,00000	298,60000	203,30000	2,30000	2,83000	10,87000	4(1)		
5	Generali	1,065242	17,80247	452,79012	57,57531	15,50000	4,28568	4,15444	1,17444	3(.951)	12(.049)	
6	ING PF	1	346,00000	9767,00000	289,10000	221,70000	4,00000	4,27000	0,26000	6(1)		
7	CP PF I	1	225,00000	6348,00000	290,70000	184,70000	3,34000	3,65000	6,83000	7(1)		
8	CP PF II	1,045523	232,51102	6553,36447	291,45110	186,46842	3,24112	3,57204	7,21411	4(.095)	7(.905)	
9	CS PF	1,284129	129,81710	3711,92530	158,08795	92,44607	4,00651	4,25047	1,41254	6(.349)	12(.651)	
10	KB PF	1	285,00000	11776,00000	441,60000	166,00000	3,40000	4,14000	6,40000	10(1)		
11	PF Ostrava	1,56136	16,46914	458,12346	68,20864	18,20000	4,18901	4,18444	1,47444	3(.617)	12(.383)	
12	PF Zemsky	1	14,00000	468,00000	87,90000	23,20000	4,01000	4,24000	2,03000	12(1)		

Fig. 4. A brief output information sheet

CONCLUSIONS

The DEA Excel Solver described in the previous sections can be downloaded from the download section of the web page http://nb.vse.cz/~jablon and used by any interested professionals. The application can solve problems up to 200 decision making units and 20 inputs and the same number of outputs. This size is sufficient for most of the real-world problems. The application will be extended in the future by other DEA models. The advantage of the system is that it does not assume any specialized software products except MS Excel including the built-in Excel optimization solver that is available on almost all computers. The functionality of the system was illustrated on a simple case study – the evaluation of the efficiency of pension funds in the Czech Republic. Even though this study contains only twelve decision making units, the system can solve any other problem (of limited size) within several seconds.

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IZAR – THE CONCEPT OF UNIVERSAL MULTICRITERIA DECISION SUPPORT SYSTEM

Abstract

Many real decision making problems are evaluated by multiple criteria. To apply appropriate multicriteria methods it is necessary to have software support. The paper presents the universal multicriteria decision support system IZAR. The main component of the system IZAR is an expert system helping the user with choosing the most suitable method for available information about the problem. IZAR not only suggests the most suitable method, but it also applies it immediately to the problem being solved. By means of properly formulated questions the expert system selects the right procedure for solving the problem, taking into account its peculiarities regarding entries and additional information which can be assigned to the system by the user. An appropriate classification of multicriteria models and methods is needed because of universality of the system. The system solves discrete and continuous problems. Basic models for multiattribute evaluation and multiobjective optimization problems are included. Methods for multiattribute evaluation problems are classified by means of types of preference information and by calculation principles. Preference information is given as aspiration levels, ordinal information, and cardinal information. Basic calculation principles are utility maximization, minimization of distance from the ideal alternative, and evaluation by preference relation. Methods for multiobjective optimization problems are classified by means of types of user information, as a priori information, a posteriori information, and progressive information. The system is available on web pages for all interested users and it can be also distributed on CD for users without Internet connection.

Keywords

Multiple criteria, multiattribute evaluation, multiobjective optimization, models, methods, decision support system.

INTRODUCTION

IZAR[1] is a non-commercial software package for students of decision theory. The final version of IZAR should be the universal system for the solution of discrete and continuous single objective and also multiobjective optimization problems. The first part of this software package is common background with an expert system of IZAR. This part has already been implemented and at present the authors focus on the second part of IZAR: continuous problems. The idea is concentrated on linear programming models, but in the future we will probably expand IZAR to include some non-linear methods. Also the third part – discrete problems – is now in the development phase.

The entire IZAR system is implemented in Smalltalk/X, a dialect Smalltalk-80. A Smalltalk/X virtual machine and runtime environment is available for both MS Windows and Linux, and also for FreeBSD.

The IZAR system is divided into three main parts:

– a core mathematical library consisting of a set of basic mathematical types and operations (matrixes and matrix operations such as matrix multiplication),
– a user interface for convenient communication with the user,
– a set of implemented methods.

All methods are implemented as external independent programme units that can be dynamically loaded to or unloaded from the running system. The methods are implemented in a slightly modified version of Pascal [4] which is interpreted by a specialized build-in interpreter. Pascal is also used as a scripting language, so the user can work with IZAR system non-interactively.

This design gives an opportunity to study and explore the IZAR system, especially by means of the implemented methods, by the user. This is one of the most important features of the system.

1. SINGLE OBJECTIVE CONTINUOUS PROBLEMS

The simplest problem that can be solved by IZAR is a continuous problem with the set of feasible solutions given by linear constraints in the form of inequalities and with only one objective function. The problem can be written as:

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[1] This project is supported by grant agency of Czech Republic GA ČR 402/07/0166.

Where *f(x)* and *g(x)* are linear functions, *n* is a number of variables and *m* is a number of constraints. Constraints can be written in the form of equations or inequalities. All constraints with non-negativity conditions comprise the feasible set.

Single objective problems can be solved by the simplex method. This method is implemented in IZAR and is used for searching for the optimum of minimum or maximum objective functions on the feasible set. In the case of constraints of the type "less than or equal to" (with non-negative right hand sides) one-phase simplex method is used. In the case of a feasible set with at least one constraint of the type "equal" or "greater than or equal to" twophase simplex method is used. Algorithms for both methods are described in [3] and are implemented in the modified simplex method form. For now, a multiplicative version of the simplex algorithm is used, but this can be changed to achieve better performance, since the simplex algorithm is a basic building block for most methods implemented.

IZAR solves the problem by this method; the result window shows results and simplex tables. All non-integer coefficients are presented in fraction form.

2. MULTIOBJECTIVE CONTINUOUS PROBLEMS

Multiobjective continuous problems differ from the single objective ones as regards the number of objective functions [1]. These functions can be minimal or maximal and the model can be written in the form:

Where *f(x)* and *g(x)* are linear functions, *n* is the number of variables, *m* is the number of constraints and *v* is the number of objective functions. As previously, constraints can be written in form of equation or inequality.

2.1. Data sources

We want IZAR to be a very friendly software package and consequently we focus on easy data input. The data can be loaded from a data file in CPLEX format (for single criteria problems) or in augmented CPLEX format for multicriteria problems (the format has to correspond to the model described in the second and third parts of this paper). Data can be also entered manually. The window for data dimension has four parts. The first window shows the name of the problem (named by default "New problem"). Figure 1 displays the problem of furniture factory that produces tables and chairs; time, wood, and chrome are needed for production. The factory maximizes profit and minimizes total costs at the same time. The second part of the window deals with variables. The user can change the number of variables and their names. The default names of variables are x1, x2, etc. The third part makes it possible to type in the number of constraints and their names (named by default Constraint 1, Constraint 2, etc.). The last part deals with objectives functions. The user can change their number and names. Note that the numbers of variables, constraints, and objective functions are unbounded. They are limited only by the computational power of CPU and the available memory.

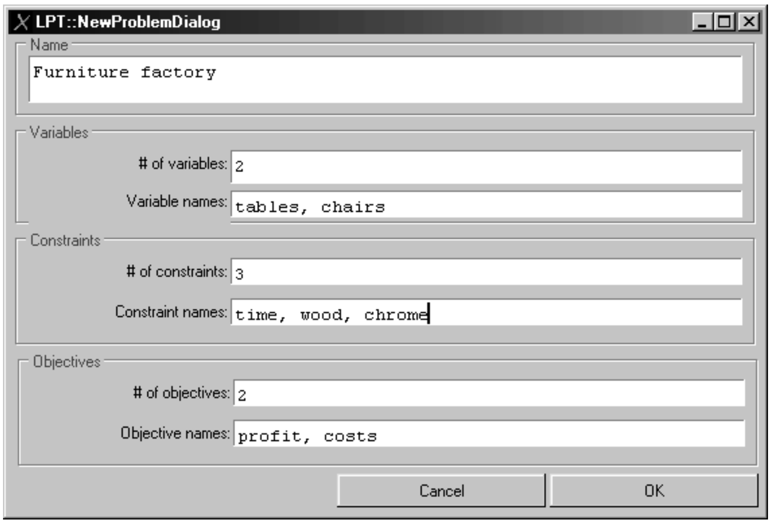

Fig. 1. Problem dimension

Next window (Figure 2) shows the subwindow for data submission. The upper part of the window is designed to input objective functions. The user should enter the prices and types of the extremes of objective functions. The next subwindow is intended for structural coefficients and right hand sides of constraints described previously. Here the user can change the sign of inequalities. Below is a place for method selection. A menu lists the methods; there is also a possibility of selecting an expert system for automatic selection of method. When the method is selected, IZAR finds a compromise solution and displays the results.

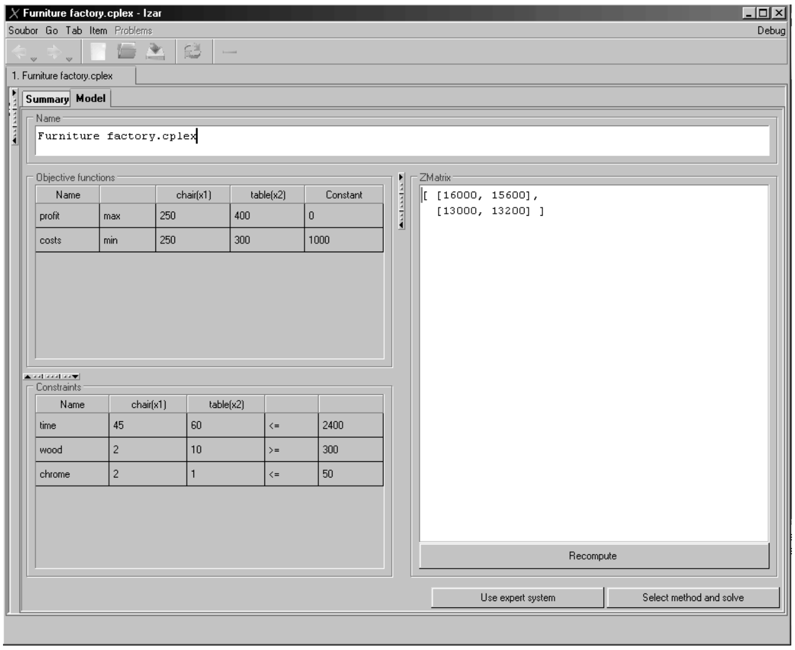

Fig. 2. Selection of functions and method (or expert system)

2.2. The expert system of IZAR for continuous problems

An expert system can be very useful. We propose a similar expert system for multiobjective optimization problems. The main menu for method selection offers eleven methods for these problems:

1) maximal utility method,
2) minimal distance from the ideal solution method,
3) lexicographic method,
4) goal programming,
5) maximal probability method,
6) minimal component method,
7) multicriteria simplex method,
8) GDF method,
9) Zionts-Wallenius method,
10) STEM,
11) Steuer method.

The main component of the IZAR system is an expert system helping the user with choosing the most suitable method for the available information about the problem. By means of properly specified questions the expert system selects the right procedure for solving the problem, taking into account its peculiarities regarding entries and additional information, which user can assign to the system. The choice of the suitable method is based on the following classifications and questions.

Methods are classified by means of setting user information:

- a priori information (methods 1-6),
- a posteriori information (method 7),
- progressive information (interactive methods 8-11).

The user chooses the way of specification of the importance of each criterion:

- weights,
- order of the criteria,
- goal (ideal) values.

Weights can be assessed:

- directly,
- by ordinal ordering of the criteria,
- by cardinal evaluation of criteria.

The information, which can affect the choice of the method is a calculation principle:

- maximization of the utility,
- minimization of the distance from goal (ideal) values.

The form of substitution information:

- explicitly expressed value of substitution (rates of substitution),
- implicitly expressed value of substitution.

IZAR not only suggests the most suitable method, but also applies it immediately to the problem being solved.

2.3. Compromise solution

Methods for multiobjective optimization problems are classified using types of user information. A priori information is given before the start of problem solving, a posteriori information is given after the computation, and progressive information is given during the calculation process.

After the method has been selected, the calculation process is started and additional information is required. In the case of maximal utility method and minimal component method weights are needed, while goal values have to be given for goal programming. Minimal distance from the ideal solution method works with both (weights and ideal values), but ideal value is calculated automatically. The order of the criteria is required for the lexicographic method, while for the maximal probability method and the multicriteria simplex method no information is needed. Progressive information is given in the case of all four interactive methods: GDF, Zionts-Wallenius, STEM, and Steuer methods.

Each method provides a compromise solution. The user can solve the same problem by another method or by the same method, but with different additional information. The user can also change the model or exit IZAR. The results of all methods are saved in a table for easy comparison (Figure 3).

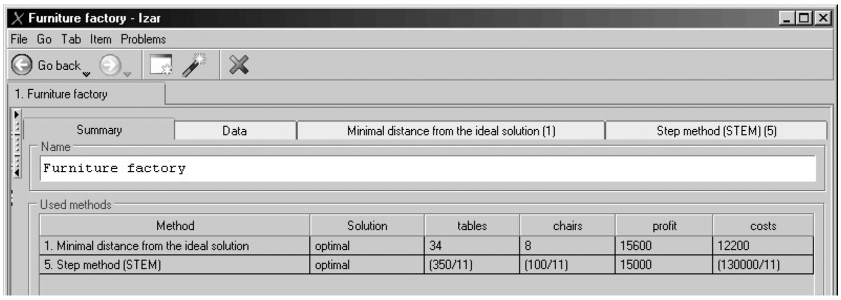

Fig. 3. Summary of results for comparison

3. MULTIATTRIBUTE EVALUATION (DISCRETE PROBLEMS)

The discrete problem with several criteria is given by the list of p alternatives, the list of k criteria, and also by the evaluation of each alternative by each criterion. Let a_i be the i-th alternative ($i = 1,2,\ldots,p$) and f_j be the j-th criterion ($j = 1,2,\ldots,k$). Then the evaluation of the alternative a_i by the criterion f_j is written as $f_j(a_i)$ or shortly as y_{ij}. Each criterion f_j can be minimized or maximized. The user searches for the most appropriate alternative from the list of alternatives; evaluation is based on the values y_{ij}. These values can be represented by the criteria matrix Y where each element y_{ij} denotes the evaluation of the alternative a_i by the criterion f_j for all $i = 1,2,\ldots,p$ and $j = 1,2,\ldots,k$.

3.1. Data sources

The data for discrete problems can be also loaded from a data file which is in a special IZAR format: *.idm (Izar Discrete Model). Data can be also entered manually. The window for data dimension (see Figure 4) displays the name of the problem, the number of criteria and their names, and the number and names of alternatives. All these information can be changed in the same way as in the continuous case. Note that the numbers of criteria and of alternatives are unbounded. They are limited only by computational power of CPU and the available memory.

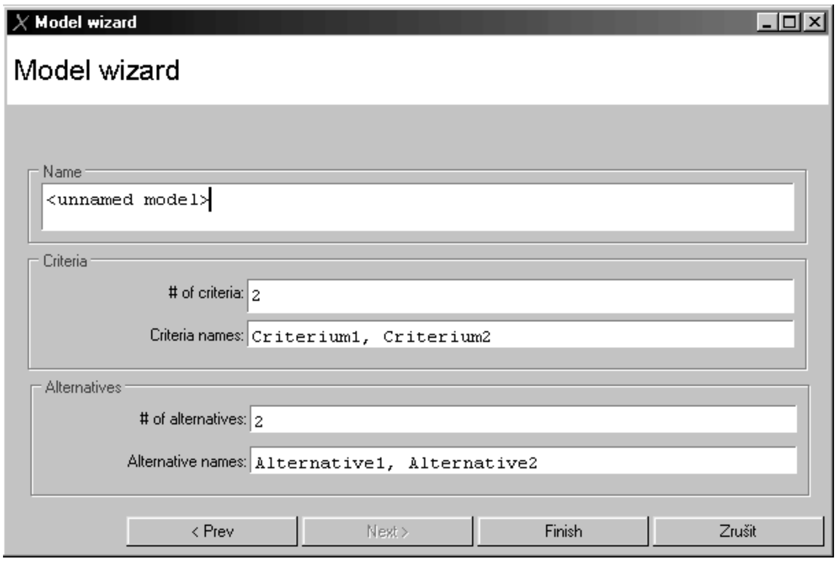

Fig. 4. Problem dimension for discrete problems

The next window (Figure 5) shows the subwindow for data input. The user should enter the evaluation of each alternative by each criterion. Below is a box for method selection. The methods are listed in the menu together with the possibility to choose an expert system for the automatic selection of method. Next, IZAR finds a compromise alternative and displays the results.

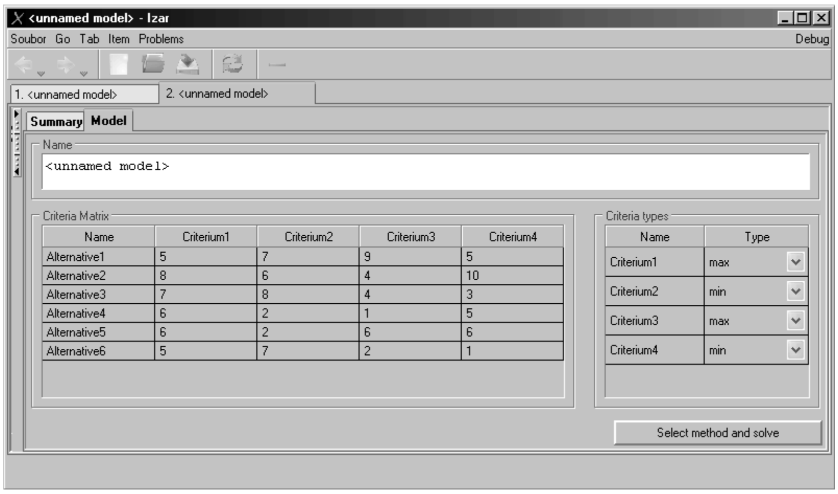

Fig. 5. Evaluation and method (or expert system) selection

3.2. Expert system of IZAR for discrete problems

An expert system for discrete problems is similar to that for continuous problems and was proposed in [2]. The choice of the suitable method is based on the following classifications and questions.

Methods are classified by the goal of multicriteria analysis:

– selection of the set of “good” alternatives,
– selection of the best alternative,
– ranking of all alternatives.

The user chooses the manner of specification of the importance of each criterion:

– weights,
– order of the criteria,
– aspiration levels.

Weights can be assessed:

– directly,
– by ordinal ordering of the criteria,
– by cardinal evaluation of the criteria.

The information which can be assigned to the alternatives:

– criteria matrix values,
– the order of the alternatives according to a single criterion,
– pairwise preferences of the alternatives according to a single criterion.

The pairwise preferences are of two types:

- the ordinal preferences of the alternatives,
- the cardinal preferences of the alternatives.

The information which can affect the choice of the method is a calculation principle:

1. Maximization of the utility – this principle is based on measuring the utility which is related to the alternatives regarding each of criterion; for this purpose it is necessary to carry out the transformation of the original data by:
   - linear normalization – simple scalarization of the multicriteria task where the transformed data resembles normalized values,
   - utility function – piecewise linear utility function can be constructed.
2. Minimization of the distance from the ideal criteria values – this principle is based on the choice of an alternative with the lowest distance from the ideal alternative according to various metrics.
3. Preference relation – relation between two alternatives, taking into account values reached in corresponding criteria.

IZAR not only suggests the most suitable method for discrete problems, but it also applies it immediately to the problem being solved. For users advanced in multicriteria analysis, to the extent that they do not need an expert system, there exists a possibility of the direct choice of the solving method.

7.3. Compromise alternative

After the method has been chosen, the calculation process is started and additional information is required. Aspiration levels are needed for Disjunctive method, Conjunctive method, and PRIAM. The Permutation, Lexicographic, and ORESTE methods need the order of criteria (or alternatives, respectively). WSA, UFA, and AHP are based on the maximization of the utility and they need cardinal information of criteria as well as TOPSIS (minimizing distance from the ideal values). Finally, AGREPREF, ELECTRE, PROMETHEE, and MAPPAC are based on the preference relations.

The Disjunctive and Conjunctive methods provide a set of "good" alternatives, while PRIAM displays the best alternative. The Permutation method, Lexicographic method, ORESTE, WSA, UFA, AHP, TOPSIS, AGREPREF, ELECTRE III, PROMETHEE I, II, and MAPPAC provide complete ranking of the set of all alternatives and they can be used for the choice of the best alternative. ELECTRE I provides a list of efficient and inefficient alternatives.

In the case of methods with aspiration levels, IZAR needs a criteria matrix and aspiration levels (repeatedly). Methods with ordinal information use a criteria matrix or order of alternatives according to each criterion and ordinal information of criteria. Methods with cardinal information need a criteria matrix and weights of criteria.

Each method provides a compromise alternative or complete ranking of the set of all alternatives. Obviously, the user can solve the same problem by another method or by the same method, but with different additional information (such as weights, aspiration levels, etc.). The user can also change the model or exit IZAR. The results of all methods are saved in a table for easy comparison, as in the case of continuous problems.

CONCLUSION

The IZAR system is in a development stage. The continuous problems are implemented and in the near future this part of the IZAR system will be tested at Department of Econometrics in the Economic University in Prague, as part of the course in Decision Theory.

The discrete problems are prepared for implementation and now six methods (Lexicographic, TOPSIS, WSA, AGREPREF, ELECTRE I, and ELECTRE III) are included in IZAR and tested. Each discrete problem is represented by a list of alternatives, a list of criteria, and a matrix with evaluation of alternatives by criteria.

The system was created for modeling and analyzing multiobjective combinatorial auctions (see Fiala's paper in this volume), but its utilization is more general: it can be used to solve many types of multicriteria decision making problems.

The IZAR system has several important advantages. The first is the fact that IZAR is a non-commercial software package for students and its design gives the user a possibility to study and extend the IZAR system. A user with elementary knowledge of the Pascal language can read the programme code for each method and study the method in detail, and a experienced one in Pascal programming can implement his own methods or improve the existing components of the IZAR system.

The second advantage is its unlimited number precision. Most of currently available software packages represent numbers (i.e. value of variables, coefficients, constants, etc.) as floating-point numbers because floating-point arithmetic is quite fast (the majority of operations can be done in hardware). Although floating-point arithmetic is fast and quite easy, it introduces almost unsolvable problems with correctness of results because it is subject to rounding errors. For that reason IZAR represents all numbers as fractions, i.e. as pairs of two (unlimited) integers – nominator and denominator, and so it is possible to represent any rational number within the IZAR system without any precision lost. The usage of fractions is transparent to the user. The main disadvantage of the fraction number representation approach is its computational and memory complexity – the solution of large problems is quite slow. But the CPUs become faster and faster and the authors believe that it is much more important to obtain correct results after a (possibly) long time than to obtain a (possibly) imperfect result in a short time. However, the problem of solving large sets of large problems is solvable. The IZAR system provides non-interactive (batch) mode, in which no user interaction is needed. This is one solution of this problem. On the other hand, since all methods are written in the Pascal language, it is possible to compile one specialized programme using an ordinary Pascal compiler (for example) that produces highly optimized code for target platform using floating-point arithmetic. The resulting specialized programme will be much faster, but may produce incorrect results (because of rounding errors in floating-point arithmetic).

The third advantage is the unbounded dimension of problems that can be solved. Programmes for solving linear programming models are usually limited by the number of variables and the number of constraints. This programme has no limits. The user can solve problems with an arbitrary number of variables, constraint, and objective functions, limited by the computer capacity only.

The fourth advantage is the number of implemented methods. IZAR knows eleven methods for multiobjective optimization problems (they are named in Section 3.2). As it is mentioned above, the user can extend this system by introduction of additional methods. In the final version, methods for multiattribute evaluation problems will be implemented. If necessary, the system can be extended, for example, by data envelopment analysis (DEA models) or by other methods.

All methods are included in the expert system which chooses the most suitable method for the given problem. By asking specific questions the expert system selects the right procedure for the problem being solved and so the system can be used by users unfamiliar with decision theory.

Last, but not least, an advantage of the IZAR system is the existence of this system. Many systems for linear programming problems exist, but they are all focused on one objective function. The IZAR system is the first system for multiobjective optimization problems that is accessible to students of Czech universities. And not only to them: The final version placed on web sites will be open to all people interested in decision problems. At the moment, the current version of IZAR is available on http://moon.felk.cvut.cz/~vranyj1/wiki/doku.php?id=izar:download.

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ON STABILITY OF EDUCATIONAL RANKINGS

Abstract

The paper deals with the problem of decreasing the imprecision in multicriteria evaluation of objects. The evaluations of educational institutions presented in form of rankings are investigated. In this context the role of ranking and sources of vagueness are discussed. The general, mathematically based concept of stability of ranking is introduced and used to describe the stability properties of an educational ranking.

Keywords

Educational ranking, multicriteria evaluation, imprecision, stability.

INTRODUCTION

We consider an educational system as a set of elements (actors – households, constitutions, etc.) and relations (norms, property rights, dependence, etc). An educational system can be viewed as a complex of interrelated input-output subsystems. Each subsystem describes an actor on educational market. Inputs consist of information reflecting evaluations of tangible and intangible factors building actors' utility. This information influences costs (present or expected) and profits. We consider situations in which the actors face decision problems which can be framed as problems of choice. This view assumes that a subject is identified and that she or he can identify a set of actions (here, for simplicity, a finite set). The task is to identify the most suitable action in the given case (with suitability to be defined). Thus, we assume that each subject is able to identify a model enabling a comparison of actions and a selection of the final one. This model will be further called a *preference model*.

In the case of individuals, their choices can be explained using the concept of human capital [7]. Human capital influences individual utility which can be approximated by the private rate of return on investment in education. In this sample situation the task of the decision maker can be solved in two phases: first – the construction of a model of preference, and second – the use of this model to identify the solution. In a more mathematical setting this would mean the identification of the preference ρ in the product $X \times X$ (where X is the set of actions to be undertaken) and finding the maximal element of this set[1].

Other actors can be described similarly (although they consider different sets of options and preferences). The complexity of the system is related to the fact that the decisions of actors are interdependent. This interrelation can be illustrated by the following example. Educational demand (reflected in past households' decisions on types of studies) influence university decisions on design of educational. These offers in turn constrain the decision process of households in phase of description of households' options.

Knauff and Szapiro [10] consider three internal actors in an educational system: university management, candidates (households), and government. In the present consideration, we use more general setting and add also firms although in many educational processes firms are represented in models as external, exogenous subjects. Their objectives are summed up in Table 1 which extends the presentation introduced by Knauff and Szapiro [10].

Table 1 illustrates the overall use of ranking in an educational system from the macroeconomic point of view. In a microeconomic setting, the widespread use of rankings is even more convincing. The question arises: Is there a possibility to create a common methodology for rankings and to use this to optimize educational market decisions? Knauff and Szapiro [10] advocate negative answer to this question and recommend the use of a computer-based interactive decision supporting system assisting decision makers as a tool in flexible structuring of selection problems and manipulation with evaluation criteria following individual preference of different users of the system. In this paper we take a different perspective.

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[1] For definitions see e.g. [15, 24].

Table 1

Structure of goals of actors in general setting of educational system

Actor	Government	Households	Universities	Firms
Goal	Socially optimal outcomes of education	Optimal choice of the economic service	Efficient administration	Labor and intellectual assets
Tasks	– to provide funding – to allocate it – to inform – to participate in control	– to elicit preferences – to collect the data – to process information	– to include market orientation into managerial operation – to perform the comparative analyses	– co-funding – manifesting preferences
Tool	Ranking of educational institutions	Ranking of relevant universities	Ranking of competitors	Ranking of relevant universities
Criteria	– to reflect social preferences – to create a scheme for allocation of public funds addressed for universities	– to involve utility of a household – to aggregate the date – clarity of criteria and results	– to serve as reference in curricula design – to serve as reference in pricing programmes – to identify competitive (external) threats	– alumni (recruited staff) competence – innovations – training – expertize

We consider a situation where the use of a formalized, computer--supported procedure is not possible. In such a situation groups of experts are gathered (e.g. family in the case of households, ranking councils or committees in press rankings, etc., senate's or rector's committees at universities). They commonly agree on the sets of actions as well as on evaluation criteria, and individually evaluate actions with respect to the chosen criteria and then agree on the final evaluation which results in a choice. This process of compromising final evaluation can be formalized as an algorithm with use of weights and of maximization of weighted evaluations; this decision rule is used in the choice of the final decision. Even in this scenario the process of weight definition remains subject to non-algorithmic agreement. In the case of allocation of public funds such algorithms are designed by means of an extensive set of numerical indicators of effectiveness of educational institutions; see also [18] for more theoretical perspective. The values of an indicator is assigned to an object during a process which involves precise definitions of given indicators. However, values assigned to objects are frequently softened by averaging subjective scales or by introducing exceptions.

Algorithmic decision procedures can be used as a regulation tool. In the 1990s in Poland, an increase of the number of doctoral programmes offered by universities was a desirable target of the state policy which was achieved by including the number of Ph.D. students in the algorithm used to allocate state funding. The quick reaction of the market to this offer was also due to expectation of high return on education at this level. Not surprisingly, the number of Ph.D. skyrocketed and new programmes based on purely private funding were set up. The dynamics of this process depends heavily on the information processed by households, university managing teams, and regulating institutions.

In real-life situations, as in the example above, the processed information is incomplete in the sense that interrelated actors compare their options without knowledge of the preferences of the others. According to bounded rationality principles, see e.g. [16, 17], they simplify their description of the problem. One of the simplification strategies is related to the intuitive evaluation of the likelihood of occurrence of others choices. Such simplified evaluations are neither measured nor analytically evaluated.

The mechanisms described above lead to imprecision and may result in decisions not leading to an intended effect and thus resulting in ineffective allocation. In this paper we present a tool which can assist decision makers in the description of the range of consequences of imprecise evaluations.

The paper is organized as follows. In Section 1 typical educational rankings of importance for education are described as a background for Section 2, where we refine the remarks on subjectivity, uncertainty, and imprecision outlined above to justify the approach and the model presented in Section 3. Section 4 is focused on possible applications of the methodology introduced. The paper is concluded with remarks and references.

1. RANKINGS IN EDUCATION – THE RATIONALE AND SELECTED SCENARIOS

The relation of education to economic growth was a subject of interest of empirical economy as well as of theoretical studies. These analyses also take into account managerial perspective; for examples and reviews [13, 14, 19, 20]. Macroeconomic studies do not give clear explanation of interdependence of empirical values of variables which describe economic systems and economic growth. On the other hand, microeconomic approaches based on measurements of individual returns on educational investment do not take into account the public return on educational investment or the problem of availability of education services [5, 6, 12]. This raises the issue of public co-financing of individual educational services [4]. Public intervention in educational market influences its mechanism and requires a cautious evaluation of its impact. This turns us back to the first scenario: The public education funding allocation scenario (abbreviated further as the PEFA Scenario).

In the PEFA Scenario, we deal with the situation described partly in Introduction. In this scenario exists an administrator responsible for allocation of public funds to educational institutions according to a procedure based on an algorithm worked out by a group of experts and approved by political and social decision makers. The procedure consists of the following steps:

Step 1. The algorithm begins with the identification of a set of objects subject to evaluation and funding – this usually results from formal, legal regulations.

Step 2. The experts work out a clear understanding of educational system goals and of an implied understanding of educational effectiveness. In the next steps they construct elements of an operational procedure to evaluate the effectiveness of the system.

Step 3. The experts define a set of variables describing objects that can be measured and used to build effectiveness indicators (criteria).

Step 4. The experts define scales to be used in measurement.

Step 5. The experts recommend administrative routines to be used in the evaluation of the objects.

Step 6. Measurement of variable values.

Step 7. Data processing.

Step 8. Implementation.

In the next section this procedure will be discussed from the point of view of its reliability.

Algorithms evaluating the effectiveness of educational institutions take into account the evaluation of their academic record. The second scenario considered here – the scenario of evaluation of individual academic records (EIAR Scenario) – describes the evaluation procedure of individual academic records authored by university employees. This evaluation is based on rating articles and other research reports published in research periodicals. Rating systems used in evaluation of research use the procedure which is described below. Again, the procedure is to be worked out by a group of experts and, as previously, is to be approved by political and social decision makers. The procedure consists of the following steps:

Step 1. The experts identify research periodicals which are taken into consideration when publications are evaluated.

Step 2. The experts design a rating system for research periodicals.

Step 3. The experts define scales for rating classes.

Step 4. The experts use a system to classify periodicals in groups.

Steps 5-8 are analogous to those in the PEFA Scenario. Again, we postpone the discussion of reliability of this scenario to the next section.

Let us consider the third procedure in the rating research periodicals scenario (RRP scenario) which is crucial for EIAR Scenario and therefore also for PEFA Scenario. The rating of research periodicals is based on evaluation of their impact on the progress of the field of research. In different countries, rating classes are defined using diverse methods. They are, however, only different solution of the same problem – the problem of measurement of impact of periodicals. Usually, the number of citations is used in the measurement. In order to exclude bias, the number of citations is transformed, e.g. self citations are excluded. Another important measure relates the significance of a periodical to the number of rejection of submitted articles. Yet another important measure is related to the degree of rigor in internal procedures of acceptance of papers (e.g. blind refereeing, participation of local authors, competence of supervising committees). RRP Scenario is based on a procedure whose Steps 1-4 involve expert compromises on goals identification and method of measurement of achievement of these goals, while its Steps 5-8 follow the previous procedures and are to be approved by political and social decision makers (in this case, at the local level).

In education, other scenarios involving rankings are used. In particular, an extensive research literature deals with rankings of economics departments. Use of rankings is important for managerial reasons: they help to attract young researchers and to retain mature ones and are an important hint in solving problem of funding allocation. Rankings results build the reputation of university departments and institutes. An interesting survey of this literature was presented by Kalaitzidakis et al. [8, 9].

Iterative procedures aimed at reduction of bias impact in rankings of periodicals have been presented in research literature. The rankings' methodology in this area was originated by seminar work of Liebowitz and Palmer [11]. The procedure of evaluation in educational systems depends heavily on the system of evaluation of research outlined in Figure 2. The figure illustrates two facts. Firstly, it shows the lack of stimuli to refresh the initial pool of reference journals. Secondly, it reveals a concentration of talented researchers in best departments. These departments prove to attract researchers with the best performance measure (which is constructed using the defined pool of journals). Thus, departments are not encouraged to take risk to refresh this pool. Other departments are not sufficiently influential to do this.

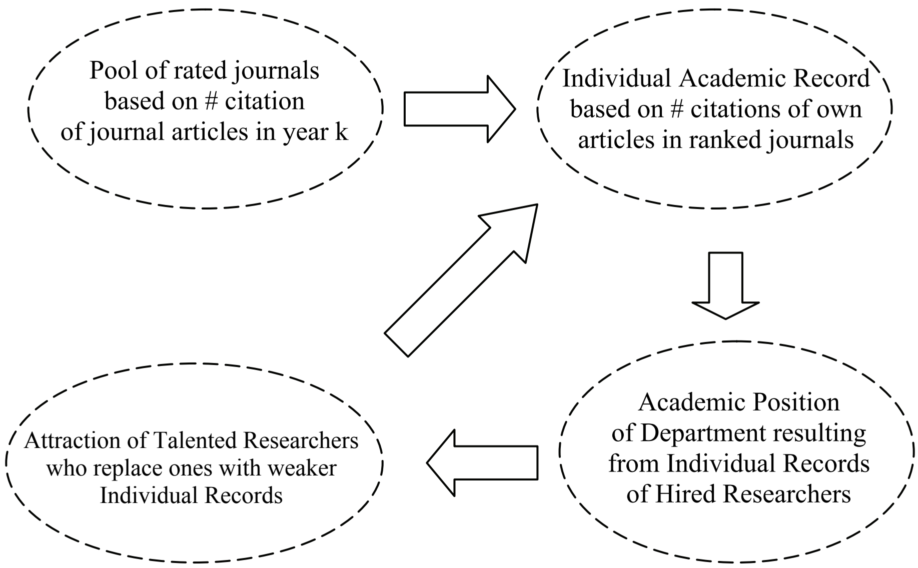

Fig. 1. Stability of traditional evaluations system

Adams et al. [1] investigate scientific influence using citations based on the data collected by the Institute for Scientific Information. They consider the top 110 US research university activities in nearly all branches of science in 1981-1999 (2.4 million papers and 18.8 million citations). Their results confirm "[…] that top institutions are more often cited by peer institutions then lower-ranked institutions are cited by their peers […]".

One of recent application of this approach together with a survey of results was presented by Amir and Knauff [2]. They rank economics departments worldwide based on the placement of their Ph.D. graduates at top-level economics departments instead of measuring the total productivity of the departments. Thus, they introduce a future orientation to the procedure of ranking of economics departments. Their results are obviously different from the earlier approaches and thus refresh results of previous rankings. However, they also show concentration of research.

As mentioned earlier, evaluation of research is an extremely important part of the evaluation of educational institutions. However, iterative analytical procedures are not used directly in such evaluations. Their results spread informally, but they form background knowledge for evaluations by experts.

A methodology (based on evaluation methods of performance in sports) of evaluation of efficiency of educational units was recently presented by Avery et al. [3]. The authors construct a ranking of US undergraduate programmes based on information about preferences of their best students. This evaluation does not use admission rate as a measure of attractiveness, since this could be manipulated by a college. Instead, the authors use independent college data, e.g. tuition discounts, alumni preferences, and use statistical inference in evaluation.

It is believed that any evaluation of teaching – programmes and methods – involves students' evaluation of their satisfaction. Weinberg et al. [2, 5] present a model used to identify the determinants of the evaluations (grades, learning measures). The model shows a weak awareness of learning effects and resulting bias.

Formal approaches are rarely used in analyses of educational processes. Complexity of these processes forces researchers to use advanced methods which are hard to communicate to wider audience. According to bounded rationality principles, precise formal descriptions and algorithms are replaced by simplification strategy and group evaluations. And so we return to rankings.

In the next section subjectivity, uncertainty, and imprecision of evaluation procedures in educational environment will be discussed in the case of PEFA, EAIR, and RRP scenarios.

2. SUBJECTIVITY, UNCERTAINTY, AND IMPRECISION IN EDUCATION

Let us now clarify the terminology. First, let us recall that we assume that solving decision problems requires its structuring. Thus, one copes with three partial problems: identification of the set of actions to be undertaken, description of one's own preferences, and finally, determination and implementation of the final decision. We consider problems of choice with the finite set of options assumed to be identified. It is assumed that the solution of the second task – the description of preferences – is solved through identification of goals of the decision maker and construction of respective set of criteria. By "criteria" we mean here the methods of evaluation of options (Figure 2).

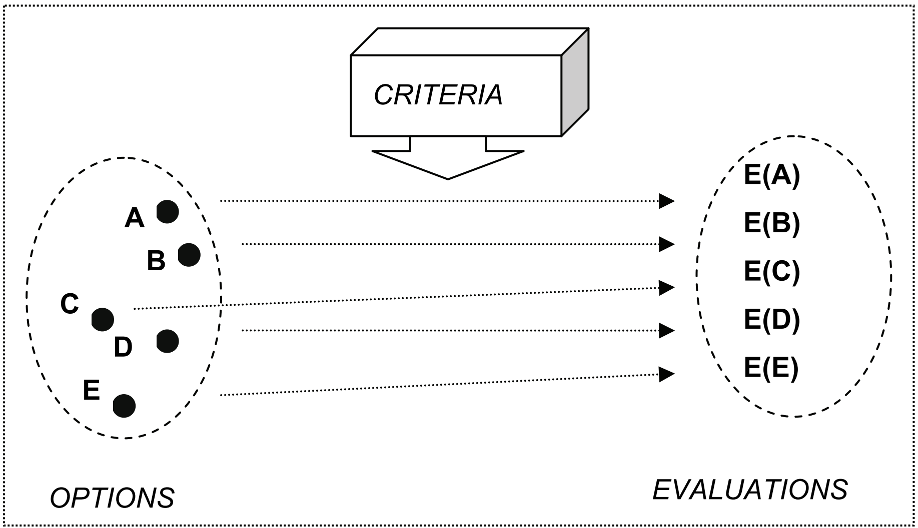

Fig. 2. The initial structure of a selection problem

In the next section we consider the situation in which evaluations are assumed to be numerical (ordered) and to measure the achievement of goals of the decision maker. In this phase of structuring, the decision space is represented by an m-dimensional (where m is the number of criteria used) vector space (of evaluations) with a quasi-order defined by single criteria values (in the weakly ordered space (R,≥)). The problem boils down to determination of non-dominated evaluation; the final solutions are options with best evaluations.

Subjectivity occurs when there is no method (unique and independent on the decision maker) to define an evaluation (in numerical cases: when there is no unique definition of criteria functions or numerical scales). *Uncertainty* occurs when an environmental variable determines a set of evaluations of an option, but the determination of the proper evaluation is not possible. A problem with uncertainty is transformed into a *risk problem* when the set of environmental values is supplied with the structure of a probability space (Ω, F, P), i.e. a measure space with a measure P that satisfies the probability axioms and the probability theory is used to describe uncertainty of outcomes of the option.

Imprecision occurs when the options or their evaluations can be described numerically only as subsets (usually: intervals). A problem with imprecision is transformed into a *fuzzy problem* when imprecision is described using fuzzy set theory.

Table 2 illustrates applications of these concepts for the PEFA, EAIR, and RRP scenarios presented in Section 2.

Table 2

Subjectivity, uncertainty/risk, and imprecision/fuzziness in educational scenarios

	Subjectivity	Uncertainty/risk	Imprecision/Fuzziness
PEFA Scenario	Different expert choices of institution effectiveness criteria	Different expert valuations with respect to agreed criteria	Quantification of evaluations on qualitative scales, use of evaluation dispersion
EAIR Scenario	Different expert choices of research performance criteria	Different expert valuations with respect to agreed criteria	Quantification of evaluations on qualitative scales, use of evaluation dispersion
RRP Scenario	Different expert choices of periodical impact criteria	Different expert valuations with respect to agreed criteria	Quantification of evaluations on qualitative scales, use of evaluation dispersion

Several sources of non-deterministic factors influencing evaluations have been discussed in research literature. For example, Kalaitzidakis et al. [9] recall that rankings of periodicals are based on evaluation of past achievements while they are intended to influence future actions. Also, different periods of evaluation may result in biased comparison of rankings' results. In most rankings new periodicals and innovative research stand at lost positions.

Sample situations in Table 2 show the necessity to use rankings in scenarios outlined in the previous section and lead to the question regarding the impact of subjectivity, uncertainty/risk, and imprecision/fuzziness in educational ranking procedures. In this paper, we attempt to investigate impact of imprecision of evaluation on the final ranking; it is called the *Preference Stability Problem Analysis*.

3. SUBJECTIVITY, UNCERTAINTY, AND IMPRECISION IN RANKING DECISIONS

In this section a basic analysis of rankings is presented and then the concept of rankings stability is introduced.

Let us consider the set $\mathbf{O}$ called in the sequel the set of *objects*. Every element $\mathbf{o} \in \mathbf{O}$ is called an *object*. Let us consider m objects $\mathbf{o}_i \in \mathbf{O}$, i = 1,..., m.

Let us also consider a mapping $\mathbf{c} : \mathbf{O} \rightarrow \mathbf{R}^{n+}$ which assigns a vector $\mathbf{c}(\mathbf{o})$, $\mathbf{c}(\mathbf{o}) = (c_1(\mathbf{o}), c_2(\mathbf{o}),..., c_n(\mathbf{o}))$, to each object $\mathbf{o} \in \mathbf{O}$. $\mathbf{R}^{n+}$ denotes a space of vectors with non-negative coordinates. Each vector $\mathbf{c}(\mathbf{o}) = (c_1, c_2,..., c_n)$ is called a *vector of characteristics* of an object $\mathbf{o}$. If $\mathbf{c}(\mathbf{o}) = \mathbf{x}$, we also say that $\mathbf{x}$ is (represents) an object $\mathbf{o}$. Therefore two objects with the same vector of characteristics are considered the same object. Let $(\mathbf{R}^n, < , >)$ be a Euclidean space with a standard scalar product $< , >$, $< , > : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}$.

Definition 3.1

Let $\mathbf{A} = \{\mathbf{a} \geq \mathbf{0} : <\mathbf{a}, \mathbf{1}> = 1\}$, where $\mathbf{1}$ is a unit vector in R^n. The set $\mathbf{A}$ is called a *set of weights*. The vectors $\mathbf{a} \in \mathbf{A}$ are called *weighting vectors*.

Weighting vectors have non-negative coordinates which sum up to 1.

Definition 3.2

A function $E : \mathbf{A} \times \mathbf{O} \rightarrow \mathbf{R}$ is said to be an *evaluation function*.

An evaluation function assigns to each object a value which depends on the weighting vector and on the object itself.

Throughout this paper, we assume that for every $\mathbf{a} \in \mathbf{A}$ and $\mathbf{o} \in \mathbf{O}$ the evaluation function is given by $E(\mathbf{a}, \mathbf{o}) = <\mathbf{a}, \mathbf{x}>$, where $\mathbf{x} = \mathbf{c}(\mathbf{o})$.

Definition 3.3

Let $\mathbf{a} \in \mathbf{A}$, $\mathbf{o}_1, \mathbf{o}_2 \in \mathbf{O}$ and let E, $E : \mathbf{A} \times \mathbf{O} \rightarrow \mathbf{R}$, be an evaluation function. Let $\mathbf{x} = \mathbf{c}(\mathbf{o}_1)$ and $\mathbf{y} = \mathbf{c}(\mathbf{o}_2)$. The relation ρ defined by the condition: $\mathbf{o}_1 \rho \mathbf{o}_2 \Leftrightarrow E(\mathbf{a}, \mathbf{o}_1) \geq E(\mathbf{a}, \mathbf{o}_2)$ is called a *preference*.

Definition 3.4

Let $\mathbf{o}_i \in \mathbf{O}$, $i = 1,...,m$, and let ρ be a preference. Given the set $\{\mathbf{o}_1, \mathbf{o}_2,\ldots, \mathbf{o}_m\}$ of m objects, an ordered set $R = (\{\mathbf{o}_1, \mathbf{o}_2,\ldots, \mathbf{o}_m\}, \rho)$ is said to be a *ranking* of objects $\mathbf{o}_i$, $i = 1,...,m$.

From Definition 3.3 it follows that the preference ρ is a *linear order*. A *strict* order induced by ρ will be denoted by ». Therefore $\mathbf{x}»\mathbf{y} \Leftrightarrow \mathbf{x}\rho\mathbf{y} \wedge \neg\mathbf{y}\rho\mathbf{x}$. If $\mathbf{x}»\mathbf{y}$ then $\mathbf{x}$ is preferred to $\mathbf{y}$. If $\mathbf{x}\rho\mathbf{y} \wedge \mathbf{y}\rho\mathbf{x}$, then $\mathbf{x}$ is equivalent to $\mathbf{y}$, and the notation $\mathbf{x}\approx\mathbf{y}$ is used in this case. The relation $\approx$ is called the *indifference*.

Let $\mathbf{x},\mathbf{y} \in \mathbf{R}^{n+}$ represent two objects and let $\mathbf{a} \in \mathbf{A}$ (i.e. $\mathbf{a}$ is a weighting vector). For every set $X \subset R^n$, the set $X^\perp$ is given by the following expression:

$$X^\perp = \{\mathbf{y}\in\mathbf{R}^n : \forall\mathbf{x}\in X <\mathbf{y},\mathbf{x}> = 0\}\equiv\{\mathbf{x}\}^\perp$$

Remark 3.5

The indifference is characterized by the following algebraic property:

$$\mathbf{x} \approx \mathbf{y} \Leftrightarrow \mathbf{y} - \mathbf{x} \in \{\mathbf{a}\}^\perp$$

Proof

From the definitions we have:

$$\mathbf{x} \approx \mathbf{y} \Leftrightarrow <\mathbf{a},\mathbf{y}> = <\mathbf{a},\mathbf{x}> \Leftrightarrow <\mathbf{a},\mathbf{y} - \mathbf{x}> = 0 \Leftrightarrow \mathbf{y}-\mathbf{x}\in \{\mathbf{a}\}^\perp$$

■

Remark 3.6

The indifference relation $\approx$ is an equivalence relation.

Proof

For all $\mathbf{x},\mathbf{y}\in\mathbf{R}^n$ we have the following implication:

$$<\mathbf{a},\mathbf{x}-\mathbf{x}> = 0 \Rightarrow \mathbf{x} - \mathbf{x}\in \{\mathbf{a}\}^\perp \Rightarrow \mathbf{x}\approx\mathbf{x}$$

Now, if $\mathbf{x} \approx \mathbf{y}$, then $\mathbf{y} - \mathbf{x}\in \{\mathbf{a}\}^\perp$, then $\mathbf{x} - \mathbf{y} = -(\mathbf{y} - \mathbf{x})\in \{\mathbf{a}\}^\perp$, hence $\mathbf{y} \approx \mathbf{x}$.

Finally, if $\mathbf{x} \approx \mathbf{y}$ and $\mathbf{y} \approx \mathbf{z}$, then $\mathbf{x} - \mathbf{y}\in \{\mathbf{a}\}^\perp$ and $\mathbf{y} - \mathbf{z}\in \{\mathbf{a}\}^\perp$ then $\mathbf{x} - \mathbf{z} = \mathbf{x} - \mathbf{y} + \mathbf{y} - \mathbf{z}\in \{\mathbf{a}\}^\perp$, hence $\mathbf{x} \approx \mathbf{z}$.

Since $\{\mathbf{a}\}^\perp$ is a vector space, then for every $\mathbf{x}\in R^{n+}$ the affine space $W_\mathbf{x}$,

$$W_\mathbf{x} \equiv \{\mathbf{x}+\alpha : \alpha\in \{\mathbf{a}\}^\perp\} \equiv \mathbf{x}+\{\mathbf{a}\}^\perp$$

is a layer of $\{\mathbf{a}\}^\perp$ in $\mathbf{R}^n$. Hence Remark 3.5 implies the following corollary.

Corollary 3.7

The indifference is algebraically characterized by the following condition:

$$\mathbf{x} \approx \mathbf{y} \Leftrightarrow W_{\mathbf{x}} = W_{\mathbf{y}}$$

Proof

By definition, the tangent space $W_{\mathbf{x}}$, i.e. $\{\mathbf{a}\}^{\perp}$, is the same as in case of $W_{\mathbf{y}}$. If $W_{\mathbf{x}} = W_{\mathbf{y}}$, i.e. $\mathbf{x}+\{\mathbf{a}\}^{\perp}=\mathbf{y}+\{\mathbf{a}\}^{\perp}$, then $\mathbf{x} = \mathbf{y} + \alpha$ for some $\alpha \in \{\mathbf{a}\}^{\perp}$. Hence $\mathbf{x} - \mathbf{y} \in \{\mathbf{a}\}^{\perp}$, which means, from Remark 3.5, that $\mathbf{x} \approx \mathbf{y}$. If $\mathbf{x} \approx \mathbf{y}$ then, from Remark 3.5, $\mathbf{y} - \mathbf{x} \in \{\mathbf{a}\}^{\perp}$, which means that $\mathbf{x} = \mathbf{y} + \alpha$ for some $\alpha \in \{\mathbf{a}\}^{\perp}$. Than, for every $\beta \in \{\mathbf{a}\}^{\perp}$, we have $\mathbf{x} + \beta = \mathbf{y} + \alpha + \beta \in W_{\mathbf{y}}$, since $\alpha + \beta \in \{\mathbf{a}\}^{\perp}$. Therefore $W_{\mathbf{x}} \subset W_{\mathbf{y}}$. Due to Remark 3.6 we have $\mathbf{x} \approx \mathbf{y} \Rightarrow \mathbf{y} \approx \mathbf{x}$, hence if we reverse the sequence of variables, then $W_{\mathbf{y}} \subset W_{\mathbf{x}}$.

■

The fact that **x** is equally preferred as **y** if and only if **x** and **y** belong to the same layer of $\{\mathbf{a}\}$ results in:

Corollary 3.8

The layers of the space $\{\mathbf{a}\}^{\perp}$ constitute classes of abstraction of the indifference relation $\approx$.

Proof

In Remark 3.6 it is noticed that $\approx$ is an equivalence. Furthermore $\mathbf{x} + \{\mathbf{a}\}^{\perp}=\{\alpha \in R^n : \alpha = \mathbf{x} + \beta \quad \text{for some} \quad \beta \in \{\mathbf{a}\}^{\perp}\} = \{\alpha \in \mathbf{R}^n : \alpha - \mathbf{x} \in \{\mathbf{a}\}^{\perp}\}$ – is the class of abstraction of **x** with respect to relation $\approx$.

■

If $\mathbf{x} » \mathbf{y}$ then the layer $W_{\mathbf{x}}$ is said to be positioned above layer $W_{\mathbf{y}}$. Let $\cos(\mathbf{a},\mathbf{b})$ denote the cosine of the angle between the vectors **a** and **b** in $(\mathbf{R}^n, < , >)$ and let $|\mathbf{a}|$ denote a canonical norm in $(\mathbf{R}^n, < , >)$, i.e. $|\mathbf{a}| = <\mathbf{a},\mathbf{a}>^{1/2}$. Remark 3.5 and Corollary 3.7 can be rephrased as follows:

Remark 3.9

The indifference is geometrically characterized by the following condition:

$$\mathbf{x} \approx \mathbf{y} \Leftrightarrow \cos(\mathbf{a},\mathbf{y}-\mathbf{x}) = 0$$

Proof

$$\mathbf{x} \approx \mathbf{y} \Leftrightarrow <\mathbf{a},\mathbf{y} - \mathbf{x}> = 0 \Leftrightarrow |\mathbf{a}|\,|\mathbf{b}|\cos(\mathbf{a},\mathbf{y}-\mathbf{x}) = 0$$

■

In the same manner one shows that $\mathbf{x} \gg \mathbf{y} \Leftrightarrow \cos(\mathbf{a},\mathbf{y}-\mathbf{x}) < 0$. Therefore W_x is positioned above W_y if and only if $\cos(\mathbf{a},\mathbf{y}-\mathbf{x}) < 0$. If we assign to each layer W_x a number $\xi = <\mathbf{a},\alpha>$ for some $\alpha \in W_x$, then higher positioned layers are associated with bigger numbers ξ. The number associated with a given layer is equal to the evaluation of objects forming this layer.

Let us consider two objects $\mathbf{x} \in W_x$ and $\mathbf{y} \in W_y$. The positioning of layers in $\mathbf{R}^n$ and therefore the positioning of objects **x** and **y** in a ranking depends on the weighting vector **a**. Let us assume that **a** is such, that $\mathbf{x} \gg \mathbf{y}$. The following question arises: how can the weighting vector **a** be changed (to **a**'), so that for **a**' the relation $\mathbf{x} \gg \mathbf{y}$ still holds. First, let us notice that changes of the form: $\mathbf{a}' = q\mathbf{a}$, $q > 0$, $q \neq 1$ are not feasible, since it was assumed that $<\mathbf{a},\mathbf{1}> = 1$ and $<q\mathbf{a},\mathbf{1}> = q<\mathbf{a},\mathbf{1}> \neq 1$ for $q \neq 1$. Additionally, such changes would preserve the space $\{\mathbf{a}\}^{\perp}$, thus $\{\mathbf{a}\}^{\perp} = \{\mathbf{a}'\}^{\perp}$, which results in the same positioning of layers of $\{\mathbf{a}\}^{\perp}$ and therefore the same positioning of objects as in the initial ranking.

If $\mathbf{x} \gg \mathbf{y}$ then, due to Remark 3.9, $\cos(\mathbf{a},\mathbf{y}-\mathbf{x}) < 0$. If after the change of **a** to **a'** this inequality still holds, i.e. $\cos(\mathbf{a}',\mathbf{y}-\mathbf{x}) < 0$, then we still have $\mathbf{x} \gg \mathbf{y}$. Therefore any change of **a** to **a'** that preserves the sign of $\cos(\mathbf{a}',\mathbf{y}-\mathbf{x})$, i.e. keeps the angle between vectors **a**' and $\mathbf{y}-\mathbf{x}$ within the interval $(\pi/2,\pi]$ and thus preserves the relation $\mathbf{x} \gg \mathbf{y}$. The boundary case is $\cos(\mathbf{a}',\mathbf{y}-\mathbf{x}) = 0$, i.e. $\mathbf{x} \approx \mathbf{y}$. Assume **a**' is chosen in such a way that the boundary case applies.

Corollary 3.10

For any weighting vector **a**'' given by $\mathbf{a}'' = p\mathbf{a} + q\mathbf{a}'$, where $p,q \geq 0$, $p + q = 1$, $q \neq 1$, the order of **x** and **y** is preserved, i.e. $\mathbf{x} \gg \mathbf{y}$.

Proof

$$\cos(\mathbf{a}'',\mathbf{y}-\mathbf{x}) = (p<\mathbf{a},\mathbf{y}-\mathbf{x}> + q<\mathbf{a}',\mathbf{y}-\mathbf{x}>)/(|\mathbf{a}''|\,|\mathbf{y}-\mathbf{x}|) =$$
$$= p<\mathbf{a},\mathbf{y}-\mathbf{x}>/(|\mathbf{a}''|\,|\mathbf{y}-\mathbf{x}|) < 0$$

■

Corollary 3.10 states that if **a** is a current weighting vector for which $\mathbf{x} \gg \mathbf{y}$, and **a**' makes **x** and **y** equivalent, then for any vector of the form **a**'' the angle between the vectors **x** and **y** is in $(\pi/2,\pi]$. Therefore the angle between $\mathbf{y}-\mathbf{x}$ and **a** or, more specifically, between $\mathbf{y}-\mathbf{x}$ and $\{\mathbf{a}\}^{\perp}$ represents area where changes of **a** preserve ranking. This observation suggests that the angle between the vectors $\mathbf{y}-\mathbf{x}$ and **a**, or between $\mathbf{y}-\mathbf{x}$ and its orthogonal projection on $\{\mathbf{a}\}^{\perp}$ could be used as an indicator of the stability of rankings outcomes. The investigation of this issue is left for future studies.

Two other notions of rankings stability will now be introduced: the notion of k-stability and k-ε stability. In the next section a numerical procedure will be implemented which, for a given ranking, calculates the numbers *k* and ε for which the ranking is k-ε stable.

Let us recall that the positioning of objects in a ranking R depends on the chosen weighting vector $\mathbf{a}\in\mathbf{A}$. This relation will be denoted by $R_{\mathbf{a}}$.

Definition 3.11

Let $\mathbf{a}\in\mathbf{A}$ and let $R_{\mathbf{a}}$ be a ranking. The set $D_R \subseteq \mathbf{A}$ defined as: $D_{R\mathbf{a}} = \{\mathbf{d}\in\mathbf{A} : R_{\mathbf{d}} = R_{\mathbf{a}}\}$ is said to be *a stability set* of the ranking $R_{\mathbf{a}}$.

The stability set of a ranking $R_{\mathbf{a}}$ consists therefore of the weighting vectors $\mathbf{d}\in\mathbf{A}$ for which the order of objects in $R_{\mathbf{d}}$ is the same as in $R_{\mathbf{a}}$. Let $R_{\mathbf{a}}|_k$ denote a ranking consisting of k objects from $R_{\mathbf{a}}$ which have highest evaluations in $R_{\mathbf{a}}$.

Definition 3.12

Let $R_{\mathbf{a}}$ be a ranking. If for every $\mathbf{d}\in\mathbf{A}$ we have $R_{\mathbf{d}}|_k = R_{\mathbf{a}}|_k$ then the ranking $R_{\mathbf{a}}$ is said to be k-*stable*.

Note that if $D_{R\mathbf{a}} = \mathbf{A}$ then the order of objects in $R_{\mathbf{a}}$ is the same regardless of which $\mathbf{a} \in \mathbf{A}$ is chosen. If, however, only the order of the first k objects in $R_{\mathbf{a}}$ does not depend on $\mathbf{a}$ then $R_{\mathbf{a}}$ is k-stable.

Definition 3.13

Let $R_{\mathbf{a}}$ be a ranking. A set $D_{R\mathbf{a}}|_k \subseteq \mathbf{A}$ defined as: $D_{R\mathbf{a}}|_k = \{\mathbf{d} \in \mathbf{A} : R_{\mathbf{d}}|_k = R_{\mathbf{a}}|_k\}$ is said to be the *k-stability set* of a $R_{\mathbf{a}}$.

The k-stability set $D_{R\mathbf{a}}|_k$ of a ranking $R_{\mathbf{a}}$ consists therefore of the weighting vectors $\mathbf{d}\in\mathbf{A}$ for which the order of the first k elements in $R_{\mathbf{d}}$ is the same as in $R_{\mathbf{a}}$.

Let $\varepsilon \geq 0$. Let $\mathbf{J}(\varepsilon) = \{\varepsilon_1, \varepsilon_2,..., \varepsilon_n\}$ denote an n-element set whose elements are equal to ε or $-\varepsilon$, i.e. $\varepsilon_j \in \{\varepsilon,-\varepsilon\}$, $j = 1,...,n$. Additionally, let us assume that the elements of $\mathbf{J}(\varepsilon)$ sum up to 0. If $\varepsilon > 0$ then such combination of elements ε_j exists for even numbers n only. Therefore for odd numbers n any element of $\mathbf{J}(\varepsilon)$ is assumed to be 0.

Let $\varepsilon \geq 0$. Let us consider the *group* $P_{\mathbf{J}(\varepsilon)}$ *of permutations* of $\mathbf{J}(\varepsilon)$, i.e. the set of all bijective functions $\sigma_i : \mathbf{J}(\varepsilon)\to\mathbf{J}(\varepsilon)$. We have $P_{\mathbf{J}(\varepsilon)} = \{\sigma_i, i = 1,2,...,n!\}$.

From now on, the set $\mathbf{J}(\varepsilon)$ and the elements of $P_{\mathbf{J}(\varepsilon)}$ will be represented as vectors, so $\sigma_i\in P_{\mathbf{J}(\varepsilon)}$ is the i-th permutation of vectors' $\mathbf{J}(\varepsilon)$ co-ordinates.

Definition 3.14

Let $\mathbf{a}\in\mathbf{A}$ and let $P_{\mathbf{J}(\varepsilon)}$ be the group of permutations of $\mathbf{J}(\varepsilon)$. A number $\varepsilon \geq 0$ satisfying $\mathbf{a}+\sigma\in\mathbf{A}$ for every $\sigma\in P_{\mathbf{J}(\varepsilon)}$ is said to be *feasible*.

Definition 3.15

Let $\varepsilon \geq 0$ be feasible and let $P_{\mathbf{J}(\varepsilon)}$ be the group of permutations of $\mathbf{J}(\varepsilon)$. Each vector $\sigma\in P_{\mathbf{J}(\varepsilon)}$ is said to be an *ε-perturbation*. Let $\mathbf{a}\in\mathbf{A}$. For every ε-perturbation $\sigma\in P_{\mathbf{J}(\varepsilon)}$ the vector $\mathbf{a}' = \mathbf{a} + \sigma$ is said to be an *ε-perturbed weighting vector* **a**.

Definition 3.16

Let $\mathbf{a}\in\mathbf{A}$ and let $R_{\mathbf{a}}$ be a ranking. Let $\mathbf{a}'$ be an ε-perturbed weighting vector **a**. If every $\mathbf{a}'\in D_{R\mathbf{a}}$ then $R_{\mathbf{a}}$ is called *ε-robust*.

Let us note that a ranking $R_{\mathbf{a}}$ is ε-robust if the ranking $R_{\mathbf{a}+\sigma}$ preserves the order of objects for all ε-perturbations $\sigma\in P_{\mathbf{J}(\varepsilon)}$.

Corollary 3.17

Let $\mathbf{a}\in A$ and let $R_{\mathbf{a}}$ be a ranking. Let $\varepsilon \geq 0$ be feasible. If $R_{\mathbf{a}}$ is ε-robust then $R_{\mathbf{a}}$ is δ-robust for all feasible $0 \leq \delta \leq \varepsilon$.

Proof

Let $\mathbf{x},\mathbf{y}\in R_{\mathbf{a}}$ and let $\mathbf{x} \gg \mathbf{y}$. Let $\sigma\in P_{\mathbf{J}(\varepsilon)}$. If $R_{\mathbf{a}}$ is ε-robust then $\mathbf{a}' = \mathbf{a} + \sigma\in D_{R\mathbf{a}}$. Let $\mathbf{I} = \sigma/\varepsilon$. If $\mathbf{a}'\in D_{R\mathbf{a}}$ then $\mathbf{x}\gg\mathbf{y}$ holds for $R_{\mathbf{a}'}$, hence $<\mathbf{a}',\mathbf{y}-\mathbf{x}> = <\mathbf{a},\mathbf{y}-\mathbf{x}> + \varepsilon<\mathbf{I},\mathbf{y}-\mathbf{x}> < 0$. Let $\phi = \mathbf{I}\delta$ for some feasible $0 \leq \delta \leq \varepsilon$. Note that $\phi\in P_{\mathbf{J}(\delta)}$. Let $\mathbf{a}'' = \mathbf{a} + \phi$. Note that $<\mathbf{a}'',\mathbf{y}-\mathbf{x}> = <\mathbf{a},\mathbf{y}-\mathbf{x}> + <\phi, \mathbf{y} - \mathbf{x}> = <\mathbf{a},\mathbf{y}-\mathbf{x}> + \delta<\mathbf{I},\mathbf{y}-\mathbf{x}> < 0$. This is because $<\mathbf{a},\mathbf{y}-\mathbf{x}> < 0$ and $\delta \leq \varepsilon$. If $<\mathbf{I},\mathbf{y}-\mathbf{x}> \leq 0$ then $<\mathbf{a},\mathbf{y}-\mathbf{x}> + \delta<\mathbf{I},\mathbf{y}-\mathbf{x}> < 0$ for all $\delta \geq 0$. If $<\mathbf{I},\mathbf{y}-\mathbf{x}> > 0$ then $\delta<\mathbf{I},\mathbf{y}-\mathbf{x}> \leq \varepsilon<\mathbf{I},\mathbf{y}-\mathbf{x}>$, hence $<\mathbf{a},\mathbf{y}-\mathbf{x}> + \delta<\mathbf{I},\mathbf{y}-\mathbf{x}> \leq <\mathbf{a},\mathbf{y}-\mathbf{x}> + \varepsilon<\mathbf{I},\mathbf{y}-\mathbf{x}> < 0$. This means that $\mathbf{x} \gg \mathbf{y}$ holds for $\mathbf{a}' = \mathbf{a} + \phi : \phi = \mathbf{I}\delta$. As noted above, $\phi\in P_{\mathbf{J}(\delta)}$. $\sigma\in P_{\mathbf{J}(\varepsilon)}$ was arbitrary and we required $\phi = \mathbf{I}\delta = \sigma(\delta/\varepsilon)$, hence if σ is any permutation of $\mathbf{J}(\varepsilon)$ then ϕ is the corresponding permutation of $\mathbf{J}(\delta)$. This means that $\mathbf{x} \gg \mathbf{y}$ holds for all $\mathbf{a}' = \mathbf{a} + \phi : \phi\in P_{\mathbf{J}(\delta)}$, that is, that $R_{\mathbf{a}}$ is δ-robust.

■

Remark 3.18

The requirement for δ to be feasible in Corollary 3.18 is not necessary.

Proof

It was assumed in Corollary 3.14 that $\varepsilon \geq 0$ is feasible and that $0 \leq \delta \leq \varepsilon$. Let $\mathbf{a} \in \mathbf{A}$, $\varepsilon \geq 0$ and $\sigma \in P_{\mathbf{J}(\varepsilon)}$. Let $\mathbf{a}' = \mathbf{a} + \sigma$. A number $\varepsilon \geq 0$ is feasible if for all $\sigma \in P_{\mathbf{J}(\varepsilon)}$ the following two conditions hold: $<\mathbf{a} + \sigma, \mathbf{1}> = 1$ and $\mathbf{a} + \sigma \geq 0$. Let us observe that the first condition holds by definition of $\mathbf{J}(\varepsilon)$. The second condition holds if $\varepsilon \leq \min(\min(a_1,...,a_n), 1 - \max(a_1,...,a_n)) = r$, where $(a_1,...,a_n) = \mathbf{a}$. If $0 \leq \delta \leq \varepsilon$ and $\mathbf{a}' = \mathbf{a} + \phi : \phi \in P_{\mathbf{J}(\delta)}$, then, the analogous first condition for δ holds by definition of $\mathbf{J}(\delta)$ and the second one follows from the fact that $\delta \leq \varepsilon \leq r$.

■

The notion of k-stability and ε-robustness leads to their following composition.

Definition 3.19

Let $\mathbf{a} \in \mathbf{A}$ and let $R_{\mathbf{a}}$ be a ranking. Let $\mathbf{a}'$ be an ε-perturbed weighting vector $\mathbf{a}$. If every $\mathbf{a}' \in D_R|_k$ then $R_{\mathbf{a}}$ is called *k-ε-stable*. Let us note that a ranking $R_{\mathbf{a}}$ is k-ε-stable if the ranking $R_{\mathbf{a}+\sigma}$ preserves the order of k objects with the highest evaluations for all ε-perturbations $\sigma \in P_{\mathbf{J}(\varepsilon)}$.

In this section two concepts of ranking stability have been introduced. On the basis of these concepts the stability of an educational ranking will be investigated in the following section.

4. CASE STUDY AND SIMULATION OF PREFERENCE STABILITY

In this section an empirical study of a ranking of economic and business schools (both public and private) in Poland is conducted. The ranking under consideration was published as a cover story in 2004 by the nationwide periodical "Polityka" and hundred schools were investigated. The ranking procedure was constructed by a panel of experts invited by "Polityka". The data on universities were taken from Ministry of Education and National Research Committee and from own surveys.

"Polityka's" ranking is of the same form as the rankings considered in Section 3. For the comparison of universities six aggregate criteria were taken into account: *academic position*, *academic staff potential*, *pro-student orientation*, *contacts with social and business environment including international relations*, *selectivity,* and *infrastructure*. Therefore every object

o (university) was characterized by a six-element vector **x** = **c**(**o**). For each university, appropriate values were assigned to each of the six criteria. These values constituted co-ordinates of **x**. The team of experts agreed on a weighting vector **a** whose co-ordinates reflected, as far as experts' subjective perception is concerned, the relative importance of criteria. The employed weights were: 25%, 20%, 20%, 15%, 10%, and 10%, respectively, i.e. **a** = (0.25, 0.2, 0.2, 0.15, 0.1, 0.1).

The declared mission of the "Polityka", ranking was created to assist households (and candidates) in choice of university which would closely match their expectations. "Polityka" informed that weights used to rank universities did not follow any scientific survey, but were a compromise of subjective experts' judgements. In order to enable readers to individually process the data and to modify published results, "Polityka" provided its audience not only with final results, but also with criteria, values of experts' measurements and with the weight system agreed and used by them. The readers were also encouraged to modify themselves experts' weights in order to better reflect own subjective preferences and thus to arrive to own results. "Polityka" warned however that a reader who is inexperienced in the field may find the comparison of universities difficult. The difficulty results from several reasons: many schools use similar names, their offer is difficult to evaluate, and the concepts used in criteria are not always easy to understand for the wide audience (the cognitive barrier). A user who has problems with strict numeric correction of experts' weights turns to interval preferences in order to better elicit her preference. This means that although she or he cannot strictly determine the weights, she or he can define intervals in which weights can be found. If the ranking remains the same for all weights from this interval (robustness of a ranking) then users' weighting vector is equivalent to the one used by experts. In our parlance this means that the order of objects in the ranking is *robust* with respect to *perturbations* in weights from user-defined interval.

The concept of stability presented in Section 3 allows assist users of rankings who may experience cognitive barriers in evaluations (e.g. resulting from the use of technical language in criteria description) and who attempt to elicit interval preferences. It is possible to provide the user with the answer to the question: How big the difference between his weights and the experts' weights (as described in Section 3) can become and still preserve the ordering of universities by the experts?

An application which for given $k > 0$ finds maximal ε such that the considered ranking $R_{\mathbf{a}}$ is k-ε stable was implemented in the Matlab environment. This application generates, for a given weighting vector $\mathbf{a}\in\mathbf{A}$ (experts' weights), feasible ε-disturbed weighting vectors, i.e. vectors of the form: $\mathbf{a'} = \mathbf{a} + \sigma : \sigma\in P_{\mathbf{J}(\varepsilon)}$, $\varepsilon \geq 0$ – feasible, and checks if for all[2] $\sigma\in P_{\mathbf{J}(\varepsilon)}$ the order of objects in the ranking (universities) is preserved, that is if $\mathbf{a'}\in D_{R\mathbf{a}}|_k$.

A direct application of definitions from Section 3 shows that the ranking of "Polityka" is 2-ε stable for any feasible ε (this is denoted in the Table 3 by ε-robust). This result shows that for every feasible[3] change of experts' weights (according to definitions from Section 3) the sequence of the first two universities in the ranking remains the same. In this ranking the relative positioning of the first two universities does not change: the first university will always be better ranked then the second one. However, these statements are not true for *every* weighting vector.

Another use of the procedure is recommended in search of a leader in the pool of ranked objects when visible disagreement of experts on weights' definitions occurs. This situation puts credibility of weighting in question and may lead to failure of search of a leader. The 2-ε stability of ranking means that if their deviation from the weighting vector identifying the leader is feasible then the order of the first two schools is correct.

For $k = 3$ the ranking turns out to be 3-0.077 stable, which means that if the weighting vector is in the 0.077-neighborhood of experts' weights then the ranking remains the same.

In the case of $k = 4$ the ranking is not k-ε stable for any $\varepsilon \geq 0.01$. This is due to the fact that the third and the fourth schools have equal evaluations for the weighting vector assigned by the experts. In this case the result is not stable. If, however, the fourth university drops out then the ranking becomes 4-0.052 stable. The interpretation is the same as in the case of 3-0.077 stability. For $k = 5$ the ranking is still stable for $\varepsilon = 0.052$, that is it is 5-0.052 stable. For $k = 6$ we have 6-0.034 stability and for $k = 7$ we have 7-0.033 stability.

In general, stability drops as new objects arrive, but up to five objects the weights can be distorted by up to 5 pp, which in the case of a 10% weight constitutes half of this value. A summary of the results is presented in Table 3 (the left part).

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[2] In fact, not all permutations are considered, but only these, which imply different ε-perturbed vectors.

[3] Notice, that feasible ε must satisfy $\varepsilon \leq 0.1$, since 10% is the minimal weight assigned by experts (otherwise we end up with negative weights).

Apart from the evaluation of stability of the whole ranking, calculations were done for universities located in Warsaw only. Table 3 (the right part) presents the results for Warsaw schools. In this case, contrary to the overall results, the ranking is far more stable.

Table 3

k-ε stability of "Polityka" educational ranking for $k = 1,2,\ldots,7$

K	ε	k	ε
1	ε-robust (10%)	1	ε-robust (10%)
2	ε-robust (10%)	2	ε-robust (10%)
3	0.077 (7.7%)	3	ε-robust (10%)
4	0.052 (5.5%)*	4	ε-robust (10%)
5	0.052 (5.2%)	5	ε-robust (10%)
6	0.034 (3.4%)	6	ε-robust (10%)
7	0.033 (3.3%)	7	ε-robust (10%)

Results for whole ranking (left) and for Warsaw only (right).
* Fourth object dropped out.

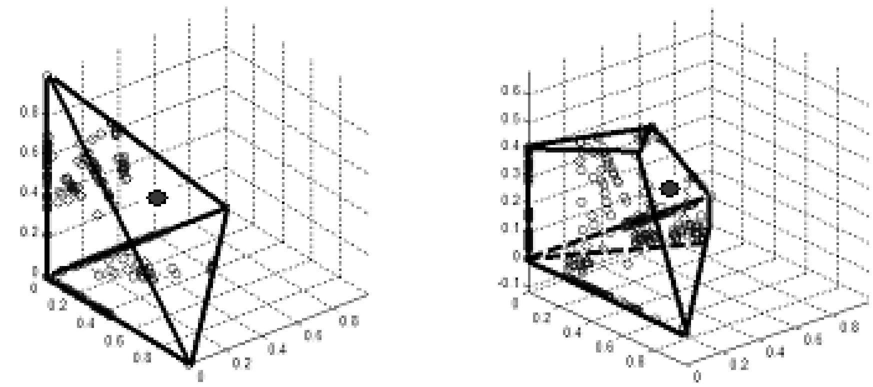

Fig. 3. Plot of ε-perturbed weighting vectors for which the order of objects in the ranking is preserved in the case of a less (left) and more (right) stable ranking (black lines are shown for visualization purposes)

Let us now present the graphical interpretation of the concept of ranking stability. Assume that for a given n-dimensional weighting vector **a**, say **a** = (1/3,1/3,1/3), the ranking is k-ε stable for some k. We take all ε-perturbed weighting vectors **a**' (for these vectors the order of objects in the considered ranking is preserved) and plot them. If $n > 3$, we also have to choose three out of n available co-ordinates and project all ε-perturbed weighting vectors **a**' on a corresponding 3-dimensional space. If $n \leq 3$, as it is in our case, we simply plot these vectors. Figure 3 presents the perturbed vectors in the case of a more (to the left) and a less (to the right) stable ranking. The red dot represents the vector **a**. One can see that in the case of a less stable ranking, the ε-perturbed weighting vectors **a**' are concentrated in greater degree than in the case of a more stable ranking. For $n > 3$, one can observe projections of ε-perturbed weighting vectors **a**' on axes and the conclusion remains the same. This observation allows us to visualize the concept of ranking stability in a simple way.

CONCLUDING REMARKS

In the paper educational rankings were discussed and mathematically described. First, the great importance of such rankings was shown: three scenarios were discussed in detail. It appears that educational decisions deal necessarily with factors which are difficult to represent in formal, mathematical way such as subjectivity, uncertainty, and imprecision. Ranking is a widely used analytical method which can tackle soft properties in a procedural way. Rankings are simple and admit easy interpretations, but they also involve a good deal of subjectivity. Thus, they can lead to different results for different users depending on the choice of weighting values.

A mathematical description of rankings allows for construction of an algorithm facilitating a search of the preferred object – an educational institution or a programme. The case study with real-life data shows that the methodology presented provides the user with meaningful information assisting him in overcoming cognitive barriers and in her preference elicitation.

Without loss of generality and for the sake of simplicity of demonstrations the argument was kept at a simplified level. However, the methodology presented here can be extended, in a natural way, in several directions. Firstly, more sophisticated mathematical models can be developed, e.g. models including continuous variables or based on set theoretical analysis of interval preferences. Secondly, the concept of 2-ε-stability can be extended to include additional questions related to management of educational institutions, e.g. what are conditions for an educational unit sufficient to remain in a group of ranking leaders? Finally, lack of space forced us to leave for future presentations the reverse problem of finding the range of *forced preference change* which would lead to ranking result compatible with university preferences.

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COMPUTER-BASED SUPPORT OF MULITCRITERIA COOPERATIVE DECISIONS – SOME PROBLEMS AND IDEAS

Abstract

The paper deals with a class of cooperation problems related to the decision situations in which several parties consider participation in a joint enterprise. Typical questions arise: When is cooperation beneficial? What should be the fair engagement of the parties in the enterprise and the fair allocation of benefits among them? Problems related to construction of a computer-based system supporting the decisions analysis made by the parties are discussed.

To construct the system, first a substantive model describing the decision situation has to be built. The model includes specifications of decision variables, exogenous variables, output quantities, criteria of the parties, constraints defining the set of admissible decisions, mathematical relations enabling derivation of the output quantities, and the criteria as dependent on given decision and exogenous variables. It is assumed that each party has given its own set of criteria, in general different, and has independent preferences over the criteria.

The sovereignty of the decision makers representing the parties is assumed, i.e. the decision makers are fully responsible for the decisions they make. The computer-based system is only a tool aiding the analysis of decision situations and a tool facilitating the consensus seeking.

The bargaining game model extended to the multicriteria case is used to describe the cooperation problems. Solution concepts formulated for the classical bargaining problem, such as the egalitarian solution and those proposed by Nash, Raiffa-Kalai-Smorodisky, and Imai are considered. Extensions of the solutions to the multicriteria bargaining problem are presented. Properties of solutions are discussed. The solution concepts can be used to derive mediation proposals generated in the system and presented to the parties for analysis in iterative procedures.

The application area includes, among others, the analysis of cooperation in the case of innovative activities, education systems, and cost allocation problems.

Keywords

Cooperative decisions, multicriteria analysis, mediation support, computer-based systems.

INTRODUCTION

The paper deals with cooperation problems related to the decision situations in which several parties consider participation in a joint enterprise. Questions arise: When the cooperation is beneficial? What should be the fair engagement of the parties in the enterprise and the fair allocation of benefits among them? In the paper problems related to the construction of computer--based systems supporting the decisions analysis made by the parties are discussed. Such systems can be built with the application of the control theory methods, the mathematical modeling technics, the optimization procedures, and the modern advanced information technology.

It is assumed that each party has a given set of criteria, in general different, and independent preferences over the criteria. Sovereignty of the decision makers representing the parties is assumed, i.e. the decision makers are fully responsible for the decisions they made. The computer-based system is only a tool aiding the analysis of the decision situations and facilitating the consensus sought.

In practice, cooperation problems are solved through a negotiation process. Before the negotiations each party should derive its Best Alternative to Negotiated Agreement (abbreviated further as BATNA) – a concept introduced by Fisher and Ury [1]. In the negotiations a party can compare the analyzed proposals to the derived BATNA and can evaluate its possible benefit from the cooperation.

The cooperation situations are modeled in the game theory: as the so--called bargaining problem for two and more players. The classical axiomatic theory of bargaining has been developed by Nash [23], Raiffa [27], Kalai and Smorodinsky [3], Roth [29], Thomson [31], and many others. The classical bargaining problem in the case of two and many issues is formulated theoretically in terms of utilities as a pair (S, d). Two parties (players) can reach any of the payoffs from the agreement set S, if they agree. The disagreement point d defines the payoffs of the players in the case when they do not reach such an agreement. It is derived on the basis of the BATNA concept; in particular it can be the status quo point.

A solution of the bargaining problem is considered as a method of chosing a point from the set S, accepted by rational players. Different solution concepts are proposed under different sets of axioms (assumed properties describing the feeling of fairness) the solution should fulfill. The argumentation for acceptation of the solution concept by the players is the following: If rational players agree on a selected set of axioms-principles and accept them as fair, why they should not accept the solution concept which fulfills the axioms?

In the paper the cooperative game model is used to describe the cooperation problems in the case of multicriteria payoffs of players. Solution concepts for the classical bargaining problem, like the egalitarian solution and those proposed by Nash and Raiffa, Kalai, and Smorodisky are considered. The solution concepts extended to the multicriteria case can be used to derive mediation proposals generated in the system and presented to the parties for analysis in iterative procedures. The presented mediation support with use of the computer-based system has been inspired by the single negotiation procedure frequently applied in international negotiations (see [28]).

The application area includes among others analysis of cooperation in the case of innovative activities, educational systems and cost allocation problems. The references include selected papers dealing with computer-based support in negotiations [2, 5-9, 11-18], related to the multicriteria decision analysis [10-13, 16, 18, 25, 26, 34-36], to the utility function approach [4, 19-22, 29, 30, 32, 33] and to the game theory, as mentioned above.

1. THE IDEA OF A COMPUTER-BASED SYSTEM

The proposed system includes a model representation, modules supporting unilateral analysis made by decision makers (DMs), a module generating mediation proposals, as well as modules including an optimization solver, respective data bases, procedures enabling interactive sessions realizing the mediation procedure, and a graphical interface.

The model describing the decision situation of the parties is the basis for decision analysis and is constructed by a system analyst with use of the gathered information according to the rules of system sciences. It includes the specification of decision variables, exogenous variables, output quantities, criteria, and model relations. The model parameters are identified on the basis of the collected data and should be verified as well as validated. Therefore modules containing respective data base, model editor, procedures for estimation of model parameters, and for model verification are included in the system.

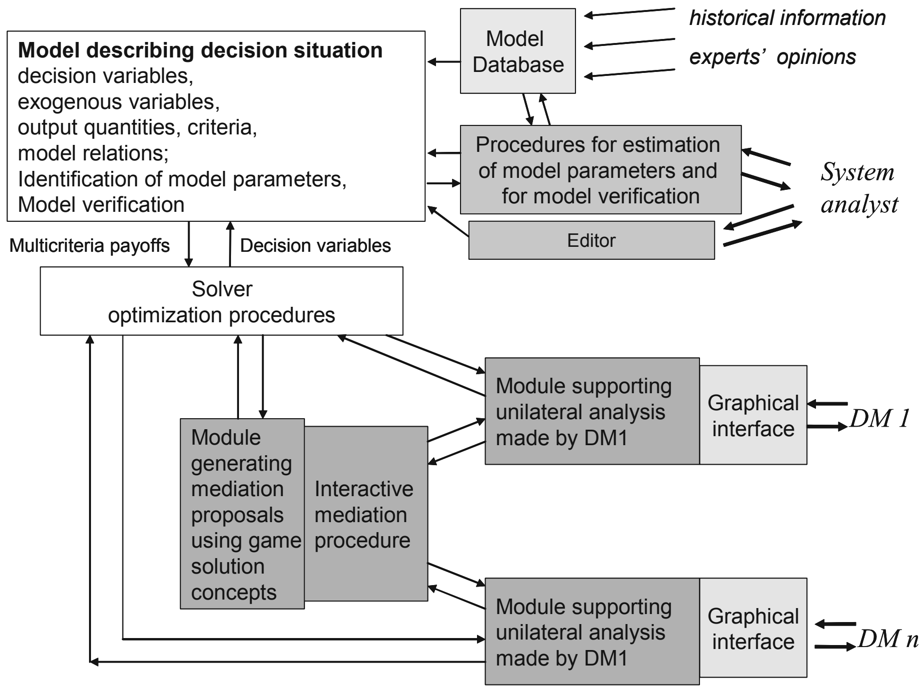

Fig. 1. General idea of a computer-based system supporting cooperative decisions

The module supporting unilateral analysis enables each DM to obtain independently information about possible multicriteria payoffs for the assumed scenario and to look for the preferred option. The analysis is made interactively.

The system generates also mediation proposals. The mediation proposals are derived with use of selected solution concepts of the theory of cooperative games and on the basis of the preferences expressed by the DMs. The mediation proposals are generated and presented to the DMs within a special mediation procedure.

Optimization technics are used in the system: in the procedures of multicriteria analysis, in the modules supporting individual unilateral analysis, and in the module generating mediation proposals to calculate game solution concepts. The respective optimization procedures are included in the solver module.

2. MODEL

To describe the cooperation situation an extension of the classical bargaining problem is considered in the case of n decision makers (DMs) further called players. For each DM (player), $i = 1,...,n$, we define:

A vector of decision variables $x_i \in \boldsymbol{R}^{k_i}$, where k_i is the number of variables of the player i,

– a vector of criteria (to be maximized) $y_i \in \boldsymbol{R}^{m_i}$, where m_i is the number of criteria of the player i.

A mathematical model describing the decision situation is given with:

– a set of admissible decisions $X^0 \subset R^K$, where $R^K = R^{k_1} \times ... \times R^{k_n}$ is the space of decisions of all the players,
– a space of payoffs of all the players $R^M = R^{m_1} \times ... \times R^{m_n}$, it is the Cartesian product of the multicriteria spaces of the players' payoff,
– a function $F : X^0 \rightarrow R^M$ defining vectors of the players' payoffs for given values of decision variables. If the function F is continuous and the set X^0 is compact, then the set of attainable payoffs $Y^0 = F(X^0)$ is also compact.

Let each player have his own reservation point $d_i \in R^{m_i}$ assumed in his multicriteria space on the basis of the BATNA concept. Then the Multicriteria Bargaining Problem (MBP) can be defined by the disagreement point $d = (d_1, ...d_n) \in R^M$ and the agreement set S consisting of the points of the set $Y^0 \subset R^M$ dominating the point d. Each point of the agreement set can be reached if all the players agree, i.e the problem consists in the selection of the point from the set S, which could be accepted by all the players.

Remarks to the problem formulation:

1. Each DM (player) has his own set of criteria, in general different.
2. A set of attainable payoffs is considered in the space R^M which is the Cartesian product of individual multicriteria spaces of the players.
3. The set of attainable payoffs $Y^0 \subset R^M$ is in general not given explicitly.
4. The multicriteria payoffs of each player can be derived by means of a computer-based system for the given values of the decision variables of all the players, using model relations.

An example of the multicriteria bargaining problem is presented in Figure 2 in the case of two players. Player 1 has criteria y_{11} and y_{12}, player 2 has only one criterion y_{21}. In the three-dimensional space of criteria an agreement set S and a disagreement point d are shown.

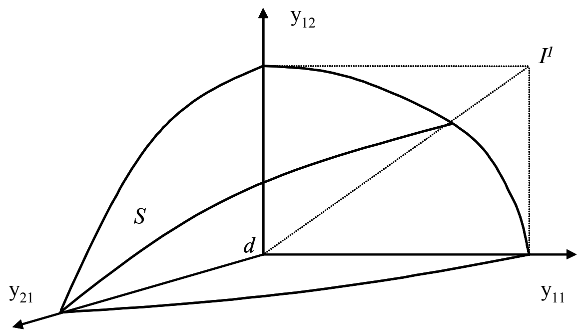

Fig. 2. An example of the multicriteria bargaining problem

The disagreement point d is based on BATNA of each player. In general case the derivation of the disagreement point may also require additional multicriteria analysis performed by each player. The agreement set S is defined by model relations, and in general is not known explicitly. The ideal point in the criteria space of the player 1 is also shown denoted by I^1.

3. UNILATERAL ANALYSIS

Each player starts from unilateral interactive multicriteria analysis of the problem. During the analysis he can obtain information about possible outcomes for different assumptions about his preferences. He has also to make assumptions about the counterplayers' outcomes or counterplayers' preferences. The analysis can be made by applying the reference point approach developed by Wierzbicki [31-33] with use of the order approximation functions. According to the approach, each player assumes reference points in the space of his criteria and the system generates the respective outcomes which are Pareto optimal in the set S. For some number of reference points assumed by a player, a characterization of the Pareto frontier of the set S can be obtained.

The outcomes characterizing the Pareto frontier in the case of the i-th player are derived by:

$$\max_{x \in X_0}[s\ \ (y_i(x), y_i^{*})] \tag{1}$$

where:

y* is a reference point assumed by the player in the space R_{m_i},

$y_i(x)$ defines the vector of criteria of the i-th player, which are dependent on the vector x of decision variables, by the model relations,

s(y,y*) is the order approximating achievement function.

The function:

$$s(y_i, y_i^*)= \min_{1\leq j\leq mi} [a_j(y_{ji} - y^*_{ji}) + a_{m_i+1}\sum_{i=1}^{m_i} a_j(y_{ji} - y^*_{ji})] \qquad (2)$$

is an example of the achievement function suitable in this case, where $y_i^* \in R^{m_i}$ is a reference point, a_j, $1 \leq j \leq m_i$, are scaling coefficients, and $a_{m_i+1} > 0$ is a small parameter.

The assumed reference points and the obtained Pareto outcomes are stored in a database, so that a characterization of the Pareto frontier can be made and analyzed by the player.

Figure 3 presents the results of an unilateral, interactive analysis made by the player 1 in his criteria space for two different assumptions about the second player's outcomes: 1st – for the counterplayer's outcomes assumed on the level of *d*, and 2nd – for the conterplayer's aspirations assumed by the player 1.

Using the reference point approach the player can generate a number of such characterizations of the Pareto frontier. At the end, the player is asked to indicate the preferred outcome.

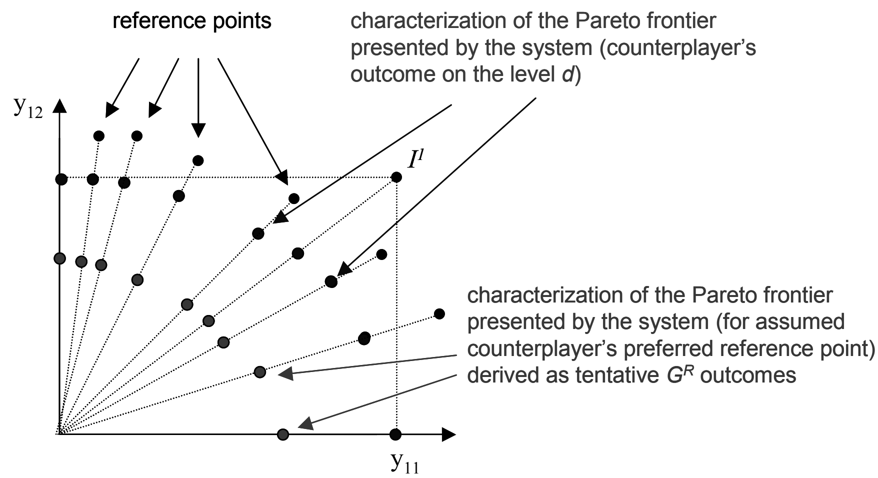

Fig. 3. Characterizations of the Pareto frontier obtained during unilateral analysis

The unilateral analysis is made by each player. Information about the indicated preferred outcomes of all players are collected.

4. SUPPORTING MEDIATION PROCESS

A procedure supporting mediation process has been proposed as inspiration of the Single Negotiation Text (SNT) procedure frequently applied in international negotiations. The SNT procedure formulated by Roger Fisher and described among others by Raiffa [28], is applied to solve crisis situations which appear in hard positional negotiations. According to the procedure, the opponents should not discuss the tasks independently nor formulate and consider counterproposals. They obtain and analyze, in consecutive rounds, proposals prepared by the mediator. In each round they work on the same text. On the basis of their opinions and suggestions, the mediator prepares an improved proposal to be analyzed in the next round.

The proposed procedure consists of a sequence of rounds $t = 1,2,\ldots T$. The rules of the procedure can be listed as follows:

- in each round each player, supported by the computer-based system, performs an interactive unilateral analysis in his criteria space and indicates a required improvement direction of his outcome according to his preferences among the criteria,
- the computer-based system generates the consecutive mediation proposals on the basis of the improvement directions indicated by all players,
- each player analyzes the proposals and introduces the required improvements of his outcome and the system generates a new improved mediation proposal.

The consecutive mediation proposal d^t is generated in the round t on the basis of the players' indications, according to the scheme:

$$d^0 = d$$

$$d^t = d^{t-1} + \alpha^t \cdot [G^t - d^{t-1}], \text{ for } t = 1, 2, \ldots, T \tag{3}$$

where $\alpha^t = \min\{\alpha^{t_1}, \ldots, \alpha^{t_n}\}$, α^{t_i} is the so-called confidence coefficient assumed by the player i in the round t, $0 < \varepsilon < \alpha^{ti} < 1$, G^t is the game solution calculated in the round t, for example the Raiffa solution, generalized in multicriteria case.

In the Cartesian product of the multicriteria spaces of the players' payoffs a point, which is a composition of the preferred outcomes indicated by the players after the unilateral analysis is found. This point denoted by U^R in Figure 4 is called the relative utopia point. It relates to the aspirations of the players. In fact it is derived according to the players' preferences expressed after the unilateral analysis. In general it is different from the ideal point defined by the maximal values of criteria in the set S.

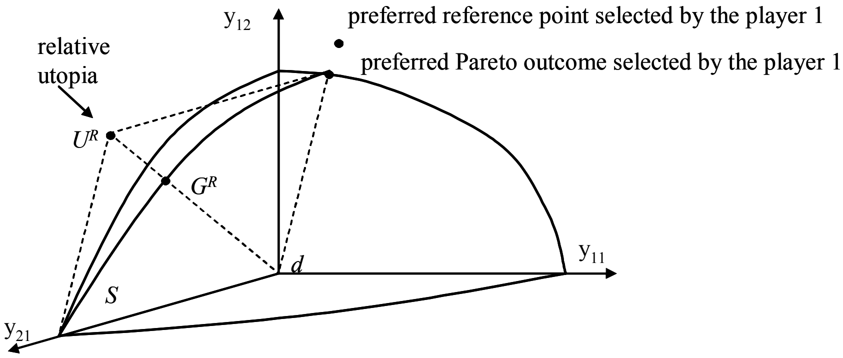

Fig. 4. Relative utopia and generalized Raiffa solution

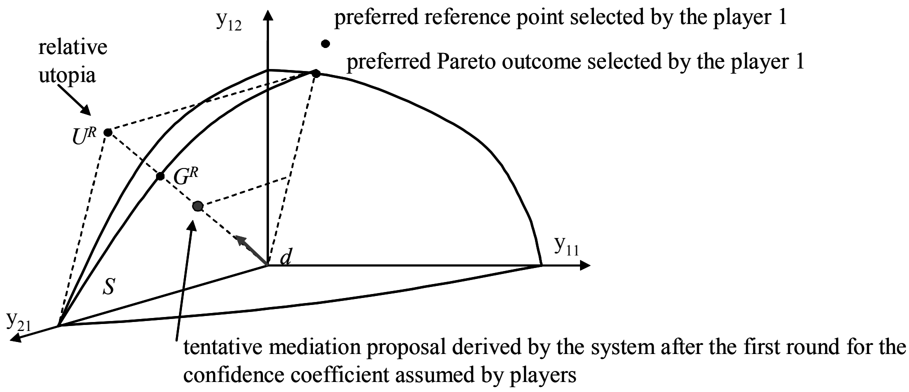

Fig. 5. Tentative mediation proposal

The generalized Raiffa solution G^R is the maximal point in S, located on the line linking the disagreement point d and the relative utopia U^R. If the confidence coefficient is relatively small (less then 1), then each player can limit the increase of the payoffs of all the players in the given round as it is presented in Figure 5. A tentative mediation proposal is derived according to the formula (3).

The tentative mediation proposal derived in the round t is treated as the disagreement point d^t in the next round $t + 1$. Next, unilateral analysis is performed by each player who explores the set of points from S and dominating the point d^t. This is illustrated in Figure 6.

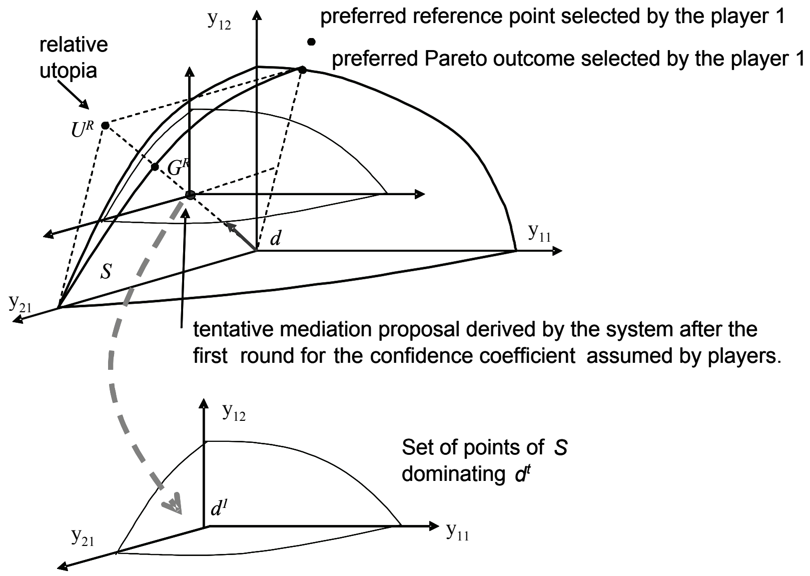

Fig. 6. The mediation proposal treated as the disagreement point in the problem analyzed in the next round

The preferred outcome selected by the player defines a direction in his multicriteria space. The directions of all players define a hyperplane in the Cartesian product of the spaces. The relative utopia and generalized Raiffa solutions lie on the hyperplane. Other theoretical solutions of the game theory lying on the hyperplane can be considered based on, for example, the egalitarian concept or the Nash cooperative solution concept.

The egalitarian solution maximizes gain of equal coordinates. It satisfies the axioms of weak Pareto optimality, symmetry, and strong monotonicity.

The Nash cooperative solution maximizes the product of the increases of the payoffs. It satisfies the axioms of Pareto optimality, symmetry, scale invariance, and independence of irrelevant alternatives [23].

The generalized Raiffa solution concept mentioned before satisfies the axioms of weak Pareto optimality, symmetry, scale invariance, and restricted monotonicity [12, 16]. The solution can be only weakly Pareto optimal even if the set S is convex. This means that the payoff of some player can be improved without decreasing the payoffs of other players. The application of the reference approach and of the achievement function of the form (2) partially solves the problem. The maximization of the function for $x \in S$ and decreasing parameter $a_{n+1} \rightarrow 0_+$ results in the lexicographic order applied to two separate terms of the function. For further discussion of the reference point method and the lexicographic ordering see [10, 26].

FINAL REMARKS

In the paper, a computer-based system supporting cooperative decisions is proposed and a model-based approach is applied. The model of the cooperation problem is formulated with use of ideas from the game theory. The system supports multicriteria analysis of the problem performed independently by parties with use of the reference point approach. Each player by assuming a reference point in his criteria space can use the system to generate a set of outcomes characterizing Pareto frontier of possible outcomes. It is made by solving maximization problems with specially constructed achievement functions. The system generates also Pareto optimal compromise outcomes. They are derived taking into account the information on the parties' preferences expressed in a special interactive procedure. The outcomes satisfy the axioms of cooperative solutions formulated in the theory of games generalized to the multicriteria case. They can be treated as mediation proposals aiding the players in looking for the consensus. The parties using the system can understand the nature of the cooperation problem, learn what their real preferences among the criteria are, analyze the possible outcomes, and make the final decision about cooperation consciously.

The paper continues the line of research presented in the references [11--18]. It is a part of the research including development of methods and computer experiments in the case of different cooperation problems. The research includes decision situations described by the multicriteria bargaining problems, but also by the multicriteria noncooperative games, the multicriteria cooperative games with and without side payments. In the research the utility function approach, which is an alternative to the direct multicriteria analysis, is also developed. In particular, the concepts proposed by R. Kulikowski [19-22] are applied to the support decision analysis taking into account the presence of risk. The concepts extend ideas developed in the papers [4, 24, 30, 32, 33] and are applied among others in the case of financial analysis [19, 21], analysis of innovative activities [15, 20], and analysis of education decisions [22].

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AFFINITY SET AND ITS APPLICATIONS*

Abstract

Affinity has a long history related to the social behavior of human, especially, the formation of social groups or social networks. Affinity has two meanings. The first is a natural liking for or attraction to a person, thing, idea, etc. The second defines affinity as a close relationship between people or things that have similar appearances, qualities, structures, properties, or features. Affinity here is simply defined as the distance/closeness between any two objects: the distance measurement could be geometric or abstract, or any type a decision maker prefers. A new forecasting method without historical memory, based on game theory and affinity set is originally proposed. The prediction performance of this new model is compared with the simple regression model for their performances on decision of buying in or selling out stocks in a dynamic market. Interestingly the qualitative model (affinity model) performs better than the quantitative model (simple regression model). Possible affinity set applications are provided as well in order to encourage readers to develop affinity models for actual applications.

Keywords

Affinity, forecasting, decision, distance.

INTRODUCTION

Affinity forms the basis for many aspects of social behavior, especially, the formation and evolution of groups or networks [6, 7, 12]. Affinity has two meanings. The first is a natural liking for or attraction to a person, thing, idea, etc. This kind of affinity is called *direct affinity* in this paper. The second defines affinity as a close relationship between people or things that have similar appearances, qualities, structures, properties, or features. This paper

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calls it *indirect affinity*. Two difficulties arise when dealing with affinity. First, affinity is, by definition, a vague and imprecise concept. Indeed, it is very difficult to precisely evaluate an affinity like friendship; it can be approximately described by linguistic terms like strong or weak. The second is that affinity often, if not always, varies with time. For example, a student may have strong affinity with the college he is studying now, but the affinity becomes weak after he graduates.

So far as we know, in literatures, there is no theory dealing with affinity as a vague and time-dependent concept, and little scholarly awareness that such a simple affinity idea could be developed for valuable models in management sciences. This paper originally proposes a theoretical framework for the affinity concept, different from fuzzy sets [13] and fuzzy relations [3]. Fuzzy set theory is the best tool for representing vague and imprecise concepts so far; however, the affinity set proposed here is not merely a fuzzy set because assuming any type of membership function here is unnecessary: the affinity concept is more general than the fuzzy concept. In the affinity set theory, allowing a decision maker uses his subjective perception of distance to from a set is possible, interesting, and innovative. Therefore, this work simply defines affinity as the closeness/distance between two objects [2]: the distance measurement could be geometric or abstract, and affinity could play various roles in a decision problem depending on decision maker achievement. Actually, the distance/closeness concept is more strongly related to Topology [4] rather than Fuzzy sets [5, 13]; however, these topology abstractions are translated/simplified into useful modeling concepts and procedures here.

The paper is organized as follows. Section 1 introduces the affinity set and related notions, formalizing indirect affinity and discussing its application. A qualitative forecasting method based on affinity set and game theory is newly presented, different from the traditional quantitative forecasting models because the historical trend is no longer necessary. The performance of this new model is compared with the simple regression model to show its value. Section 2 formalizes direct affinity. Last section concludes the paper.

1. AFFINITY SET AND INDIRECT AFFINITY

This paper refers to this type of affinity, which could be mediated by some intermediums as *indirect affinity***.** Mathematically, indirect affinity can be understood as a relation between elements of a set, the subjects, with an object or medium, the relation is the affinity itself. The traditional crisp relations cannot be used to model indirect affinity for the following two reasons. First, affinity is, by definition, a vague and imprecise concept. The second is that affinity often, if not always, varies with time, for example the affinity between a student and his studying college may become stronger or weaker or have ups and downs over time.

1.1. Affinity set and affinity

We start by presenting the meaning we give to the primitive notion of *set.* Since the objective in this section is to formalize affinity time-dependence between an element and a set, our meaning should encompass the variability of shape or content of a set.

Definition 1

By affinity set we mean any object (real or abstract) that creates affinity between objects.

Example 1

A religion is an affinity set, for it creates affinity between people that makes them live a certain way of life.

Example 2

A famous artist or scientist or singer or sportsman or sportswoman is an affinity set for he or she creates affinity between people who appreciate him or her.

From the above examples we deduce that our set notion is wider than the traditional set notion and the fuzzy set notation. Let us now give a formal definition of affinity between a subject *e* and an affinity set.

Definition 2

Let e *and* A *be a subject and an affinity set, respectively. Let I be a subset of the time axis* $[0,+\infty[$. *The affinity between e and* A *is represented by a function.*

The value $M_A^e(t)$ expresses the degree of affinity between the subject *e* and the affinity set A at time *t*. When $M_A^e(t) = 1$ this means that affinity of *e* with affinity set A is complete or at maximum level at time *t*; *but it doesn't mean that e belongs to* A, unless the considered affinity is belongingness. When

$M_A^e(t) = 0$ this means that e has no affinity with A at time t. When $0 < M_A^e(t) < 1$, this means that e has partial affinity with A at time t. Here we emphasize the fact that the notion of affinity is more general than the notion of membership or belongingness. The later is just a particular case of the former.

Definition 3

The universal set, denoted by U, is the affinity set representing the fundamental principle of existence. We have:

$$M_U^e(\,.\,): [0,+\infty\,) \to [0,1]$$

$$t \to M_U^e(t)$$

and $M_U^e(t) = 1$, for all existing objects at time t and for all times t.

In other words the affinity set defined by the affinity „existence" has complete affinity with all previously existing objects, that exist in the present, and that will exist in the future. In general, in real world situations, some traditional referential set V, such as that when an object e is not in V, $M_A^e(t) = 0$ for all t in $I \subset [0,+\infty[$, can be determined. In order to make the notion of affinity set operational and for practical reasons, in the remainder of the paper, instead of dealing with the universal set U, we will deal only with affinity sets defined on a traditional referential set V. Thus, in the remainder of the paper when we refer to an affinity set, we assume that sets V and I are given.

Definition 4

Let A be an affinity set. Then the function defining A is:

$$\begin{aligned} &F_A(.,.): V \times I \to [0,1] \\ &(e, t) \to F_A(e, t) = M_A^e(t) \end{aligned} \tag{1}$$

An element in real situations often belongs to a set at some time and not at other times. Such behavior can be represented using the affinity set notion. The behavior of affinity set A over time can also be investigated through its function $F_A(.,.)$.

Interpretation 1

1) For a fixed element e in V, the function (1) defining the affinity set A reduces to the fuzzy set describing degree variation of affinity of the element e over time.
2) For a fixed time t, the function (1) reduces to a fuzzy set defined on V that describes the affinity between elements V and affinity set A at time t. Roughly speaking it describes the shape or "content" of affinity set A at time t.
3) In addition to 1) and 2), we can't say/validate affinity set as a special fuzzy set, unless we can prove that any affinity set A is included in a fuzzy set B and vice versa.
4) Any distance/closeness could be normalized to [0, 1], however, such a normalization process is not necessarily fuzzy.

The maximum affinity $\mathrm{M}_{\mathrm{A}}^{e}(t)=1$ may not be reached at any time in real-world problems. In order to consider various situations we introduce the following definition.

Definition 5

Let A be an affinity set and $k \in [0,1]$. We say that an element e is in the t-k-Core of the affinity set A at time t, denoted by t-k-Core(A), if $\mathrm{M}_{\mathrm{A}}^{e}(\mathrm{t}) \geq k$, that is:

$$\text{t-k-Core(A)} = \left\{e \mid \mathrm{M}_{\mathrm{A}}^{e}(t) \geq k\right\}$$

when $k = 1$, t-k-Core(A) is simply called the core of A at time t, denoted by t-Core(A).

Definition 6

An observation period is defined as the period (continuous or discrete) analyzing the behavior of an element e of V with respect to an affinity set A (an illustration is given in Figure 1 below).

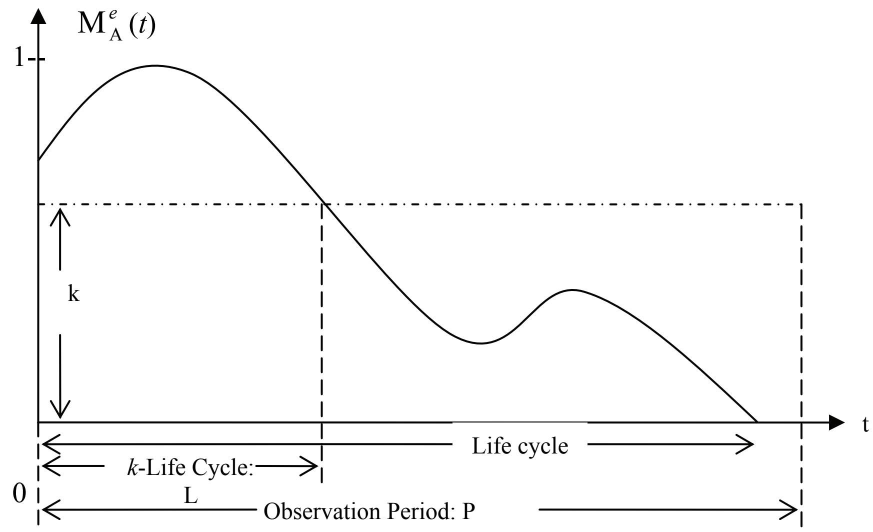

Fig. 1. Illustration of the affinity between an element e and an affinity set A over an observation period P

Definition 7

Let A be an affinity set and $k \in [0,1]$. A subset T (discrete or continuous) of I is said to be the k-life cycle of an element e with respect to A if:

$$M^e_A(t) \geq k, \text{ for all } t \in T \text{ and } M^e_A(t) < k, \text{ elsewhere in I}$$

In other words, the period T is the k-life cycle of e with respect to A if e is in the t-k-Core(A) for all t in T. It is the time period that element e keeps its affinity at least equal to k in I. The period of time $T_C = \{t \mid M^e_A(t) > 0, t \in I\}$ is called life cycle of the element e with respect to the affinity set A.

1.2. Indirect affinity

Indirect affinity occurs when affinity between subjects takes place via a medium. This section gives a formal definition of indirect affinity. The notion of harmony between objects with respect to an affinity set is also formalized.

Definition 8

Let A be an affinity set and $k \in [0,1]$. Let D be a subset of V. A k-indirect affinity degree with respect to A, at time t, between the elements of D exists, if they all belong to the t-k-Core(A), that is $D \subset t-k-Core$ (A). A k-indirect affinity degree with respect to A, during an observation period T, between the elements of D exists, if $D \subset t-k-Core$ (A) at any time t in T.

Definition 9

Let A and D be an affinity set and a subset of V, respectively. Harmony exists at time t between the elements of D with respect to A, if they all belong to t-Core(A) at time t, that is, $D \subset t-Core$ (A). In other words, harmony between the elements of D with respect to A is reached at time t when the maximum indirect affinity degree between them is $k=1$ at this time. Harmony exists during the observation period of time T, with respect to A, between the elements of D, if there is harmony with respect to A between them at any time t in T. This definition expresses the fact that harmony is the highest level of affinity.

1.3. Operations on affinity sets

This section defines basic affinity set operations. The following definitions 10-14, assume that A and B are two given affinity sets defined on I and V.

Definition 10

We say that A and B are equal at time t if $M_A^e(t) = M_B^e(t)$, for all e in V. Then we write A = B at time t. If A and B are considered in an observation period T, then A = B during this period if $M_A^e(t) = M_B^e(t)$, for all e in V and all t in T.

Definition 11

We say that A is contained in B at time t if $M_A^e(t) \leq M_B^e(t)$, for all e in V. Then we write $A \subset B$ at time t. In the case that A and B are considered in an observation period T, then $A \subset B$ during this period if $M_A^e(t) \leq M_B^e(t)$, for all e in V and all t in T.

Definition 12

The union of A and B at time t, denoted by $A \cup B$, is defined by the function $F_{A \cup B}(t, e) = M^e_{A \cup B}(t) = \text{Max}\{M^e_A(t), M^e_B(t)\}$, for all e in V. In the case that A and B are considered in an observation period T, then during this period, $A \cup B$ is defined by the function $F_{A \cup B}(t, e) = M^e_{A \cup B}(t) = \text{Max}\{M^e_A(t), M^e_B(t)\}$, for all e in V and all t in T.

Definition 13

The intersection of affinity sets A and B at time t, denoted by $A \cap B$, is defined by the function $F_{A \cap B}(t, e) = M^e_{A \cap B}(t) = \text{Min}\{M^e_A(t), M^e_B(t)\}$, for all e in V. In the case that A and B are considered in an observation period T, then during this period, $A \cap B$ is defined by the function $F_{A \cap B}(t, e) = M^e_{A \cap B}(t) = \text{Min}\{M^e_A(t), M^e_B(t)\}$, for all e in V and all t in T.

Definition 14

B is said to be the complement of A at time t if it is defined by the following function $F_B(t, e) = M^e_B(t) = 1 - M^e_A(t)$, for all e in V. In the case that A an B are considered in an observation period T, then during this period, B is defined by the function $F_B(t, e) = M^e_B(t) = 1 - M^e_A(t)$, for all e in V and all t in T.

1.4. Application of forecasting

The affinity set's potential applications are valuable in analyzing, evaluating, forecasting (predicting) the time-dependent behaviors: for example, evolving an uncertain dynamic system in a human society. In addition, predicting the demand curve with high fluctuations is also possible by an affinity set. We will give a simple example of how the affinity set can be applied in forecasting real-world problems later. In fact, any time series method can be used to predict any element e behavior in V with respect to an affinity set A based on past data, if it is possible to define affinity set A. This paper proposes a new forecasting method based on affinity set and game theory. Assume that an affinity set A and a universe V are given and some data are available at some past periods $t_1, t_2, ..., t_n$ on the behavior of elements e in V with respect to affinity set A as described in the following matrix [1]:

$$\mathbf{D} = \begin{array}{c} \\ \mathrm{A} \\ \overline{\mathrm{A}} \end{array} \begin{array}{c} \begin{array}{ccc} t_1 & \dots & t_n \end{array} \\ \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \end{pmatrix} \end{array}$$

Here we can follow the similar concept in [1], $t_1, t_2, ..., t_n$ are regarded as the multiple attributes of the decision problem, and $\overline{\mathrm{A}}$ and A are two alternatives of this problem. But we will define a new method, which is different from [1] to resolve this affinity game. Where $\overline{\mathrm{A}}$ is the affinity set complementary to A (see Definition 14), entry a_{1j} is the affinity degree of element *e* with respect to affinity set A at the period t_j and $a_{2j} = 1 - a_{1j}$ is the affinity degree of element *e* with respect to affinity set $\overline{\mathrm{A}}$ at the same period. Here a decision maker wants to forecast element *e* behavior at the next period t_{n+1}. Interestingly we can look at the situation as a game between the decision maker and Nature. The decision maker faces an uncertain situation represented by future element *e* behavior. One way to handle the situation is to adopt the maximum decision making under uncertainty principle [3] by considering the situation as a game against Nature [1]. Thus, matrix D can be considered as a matrix game between the decision maker and Nature, where the decision maker is the maximizing player who chooses between A and $\overline{\mathrm{A}}$ and Nature is the minimizing player who chooses the time periods.

Definition 15

A pair of strategies (i_0, j_0) *where* $i_0 \in \{1,2\}$ *and* $j_0 \in \{1,..,n\}$ *are said to be the Nash equilibrium* [9] *of the matrix game* D *if*:

$$a_{ij_0} \le a_{i_0 j_0} \le a_{i_0 j}, \text{ for all } i \in \{1,2\} \text{ and } j \in \{1,..,n\} \tag{2}$$

Assume that the game has a Nash equilibrium (i_0, j_0). In terms of affinity, this equilibrium can be interpreted as follows. If $i_0 = 1$, the decision maker will favor element *e* affinity with affinity set A rather than affinity with $\overline{\mathrm{A}}$, with affinity degree $a_{i_0 j_0}$. The decision maker in case $i_0 = 2$ will favor element *e* affinity with $\overline{\mathrm{A}}$ rather than with A, with affinity degree $a_{i_0 j_0}$. It may happen that matrix D has no Nash equilibrium in pure strategies, then the two players have to use mixed strategies. A mixed strategy for Nature is a probability distribution over the set of its pure strategies, that is, it is a vector $y = (y_1, y_2, ..., y_n)$ such that:

$$\sum_{j=1}^{n} y_j = 1 \text{ and } y_j \geq 0,\ j = \overline{1,n}$$

Similarly, a mixed strategy for the decision maker is a vector $x = (x_1, x_2)$ such that:

$$x_1 + x_2 = 1 \text{ and } x_i \geq 0,\ i = 1,2$$

Player payoffs become expected payoffs. Decision maker payoff is $x^T \mathrm{D} y$ and that of Nature is $-x^T \mathrm{D} y$. Any matrix game always includes a Nash equilibrium in mixed strategies [9]. A Nash equilibrium in mixed strategies is defined by:

$$x^T \mathbf{D} y^* \leq x^{*T} \mathbf{D} y^* \leq x^{*T} \mathbf{D} y$$

for all mixed strategies x and y. The mixed strategy x^* of the decision maker can be interpreted as follows. The decision maker will favor A with weight x_1 and $\overline{\mathrm{A}}$ with weight x_2. He can also use these two evaluations to rank sets A and $\overline{\mathrm{A}}$ from his point of view. The expected affinity degree of element e in the period t_{n+1} with each of the affinity sets can be defined as follows:

$$\mathrm{M}_{\mathrm{A}}^{e}\ (t_{n+1}) = \sum_{1}^{n} a_{1j} y_j^* \text{ and } \mathrm{M}_{\overline{\mathrm{A}}}^{e}\ (t_{n+1}) = \sum_{1}^{n} a_{2j} y_j^*$$

respectively. The mixed strategy y^* of Nature can be interpreted as the weights Nature assigns to the periods in order to minimize expected decision maker affinity. Let us illustrate our approach by examples.

Example 3. Decision of buy in/sale out/hold

Today, we are aware that the stock price in a market is quite unstable; in other words, the stock price curve is highly fluctuating for a company. Now we collect the actual data of Taiwan TGV Company for twenty-two periods (from October 1, 2007 to October 22, 2007) from Taiwan Stock Market [8]. Assume that a decision maker wants to predict if he can buy in or sell out his stocks in the market by updating his information and using the affinity game. The first seventeen data are used as the training base, then we predict the remaining five data. Please note that if we want to predict the eighteen data by affinity model, then the previous seventeen data will be all included in an affinity game, and if predicting the nineteen data then the previous eighteen data will be included, etc. Assume, for simplicity, that by experience he classifies his decisions into only "Buy in", "Sell out" and "Hold". These two possible states can be considered as two affinity complementary sets A (Buy in) and $\overline{\mathrm{A}}$ (Sell out), respectively. His decision will be the element *e.* And if the affinity degree of *e* to A, and that to $\overline{\mathrm{A}}$ are identical, then he chooses the "Hold" state. The price data of twenty-two dates in October 2007 are collected as in Figure 2. Assume this decision maker has recorded the affinity degrees of stock price with respect to affinity set A by the following function:

$$c_{1t} = (\frac{p_t}{p_{t-1}}), t = 2,3,\ldots,n \tag{3}$$

and

if $c_{1t} < 1$ then $a_{1t} = 1 - c_{1t}$;

if $c_{1t} > 1$ then $a_{2t} = c_{1t} - 1$;

if $c_{1t} = 1$ then $a_{1t} = a_{2t} = 0.5$, which is a "Hold" state: no buying in and no selling out.

Here $a_{1t} = 1 - a_{2t}$ is also assumed.

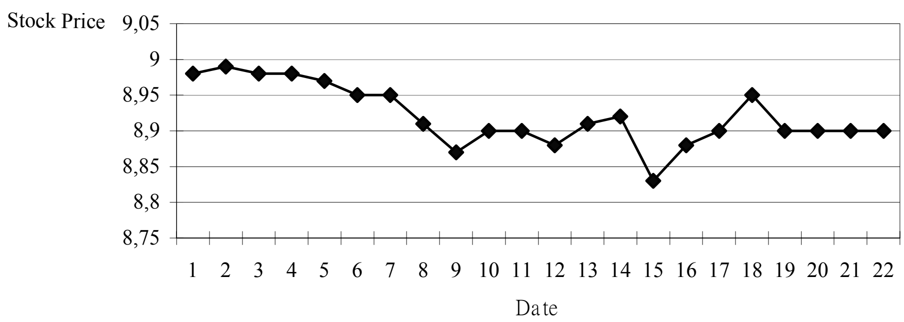

Fig. 2. Actual data of stock price (in Taiwanese Dollars per stock)

Source: [8].

Table 1

Performance comparison of affinity model and simple regression

Model\Periods	t_{18}	t_{19}	t_{20}	t_{21}	t_{22}
Affinity Game	Hold	Hold	Hold	Hold	Hold
Simple Regression	8.85	8.84	8.84	8.83	8.82
Actual Data	8.95	8.90	8.90	8.90	8.90

Source: [8].

According to the actual data, the affinity matrix is easily computed (see Appendix). The suggested decision is summarized in Table 1, which is compared with the simple regression model (only using time t as the explainable variable). Interestingly the affinity model suggests the "Hold" state, which seems to be better than predicting the declining trend by the simple regression model. Because if a declining trend is forecasted, then the action of selling out stocks will be considered by this decision maker. However, the "Hold" state suggested by the affinity model hints the decision maker makes more profits if he keeps these stocks from the time period: t_{17}. Affinity game predicts that the stock price will remain almost stable during the analyzed time. It is clear that affinity model performs better than the simple regression model in this experiment. Of course, the function (3) could be assumed by various types, the decision maker can choose any type that he prefers. The affinity spirit is eventually, a decision maker is encouraged to try/develop any possible measurement to find/explore/analyze the *special pattern* in a time-dependent data set or input/output system. And this *special pattern* is arbitrarily defined by a decision maker, just like that distance/closeness have general definitions in Topology [4]. A *special pattern* could vary with time and space, once the decision maker catches the core of this *special pattern*, he could explain the observations or predict some useful outcomes. Actually, there is an old saying: *what you measured the world filters what you see*. Thus, various measurements for modeling are natural and should be encouraged.

1.5. Affinity set depending on time and other variables

The affinity of element e with respect to affinity set A in real-world situations often depends implicitly on other variables than time. These variables generally express condition or constraint variability that affect affinity evaluation. Studying element e behavior with respect to time and other variables may be practically desirable. A decision maker may even study element e behavior at a fixed time with respect to other variables. This section extends the affinity set definition to the case where desired variables appear explicitly. This definition makes it possible to study e affinity behavior over time and with respect to other variables as well.

Definition 16

Let e and A *be an element and an affinity set, respectively. Assume that the affinity of* e *with respect to* A *depends on some variable w that takes its values in a traditional set* W. *In order to make the variable w appear in the affinity definition between e and* A, *we introduce the following affinity:*

$$\mathrm{M}_{\mathrm{A}}^{e}\ (\ .\ ): \mathrm{I} \times \mathrm{W} \rightarrow [0,1]$$

$$(t,w) \ \rightarrow\ \mathrm{M}_{\mathrm{A}}^{e}\ (t,w)$$

The value $\mathrm{M}_{\mathrm{A}}^{e}(t,w)$ expresses the degree of affinity between element e and A at time t with respect to w.

Thus, depending on the problem at hand, the decision maker can use Definitions 2 or 16 of affinity between an element e and an affinity set A.

Definition 17

Let A *be an affinity set depending on a variable* $w \in \mathrm{W}$. *Then the function defining* A *is defined by:*

$$F_{\mathrm{A}}\,(.,\,.\,,\,.): \mathrm{V}\times\mathrm{I}\times\mathrm{W} \rightarrow\ [0,1]$$

$$(e,\ t,w) \rightarrow\ F_{\mathrm{A}}\,(e,\ t,w)=\ \mathrm{M}_{\mathrm{A}}^{e}\,(t,w)$$

where V *is the traditional referential as in Section 1.*

2. DIRECT AFFINITY

Direct affinity is a natural liking for or attraction to a person or a thing, or an idea, etc. Direct affinity involves two elements: the affinity subjects and the affinity that takes place between them. Mathematically, direct affinity can be understood as a binary relation between elements of a set, where the elements are the subjects and the relation is affinity. The traditional crisp binary relations cannot be used to model direct affinity for the following two reasons: First, affinity is, by definition, a vague and imprecise concept. Indeed, it is very difficult to give a precise evaluation of affinity like friendship; it can be approximately described by linguistic terms like strong or weak;

the second is that affinity often, if not always, varies with time, for example friendship may become stronger or weaker or have ups and downs over time. Thus, the adequate way to model direct affinity is to use time-dependent fuzzy relations. Affinity can be considered as a particular case of the following general framework.

Definition 18

Let V *and* I *be a referential set and a subset of the time axis* $[0,+\infty[$, *respectively. A time dependent fuzzy relation R such that:*

$$\begin{aligned} R_{(.,.)}(.) : \mathrm{I} \times (\mathrm{V} \times \mathrm{V}) &\to [0,1] \\ (t,(e,s)) &\to R_{(e,s)}(t) \end{aligned} \tag{4}$$

is called direct affinity on the referential V.

Interpretation 2

1. *For any fixed time t the relation (2) reduces to an ordinary fuzzy relation* [3]:

$$\begin{aligned} R_{(.,.)}(t) : \mathrm{V} \times \mathrm{V} &\to [0,1] \\ (e,s) &\to R_{(e,s)}(t) \end{aligned}$$

that expresses the intensity or the degree of affinity between any couple of elements in V. *Hence affinity fuzziness between elements is taken into account in Definition 18.*

2. *For any fixed couple of elements* $(e,s) \in \mathrm{V}$, *the relation (4) reduces to a fuzzy set defined on the time-set* I*:*

$$\begin{aligned} R_{(e,s)}(.) : \mathrm{I} &\to [0,1] \\ t &\to R_{(e,s)}(t) \end{aligned}$$

that expresses affinity evolution over time between elements e *and* s.

Thus, the time-dependent fuzzy relation (4) expresses the most important characteristic of direct affinity: Fuzziness and time-dependence.

Definition 18 can be extended to affinity between groups of elements as follows.

Definition 19

Let R be a time-dependent fuzzy relation defined on a subset of time axis I *and a referential V. Let* A *and* B *be two subsets of* V. *Then the affinity between* A *and* B *can be described by the following function:*

$$R_{(\mathrm{A},\mathrm{B})}(.) : \mathrm{I} \rightarrow [0,1] \qquad (5)$$
$$t \rightarrow R_{(\mathrm{A},\mathrm{B})}(t)$$

where $R_{(A,B)}(.)$ *can be defined by many ways, depending on the decision maker. We propose the following four examples:*

1) $R_{(\mathrm{A},\mathrm{B})}(t) = \max\limits_{(e,s)\in \mathrm{A}\times\mathrm{B},\ e\neq s} R_{(e,s)}(t)$, for all $t \in \mathrm{I}$
2) $R_{(\mathrm{A},\mathrm{B})}(t) = \min\limits_{(e,s)\in \mathrm{A}\times\mathrm{B},\ e\neq s} R_{(e,s)}(t)$, for all $t \in \mathrm{I}$
3) $R_{(\mathrm{A},\mathrm{B})}(t) = \alpha \max\limits_{(e,s)\in \mathrm{A}\times\mathrm{B},\ e\neq s} R_{(e,s)}(t) + (1-\alpha) \min\limits_{(e,s)\in \mathrm{A}\times\mathrm{B},\ e\neq s} R_{(e,s)}(t)$, for all $t \in \mathrm{I}$, *where* α *is a number in* $[0,1]$ *that expresses the degree to which the decision maker prefers the maximum of affinity to its minimum.*
4) in the case A and B are finite $R_{(\mathrm{A},\mathrm{B})}(t) = \sum\limits_{(e,s)\in \mathrm{A}\times\mathrm{B},\ e\neq s} \lambda_{(e,s)} R_{(e,s)}(t)$, for all $t \in \mathrm{I}$, where $\lambda_{(e,s)} \geq 0$ is the weight assigned by the decision maker to the couple (e,s) for $e \neq s$ and $\sum\limits_{(e,s)\in \mathrm{A}\times\mathrm{B},\ e\neq s} \lambda_{(e,s)} = 1$.

Here also for practical purpose we define the *t*-*k*-affinity.

Definition 20

Let R be a time-dependent fuzzy relation defined on a subset of time axis I and *a referential* V. *Let* $k \in [0,1]$, *and* $t \in$ I. Then:

1) *we say that a couple (e, s) has k affinity degree at time t or t-k-affinity degree if* $R_{(e,s)}(t) \geq k$,
2) *a subset D of V has t-k-affinity degree if* $R_{(D,D)}(t) \geq k$. *Thus, the t-k-affinity degree of subsets depends on how affinity is defined between groups or subsets as indicated in Definition 19,* 1)-4).

Remark 1

Depending on information available for the time-dependent fuzzy relation describing affinity (2)-(3), direct affinity can be used to study networks (social or nonsocial). Indirect affinity can also be used to analyze, describe, forecast, and predict network behavior or its elements regarding the considered affinity. For example, with knowledge that network evolution over time follows a differential equation or a stochastic process, that is, the function

$t \to R_{(e,s)}(t)$ *is a solution of a differential equation or a stochastic process, then based on initial data one can predict network behavior at any time* $t \in I$ *regarding the considered affinity. Social network analysis* [6, 12] *is one area for direct affinity application. In addition, the direct affinity concept is valuable in developing network grouping or network controlling.*

CONCLUSIONS AND RECOMMENDATIONS

This paper proposes a basic framework for the affinity concept, allowing its investigation by fuzzy set tools and other nonfuzzy methods. Of course, fuzzy tools are not the only way to explore affinity. Readers should realize that the affinity model proposed in Example 3 is quite different from the fuzzy set and rough set [10, 11] because we don't need to assume any type of fuzzy membership function [10] or use the upper bound and lower bound to approximate a set [11]. Instead, the closeness or distance between any two objects within a time series data set is directly assumed, then it will form the basis of an affinity set. Numerous measurements of closeness/distance could exist in Example 3, but we only propose/assume one way here.

We studied two types of affinity: Indirect affinity and direct affinity. This work pointed out that indirect affinity requires a medium and introduces the affinity set for indirect affinity formalization, which actually represents the medium. The affinity of elements with respect to an affinity set is represented by a fuzzy set defined on the time axis. Then the affinity between elements (indirect affinity) is defined via their affinity to the affinity set. We have formalized direct affinity as a time-dependent fuzzy relation and present a new forecasting method based on affinity set and game theory. Finally, we indicate some potential areas for possible application of direct affinity and indirect affinity. Many issues are not fully discussed in this paper. One of them is the numerical determination of functions $t \to M_A^e(t)$ and $t \to R_{(e,s)}(t)$ that represent affinity in indirect affinity and direct affinity, respectively. Another issue is exploration of the affinity set notion. We believe that investigating affinity in social networks or engineering control using our framework is a worthwhile topic of research. We also hope that this paper will inspire and attract more researchers for investigating the affinity concept. The evolutionary algorithms will be beneficial when we try to find/explore the *special pattern* hidden in a large scale data set; for example, evolving the *special pattern* that maximizes a specified/predefined affinity.

Appendix

Actual data and affinity degree

Date	Traded Stocks	Average Price	+Up/-Down	$c_{1t} = (\frac{p_t}{p_{t-1}})$	a_{1t}	a_{2t}
07/Oct/01	3,252,380	8.98	-0.03			
07/Oct/02	1,058,000	8.99	+0.01	1.001114	0.998886	0.001114
07/Oct/03	1,660,201	8.98	-0.01	0.998888	0.001112	0.998888
07/Oct/04	1,018,000	8.98	+0.00	1	0.50	0.50
07/Oct/05	1,200,500	8.97	-0.01	0.998886	0.001114	0.998886
07/Oct/06	999,000	8.95	-0.02	0.99777	0.00223	0.99777
07/Oct/07	970,912	8.95	+0.00	1	0.50	0.50
07/Oct/08	1,177,913	8.91	-0.04	0.995531	0.004469	0.995531
07/Oct/09	1,696,000	8.87	-0.04	0.995511	0.004489	0.995511
07/Oct/10	1,687,000	8.90	+0.03	1.003382	0.996618	0.003382
07/Oct/11	781,000	8.90	+0.00	1	0.50	0.50
07/Oct/12	789,000	8.88	-0.02	0.997753	0.002247	0.997753
07/Oct/13	1,409,000	8.91	+0.03	1.003378	0.996622	0.003378
07/Oct/14	622,300	8.92	+0.01	1.001122	0.998878	0.001122
07/Oct/15	783,535	8.83	-0.09	0.98991	0.01009	0.98991
07/Oct/16	1,702,000	8.88	+0.05	1.005663	0.994337	0.005663
07/Oct/17	859,000	8.90	+0.02	1.002252	0.997748	0.002252
07/Oct/18	1,449,956	8.95	+0.05			
07/Oct/19	586,000	8.90	-0.05			
07/Oct/20	1,985,956	8.90	+0.00			
07/Oct/21	1,166,913	8.90	+0.00			
07/Oct/22	1,209,000	8.90	+0.00			

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Maciej Nowak

AN APPLICATION OF INTERACTIVE MULTIPLE CRITERIA TECHNIC IN LABOR PLANNING[*]

Abstract

The aim of each production or service system is to deliver final products or services in right quantities, on time and at appropriate cost. Decisions affecting such system are usually grouped in three categories: strategic, tactical, and operational. While the strategic planning decisions are mostly focused on the development of resources satisfying the external requirements, the tactical ones are concerned with the utilization of these resources. Finally, the operational decisions deal with day-to-day operational and scheduling problems and require disaggregation of the information generated on higher levels.

Labor planing, considered in this paper, is concerned with determining staffing policies that deal with employment stability and work schedules. A staffing plan is a managerial statement of time-phased staff sizes and labor-related capacities, which takes into consideration customers' requirements and machine-limited capacities. Such plan has to balance conflicting objectives involving customer service, work-force stability, cost, and profit.

In the paper, a multicriteria decision aiding procedure is proposed for labor planning problems. Simulation technic is employed for evaluating decision alternatives with respect to criteria. Demand forecasts and calendar constraints are taken into account in the simulation model. Uncertainties related to employees' accessibility are also considered. In the second phase, an interactive multiple criteria procedure is used for selecting the final solution of the problem.

A numerical example is presented to illustrate the applicability of the proposed technic.

Keywords

Labor planning, simulation, multiple criteria.

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INTRODUCTION

The aim of each production or service system is to deliver final products or services in right quantities, on time and at appropriate cost. Decisions affecting such system are usually grouped in three categories: strategic, tactical, and operational. While strategic decisions are mostly focused on the development of resources to satisfy the external requirements, tactical ones are concerned with the utilization of these resources [2]. Finally, operational decisions deal with day to day operational and scheduling problems and require disaggregation of the information generated on higher levels.

Labor planning, considered in this paper, is concerned with determining staffing policies that deal with employment stability and work schedules [3]. A staffing plan is a managerial statement of time-phased staff size and labor-related capacities, which takes into consideration customer requirements and machine-limited capacities. Such plan has to balance conflicting objectives involving customer service, work-force stability, cost, and profit.

Various technic are employed for solving labor planning problems. Linear programming and dynamic programming are used most often. However, these approaches are based on strong assumptions that often are not satisfied. Employees' attainability varies due to planned and unexpected absences. Work-force requirements are not stable as well. Often, considerable fluctuations can be noticed even in short-term. In accounts or payroll departments, for example, work-force requirements are usually higher in the early part of the month than in the latter one.

In the paper a staffing planning problem is considered. It consists in determining the number of full-time and part-time employees for a department in which work-force requirements fluctuate in a month. Three criteria are considered: yearly labor costs, number of overtime hours worked by all employees during the whole year, and the work-force utilization rate. While the first and the second criteria are to be minimized, the last one is to be maximized. The solving procedure is based on simulation and interactive multiple criteria technic. First, simulation experiments are performed in order to evaluate alternatives with respect to criteria. It is assumed that distributions of work--force requirements for each week of the month are available. For each alternative a distribution of work-force capacity is generated taking into account the probability, that employees are show up for work. The simulation model considers calendar constraints resulting from national holidays and Polish Labor

Code rules. As a result distributional evaluations are obtained for each alternative and each criterion. In the second phase, interactive technic INSDECM-II is employed for solving a multiple criteria problem.

The paper is organized as follows. The decision problem is presented in section 1. Section 2 deals with the simulation model of the problem. In Section 3 stochastic dominance rules are briefly presented. These rules are employed in the interactive multiple criteria procedure INSDECM-II described in Section 4. Section 5 provides a numerical example. The last section gives conclusions.

1. LABOR PLANNING IN ACCOUNTS DEPARTMENTS

Each economic organization has to prepare a variety of plans. Demand forecasts prepared by a manufacturer or service organization can address short-, medium- and long-time problems. Long-range forecasts deal with capacity and strategic issues. Problems, such as facility location and expansion, new product development, research funding, and investment over a period of several years have to be solved on this level.

Labor planning is a part of medium-range planning, which starts once long-term decisions have been made. Problems concerned with matching the productivity to fluctuating demand have to be solved on this level. Production and staffing medium-range plans link organization's strategic goals with the master production schedule and work-force-schedule. A planning horizon for such plans is usually one year, although it can differ in various situations.

Labor plan should provide following information:

– how many employees are needed for each department,
– should they be full-time or part-time workers,
– what the salaries should be, etc.

In the paper labor planning in accounts and payroll departments is considered. Work-force requirements in such divisions are usually higher in the early part of the month due to fixed pay day or insurance and tax forms preparation dead-lines. In order to meet requirements both full-time and part-time employees can be hired. Overtime can also be used to satisfy work-force requirements that cannot be completed in regular time. However, overtime is expensive. According to Polish Labor Code, 50 percent bonus has to be paid if overtime work is done on working day, while 100 percent bonus is to be paid for working on Saturdays, Sundays and holidays. Additionally, the number of overtime hours worked by an employee is limited to 150 per year. Moreover, in many cases workers do not want to work a lot of overtime for extended period. Finally, increased utilization of overtime may lead to decreased productivity due to employees' tiredness. If work-force requirements fluctuations are considerable, employees' working hours may not be fully utilized in some periods. Such situation is inconvenient, as it results in the labor costs increase. It is also unfavorable from psychological point of view. Balancing various objectives in order to arrive at an acceptable staffing plan involves consideration of various decision alternatives.

The decision problem considered in this paper consists in determining the number of full-time and part-time employees. Decision alternatives are evaluated with respect to three criteria:

X_1 – yearly labor costs,

X_2 – total number of overtime hours worked by all employees in the department during the year,

X_3 – work-force utilization rate measured by the contribution of regular hours effectively worked in the total number of regular hours worked by employees.

In order to solve the problem, alternatives have to be evaluated with respect to attributes. Simulation technic is an efficient and flexible tool for doing this.

2. SIMULATION MODEL FOR LABOR PLANNING

One of the most important elements of simulation modeling is identifying appropriate probability distributions for input data. Usually, this requires analyzing empirical or historical data and fitting these data to probability distributions. Sometimes, however, such data are not available and an appropriate distribution has to be selected according to the decision maker's judgment. Once the simulation model is built, verified, and validated, it can be used for generating probability distributions of output variables.

In our problem, distributions of work-force requirements in successive weeks of each month and distribution of employees' accessibility have to be identified. This requires analyzing historical data (e.g. the number of documents processed in previous periods), as well as the information about planned and unplanned absences of employees. Based on these data probability distributions of input data can be determined.

Let us assume following notation:

t – the number of the week in the year,
d_t – work-force requirements in week t,
r_t – regular time work-force capacity in week t,
w_t – regular wage per hour,
b_t – overtime bonus (%),
o_t – number of overtime hours worked by employees in week t,
u_t – number of regular-time hours that where not effectively utilized in week t,
c_t – labor cost in week t,
C – labor cost per year,
R – the number of regular-time hours worked by all employees in the year,
O – the number of overtime hours worked by all employees in the year,
U – the number of regular-time hours that were not effectively utilized

Simulation experiment is performed as follows:

1. $t = 1, X_1 = 0, X_2 = 0, X_3 = 0$.
2. Determine the work-force requirements d_t:
   - draw a random number,
   - use inverse transformation method to determine work-force requirements in week t.
3. Determine regular time work-force capacity for each working day in the week:
   - draw a random number,
   - use inverse transformation method to determine the work-force capacity for the day (number of regular hours that may be worked by all employees in this day);

   Sum daily capacities in successive days to achieve weekly capacity r_t.
4. If $r_t > d_t$ – determine the number of hours that are not effectively utilized:
$$u_t := r_t - d_t$$
5. If $d_t > r_t$ – determine the number of overtime hours worked:
$$o_t := d_t - r_t$$
6. Calculate the labor cost:
$$c_t := r_t w_t + o_t w_t \frac{b_t}{100}$$
7. $C := C + c_t$, $R := R + r_t$, $O := O + o_t$, $U := U + u_t$.
8. If $t = 52$ – go to 9, else assume $t := t + 1$ and go to 2.

9. Calculate values of criteria:

$$X_1 = C, X_2 = O, X_3 = (R - U) / R$$

Each simulation model can be classified as one of two types: terminating and non-terminating [7]. For a terminating simulation there is a natural end point that determines the length of a run. A non-terminating simulation does not have a natural end point. As the aim of our study is to analyze department's activities during a year-long period, terminating simulation is employed. In order to obtain accurate results, simulation runs should be repeated. Such experiments have to be performed for each alternative. As a result, distributional evaluations of alternatives with respect to criteria are obtained.

3. STOCHASTIC DOMINANCE RULES

The methodology used in this paper combines two methods that are frequently used for modeling the choice among uncertain outcomes: mean-risk approach and stochastic dominance. The former is based on two criteria: one measuring expected outcome and another one representing variability of outcomes; the latter one uses stochastic dominance rules. Two groups of stochastic dominance relations can be considered. The first group includes FSD, SSD, and TSD, which means first, second, and third degree stochastic dominance respectively (see Appendix for definitions). These rules can be applied for modeling risk-averse preferences. The FSD rule is for increasing utility function, i.e. for $u(x)$ such that $u'(x) > 0$ for all x. The SSD rule can be applied for concave increasing utility function: $u'(x) > 0$ and $u''(x) \leq 0$. Finally, the TSD rule is for decreasing absolute risk aversion (DARA) utility function, i.e. for a function with $u'(x) > 0$, $u''(x) \leq 0$, $u'''(x) \geq 0$ and $u'''(x) \cdot u'(x) \geq [u''(x)]^2$. The second group includes FSD and three types of inverse stochastic dominance: SISD, TISD1, TISD2 – second degree inverse stochastic dominance and third degree inverse stochastic dominance of the first and the second types. These rules can be applied for modeling risk-seeking preferences. The SISD rule is limited to a convex utility function: $u' > 0$ and $u'' \geq 0$, while TISD1 and TISD2 rules can be used in the case of increasing absolute risk aversion (INARA) utility function, i.e. function with $u'(x) > 0$, $u''(x) \geq 0$, $u'''(x) \geq 0$ and $u'''(x) \cdot u'(x) \leq [u''(x)]^2$ (TISD1) or $u'(x) > 0$, $u''(x) \geq 0$ $u'''(x) \leq 0$ (TISD2).

In the methodology presented here both approaches are employed. The decision maker defines his/her requirements by specifying minimal or maximal values of criteria measuring either expected outcome (mean) or variability of outcomes (standard deviation, standard semideviation, lower/upper standard semideviation from a target value, lower/upper mean semideviation from a target value, probability of below-target/over-target returns). However, SD relations between distributional evaluations are also analyzed in order to detect unclear situations.

4. INTERACTIVE PROCEDURE INSDECM-II

The procedure presented in this study is a modified version of INSDECM technic proposed in [6]. It also exploits some ideas used in the approach proposed in [5]. The first procedure is based on the interactive multiple criteria goal programming approach [8], the latter exploits the main ideas of the STEM technic [1].

INSDECM-II combines concepts that are used in multiple criteria goal programming and STEM method. In each iteration the ideal solution is generated. The elements of the ideal solution are the best values of mean for each criterion, which are individually attainable within the set of alternatives. Next, a candidate alternative is generated. It is the one that is closest to the ideal solution according to the minimax rule. Additionally, a potency matrix is generated. It is composed of the best and the worst values of mean with respect to all criteria. The candidate alternative and potency matrix are presented. If the decision maker is not satisfied with the data available, he/she may specify the kind of additional information that should be provided. By looking at the data, the decision maker may decide whether the proposal is satisfactory. If the answer is YES, the procedure ends, otherwise the decision maker is asked for defining additional requirements. It is assumed that such requirements specify minimal or maximal values of a specified distribution parameter. The consistency of the requirement formulated by the decision maker with stochastic dominance rules is analyzed. It is assumed that the requirement is not consistent with stochastic dominance rules if following conditions are simultaneously fulfilled:

– the evaluation of a_i with respect to criterion X_k does not satisfy the requirement,
– the evaluation of a_j with respect to criterion X_k satisfies the requirement,
– the evaluation of a_i with respect to X_k dominates corresponding evaluation of a_j under stochastic dominance rules.

The pair for which inconsistency occurs is presented and the decision maker is asked to confirm or relax the requirement. If the requirement is confirmed, the assumptions on the stochastic dominance rules that should be fulfilled are revised. As the decision maker confirms that he/she finds the distribution dominated according to stochastic dominance rules as better, it is assumed that this rule should not be used for modeling decision maker's preferences.

Let us assume the following notation:

$\mathbf{K}_1$ – the set of indices of criteria, that are defined in such a way that the larger values are preferred to smaller ones,

$\mathbf{K}_2$ – the set of indices of criteria, that are defined in such a way that the smaller values are preferred to larger ones,

$\mathbf{A}^l$ – set of alternatives considered in iteration l,

$\mathbf{I}^l$ – set of indexes i, such that $a_i \in \mathbf{A}^l$,

μ_{ik} – mean of the distributional evaluation of alternative a_i in relation to attribute k,

$\mathbf{P}_1^l$ – potency matrix:

$$\mathbf{P}_1^l = \begin{bmatrix} \underline{\mu}_1^l \cdots & \underline{\mu}_k^l & \cdots & \underline{\mu}_m^l \\ \overline{\mu}_1^l \cdots & \overline{\mu}_k^l & \cdots & \overline{\mu}_m^l \end{bmatrix}$$

where: $$\overline{\mu}_k^l = \begin{cases} \max\limits_{i \in \mathbf{I}^l}\{\mu_{ik}\} & \text{for } k \in \mathbf{K}_1 \\ \min\limits_{i \in \mathbf{I}^l}\{\mu_{ik}\} & \text{for } k \in \mathbf{K}_2 \end{cases} \qquad \underline{\mu}_k^l = \begin{cases} \min\limits_{i \in \mathbf{I}^l}\{\mu_{ik}\} & \text{for } k \in \mathbf{K}_1 \\ \max\limits_{i \in \mathbf{I}^l}\{\mu_{ik}\} & \text{for } k \in \mathbf{K}_2 \end{cases}$$

Q – number of distribution parameters chosen by the decision maker for presentation in conversational phase of the procedure,

$\mathbf{Q}_1$ – the set of indices of parameters, that are defined in such a way, that the larger values are preferred to smaller ones,

$\mathbf{Q}_2$ – the set of indices of parameters, that are defined in such a way, that the smaller values are preferred to larger ones,

v_{ip} – value of p-th parameter for alternative a_i, $i = 1, \ldots, \mathbf{I}^l$, $p = 1, \ldots, Q$,

$\mathbf{P}_2^l$ – additional potency matrix for attribute k in iteration l

$$\mathbf{P}_2^l = \begin{bmatrix} \underline{v}_1^l \cdots & \underline{v}_q^l & \cdots & \underline{v}_Q^l \\ \overline{v}_1^l \cdots & \overline{v}_q^l & \cdots & \overline{v}_Q^l \end{bmatrix}$$

where: $\overline{v}_k^l = \begin{cases} \max\limits_{i \in \mathbf{I}^l} \{v_{iq}\} & \text{for } q \in \mathbf{Q}_1 \\ \min\limits_{i \in \mathbf{I}^l} \{\mu_{iq}\} & \text{for } q \in \mathbf{Q}_2 \end{cases}$ $\quad \underline{v}_k^l = \begin{cases} \min\limits_{i \in \mathbf{I}^l} \{v_{iq}\} & \text{for } q \in \mathbf{Q}_1 \\ \max\limits_{i \in \mathbf{I}^l} \{v_{iq}\} & \text{for } q \in \mathbf{Q}_2 \end{cases}$

We will assume, that Stochastic Dominance (SD) rule is fulfilled if following relation is identified:

$$\mathbf{X}_{jk} \succ_{\text{SD}} \mathbf{X}_{ik} \Leftrightarrow \left(\left(\mathbf{X}_{jk} \succ_{\text{FSD}} \mathbf{X}_{ik} \vee \mathbf{X}_{jk} \succ_{\text{SSD}} \mathbf{X}_{ik} \vee \mathbf{X}_{jk} \succ_{\text{TSD}} \mathbf{X}_{ik}\right) \wedge k \in \mathbf{K}_1\right) \vee$$
$$\left(\left(\mathbf{X}_{jk} \succ_{\text{FSD}} \mathbf{X}_{ik} \vee \mathbf{X}_{jk} \succ_{\text{SISD}} \mathbf{X}_{ik} \vee \mathbf{X}_{jk} \succ_{\text{TISD1}} \mathbf{X}_{ik} \vee \mathbf{X}_{jk} \succ_{\text{TISD2}} \mathbf{X}_{ik}\right) \wedge k \in \mathbf{K}_2\right)$$

The operation of the procedure is as follows:

I. Initial phase:

1. Calculate means of distributional evaluations of alternatives with respect to attributes μ_{ik}, $i = 1, ..., n$, $k = 1, ..., m$.
2. $\mathbf{A}^l = \mathbf{A}$.

II. Iteration l:

3. Identify candidate alternative a_i:

   $a_i := \arg \min\limits_{j \in \mathbf{I}^l} \left\{ d_{jk}^l \right\}$

   where d_{jk}^l is calculated as follows:

$$d_{jk}^l = \max_{k=1,\dots,m} \left\{ w_k^l \left| \overline{\mu}_k^l - \mu_{jk} \right| \right\} \qquad w_k^l = \frac{1}{r_k^l} \left[\sum_{i=1}^{m} \frac{1}{r_i^l} \right]^{-1} \qquad r_k^l = \left| \overline{\mu}_k^l - \underline{\mu}_k^l \right|$$

   In the case of a tie choose any a_i minimizing the value of d_{jk}^l.

4. Present the data to the decision maker:
   - means of distributional evaluations of the candidate alternative a_i: μ_{ik}, $k = 1, ..., m$,
   - potency matrix $\mathbf{P}_1^l$.
5. Ask the decision maker whether he/she is satisfied with the data that are presented. If the answer is YES – go to 7.
6. Ask the decision maker to specify parameters of distributional evaluations to be presented; calculate distribution parameters v_{ip} for i such that $a_i \in \mathbf{A}^l$, $p = 1, \dots, Q$; calculate additional potency matrix $\mathbf{P}_2^l$; present additional potency matrix to the decision maker.

7. Ask the decision maker whether he/she is satisfied with the candidate alternative. If the answer is YES – the final solution is alternative a_i – go to 17, else – go to 8.
8. Ask the decision maker to specify an additional requirement.
9. Generate Al+1 the set of alternatives satisfying the requirement specified by the decision maker.
10. Calculate potency matrices $\mathbf{P}_1^{l+1}$ and $\mathbf{P}_2^{l+1}$; present matrices $\mathbf{P}_1^{l}$, $\mathbf{P}_2^{l}$, $\mathbf{P}_1^{l+1}$ and $\mathbf{P}_2^{l+1}$ to the decision maker; ask the decision maker whether he/she accepts the move from $\mathbf{P}_1^{l}$ and $\mathbf{P}_2^{l}$ to $\mathbf{P}_1^{l+1}$ and $\mathbf{P}_2^{l+1}$. If the answer is NO, then go to 4, else go to 11.
11. For each pair (a_j, a_i) such that $a_j \in \mathbf{A}^l \setminus \mathbf{A}^{l+1}$ and $a_i \in \mathbf{A}^{l+1}$ identify SD relation between $\mathbf{X}_{jk}$ and $\mathbf{X}_{ik}$. Generate the set of inconsistencies:

$$\mathbf{N}^l = \left\{ \left(a_j, a_i\right), a_j \in \mathbf{A}^l \setminus \mathbf{A}^{l+1}, a_i \in \mathbf{A}^{l+1}, \mathbf{X}_{jk} \succ_{\mathrm{SD}} \mathbf{X}_{ik} \right\}$$

12. If $\mathbf{N}^l = \varnothing$, then assume $l = l + 1$; go to 3, else go to 13.
13. Choose the first pair $\left(a_j, a_i\right) \in \mathbf{N}^l$; calculate:

$\Pr(X_{ik} \le s_r)$, $\Pr(X_{jk} \le s_r)$

where:

$$s_r = \min\left(\alpha_i, \alpha_j\right) + r\frac{\max\left(\beta_i, \beta_j\right) - \min\left(\alpha_i, \alpha_j\right)}{R} \quad \text{for } r = 0, 1, \ldots, R$$

α_i, β_i – lower and upper bound for evaluations of X_{ik},

α_j, β_j – lower and upper bound for evaluations of X_{jk},

R – number of observations. Initially R can be set to 10, the decision maker can increase (decrease) the value of R if he/she finds the data to be not enough detailed (too detailed).

Present the data to the decision maker pointing that a_j is to be rejected, while a_i is to be accepted. Ask the decision maker what is his/her decision – propose the decision maker:

a) accept a_i and reject a_j,
b) accept both a_j and a_i,
c) reject both a_j and a_i.

If the decision maker's decision is (a), go to 11, if the decision is (b), go to 15, otherwise go to 16.

14. $\mathbf{N}^l = \mathbf{N}^l \setminus \left\{\left(a_j, a_i\right)\right\}$; go to 12.

15. $\mathbf{A}^{l+1} = \mathbf{A}^{l+1} \cup \{a_j\}$, $\mathbf{N}^l = \mathbf{N}^l \setminus \{(a_j, a_i)\}$; go to 12.
16. $\mathbf{A}^{l+1} = \mathbf{A}^{l+1} \setminus \{a_i\}$, $\mathbf{N}^l = \mathbf{N}^l \setminus \{(a_j, a_i)\}$; go to 12.
17. End of the procedure.

III. Comments:

Step 6: The decision maker may specify various distribution parameters to be presented during the conversational phase of the procedure. Some examples are: standard deviation, lower/upper standard semideviation from a target value ψ, lower/upper mean semideviation from a target value ψ, probability of getting the outcome not exceeding a target value ψ, probability of getting the outcome not less than a target value ψ.

Step 8: The decision maker defines additional requirements by specifying minimal or maximal acceptable values of distribution parameters. Thus, the decision maker's additional requirements may be formulated as follows:
- mean not less/not greater than a specified target value ψ,
- standard deviation not greater than a specified value ξ,
- lower standard semideviation from a target value ψ not greater/not less than a specified value ξ,
- upper standard semideviation from a target value ψ not less/not greater than a specified value ξ,
- lower mean semideviation from a target value ψ not greater/not less than a specified value ξ,
- upper mean semideviation from a target value ψ not less/not greater than a specified value ξ,
- probability of getting outcome not exceeding a specified target value ψ not greater/not less than a specified value α,
- probability of getting outcome not less than a specified target value ψ not less/not greater than a specified value α.

5. ILLUSTRATIVE EXAMPLE

To illustrate the procedure let us consider a labor planning problem in a payroll department. Twenty alternative staffing plans are considered (Table 1).

Table 1

The set of alternatives **A**

Alternative	Number of employees			
	Full-time	Part-time (6/8)	Part-time (4/8)	Part-time (2/8)
a_1	4	0	0	0
a_2	3	1	0	0
a_3	3	0	1	0
a_4	3	0	0	1
a_5	3	2	0	0
a_6	3	0	2	0
a_7	3	0	0	2
$a8$	3	1	1	0
a_9	3	1	0	1
a_{10}	3	0	1	1
a_{11}	2	3	0	0
a_{12}	2	0	3	0
a_{13}	2	0	0	3
a_{14}	2	2	1	0
a_{15}	2	2	0	1
a_{16}	2	1	2	0
a_{17}	2	1	0	2
a_{18}	2	0	2	1
a_{19}	2	0	1	2
a_{20}	2	1	1	1

Three criteria are used for evaluating the performances of alternative plans:

X_1 – yearly labor costs,

X_2 – total number of overtime hours worked by all employees in the department during the year,

X_3 – work-force utilization rate measured by the contribution of regular hours effectively worked to the total number of regular hours worked by employees.

Based on past experience distributions of work-force requirements for each week of the month have been estimated (Table 2).

Table 2

Distributions of work-force requirements

Week 1 and 5		Week 2	
Work-force requirements (hours per day)	Probability	Work-force requirements (hours per day)	Probability
96	0,30	88	0,30
108	0,40	100	0,40
120	0,20	112	0,20
132	0,10	124	0,10
Week 3		**Weeks 4**	
Work-force requirements (hours per day)	Probability	Work-force requirements (hours per day)	Probability
80	0,30	88	0,30
92	0,40	100	0,40
104	0,20	112	0,20
116	0,10	124	0,10

For each alternative distribution of daily work-force's capacity has been estimated. It was assumed that the probability of an employee's absence is equal to 0,15. Table 3 presents the distribution for alternative a_1.

Table 3

Distribution of a daily work-force capacity
(man-hours)

Capacity (hours)	Probability
0	0,0005
8	0,0115
16	0,0975
24	0,3685
32	0,5220

Simulation has been applied for generating distributional evaluations of alternatives. Table 4 presents results of simulation experiments.

Table 4

Results of simulation experiments

Alternative	Mean of distributional evaluation		
	Cost (PLZ)	Overtime (hours)	Work-force utilization rate (%)
a_1	122473,14	91,9	76,18%
a_2	108509,64	131,5	80,70%
a_3	95054,28	198,1	85,19%
a_4	97601,70	314,7	89,78%
a_5	121693,85	42,5	68,36%
a_6	92769,60	86,6	76,33%
a_7	95109,00	199,2	85,39%
$a8$	107170,28	61,7	72,15%
a_9	107630,42	86,6	76,27%
a_{10}	93626,28	129,1	80,74%
a_{11}	107119,38	60,6	72,14%
a_{12}	65306,73	193,3	85,41%
a_{13}	77149,47	731,7	97,10%
a_{14}	92780,49	84,6	76,36%
a_{15}	93556,76	125,7	80,72%
a_{16}	78733,29	124,9	80,79%
a_{17}	82582,17	304,9	89,95%
A_{18}	67694,00	306,5	89,99%
A_{19}	71629,94	482,1	94,00%
A_{20}	80156,84	194,6	85,39%

Final solution is generated as follows:

I. Iteration 1:

$\mathbf{A}^1 = \mathbf{A} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}, a_{18}, a_{19}, a_{20}\}$

3. Candidate alternative is identified: $a_{17.}$
4. The data are presented to the decision maker:
   - means of distributional evaluations of the candidate alternative: μ17 1 = 82582,17, μ17 2 = 304,9, μ17 3 = 89,95%,
   - potency matrix $\mathbf{P}_1^1$ (Table 5).

Table 5

Potency matrix $\mathbf{P}_1^1$

	Criterion		
	1	2	3
$\underline{\mu}_k^1$	122473,14	731,7	68,36%
$\overline{\mu}_k^1$	65306,73	42,5	97,10%

5. The decision maker is not satisfied with data presented.
6. The decision maker is interested in the probability that the number of overtime hours (criterion No. 2) is not less then 240: $Q = 1$, $v_{i\,1} = \Pr(X_{i\,2} \geq 240)$. Additional potency matrix $\mathbf{P}_2^1$ is calculated (Table 6), the data are presented to the decision maker: $v_{17\,1} = \Pr(X_{17\,1} \geq 240) = 0{,}871$.

Table 6

Potency matrix $\mathbf{P}_2^1$

	$v_{i\,1} = \Pr(X_{i\,2} \geq 240)$
$\underline{v}_1^1$	1,000
$\overline{v}_1^1$	0,000

7. The decision maker is not satisfied with the candidate alternative.
8. The decision maker specifies additional requirement:

$$\Pr(X_{i\,2} \geq 240) \leq 0{,}2$$

9. Set of alternatives satisfying the requirement specified by the decision maker is generated:

$$\mathbf{A}^2 = \{a_1, a_2, a_3, a_5, a_6, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{14}, a_{15}, a_{16}, a_{20}\}$$

10. Potency matrices $\mathbf{P}_1^2$ and $\mathbf{P}_2^2$ are calculated (Tables 7 and 8); matrices $\mathbf{P}_1^1$, $\mathbf{P}_2^1$, $\mathbf{P}_1^2$, and $\mathbf{P}_2^2$ are presented to the decision maker who accepts the move from $\mathbf{P}_1^1$ and $\mathbf{P}_2^1$ to $\mathbf{P}_1^2$ and $\mathbf{P}_2^2$.

Table 7

Potency matrix $\mathbf{P}_1^2$

	Criterion		
	1	2	3
$\underline{\mu}_k^2$	122473,14	198,1	68,36%
$\overline{\mu}_k^2$	65306,73	42,5	85,41%

Table 8

Potency matrix $\mathbf{P}_2^2$

	$v_{i\,1} = \Pr(X_{i\,1} \le 277250)$
$\underline{v}_1^2$	0,195
$\overline{v}_1^2$	0,000

11. None inconsistencies are identified.
11. As $\mathbf{N}^1 = \varnothing$, so $l = 2$.

II. Iteration 2:

3. Candidate alternative is identified: a_{16}.
4. The data are presented to the decision maker:
   - means of distributional evaluations of the candidate alternative: $\mu_{16\,1} = 78733{,}29$, $\mu_{16\,2} = 124{,}9$, $\mu_{16\,3} = 80{,}79\%$,
   - potency matrix $\mathbf{P}_1^2$ (Table 7).
5. The decision maker is satisfied with data presented.
6. The decision maker is not satisfied with the candidate alternative.
7. The decision maker specifies additional requirement:

$$\mu_{i\,3} \ge 85{,}0\%$$

9. Set of alternatives satisfying the requirement specified by the decision maker is generated:

$$\mathbf{A}^3 = \{a_3, a_{12}\}$$

10. Potency matrix $\mathbf{P}_1^3$ is generated (Table 9); matrices $\mathbf{P}_1^2$ and $\mathbf{P}_1^3$ are presented to the decision maker who accepts the move from $\mathbf{P}_1^2$ to $\mathbf{P}_1^3$.

Table 9

Potency matrix $\mathbf{P}_1^3$

	Criterion		
	1	2	3
$\underline{\mu}_k^3$	95054,28	198,1	85,19%
$\overline{\mu}_k^3$	65306,73	193,3	85,41%

11. The set of inconsistencies is generated.
12. As $\mathbf{N}^2 = \varnothing$, so $l = 3$.

III. Iteration 3:

3. Candidate alternative is identified: a_{12}.
4. Presentation of the data to the decision maker:
   - average evaluations of the candidate alternative a_{12}: $\mu_{12\,1} = 65306{,}73$, $\mu_{12\,2} = 193{,}3$, $\mu_{12\,3} = 85{,}41\%$,
   - potency matrix $\mathbf{P}_1^3$ (Table 9).
5. The decision maker is satisfied with data presented.
7. The decision maker is satisfied with the candidate alternative – a_{22} is the final solution
10. The end of the procedure.

As the alternative a_{12} is the final solution, so 2 full-time employees and 3 part-time employees working 4 hours per day should be hired.

CONCLUSIONS

Usually, the objective of labor planning is to minimize cost over the planning period. However, other issues may be also important. Minimizing overtime and maximizing work-force utilization rate are also analyzed when a staffing plan is prepared. As these criteria are in conflict, we are faced with a multiple criteria decision problem.

The main purpose of this paper was to give comprehensive, yet simple methodology for labor planning problems. A new methodology for determining the number of full-time and part-time employees was presented. Although this approach was applied for labor planning in a payroll department, it could be easily adapted to other organizations.

The procedure uses two approaches: stochastic dominance and interactive methodology. The former is widely used for comparing uncertain prospects, the latter is a multiple criteria technic that is probably most often used in real--world applications. These two concepts have been combined in the INSDECM--II procedure.

Appendix

Notation:

$F(x)$, $G(x)$ – cumulative distribution functions

$$F(x) = \Pr(X_F \leq x)$$

$$G(x) = \Pr(X_G \leq x)$$

$\overline{F}(x), \overline{G}(x)$ – decumulative distribution functions

$$\overline{F}(x) = \Pr(X_F \geq x)$$

$$\overline{G}(x) = \Pr(X_G \geq x)$$

Definition 1:

$F(x) \succ_{\mathrm{FSD}} G(x)$ if and only if

$F(x) \neq G(x)$ and $H_1(x) = F(x) - \mathrm{G}(x) \leq 0$ for all $x \in [a,b]$

Definition 2:

$F(x) \succ_{\mathrm{SSD}} G(x)$ if and only if

$F(x) \neq G(x)$ and $H_2(x) = \int_a^x H_1(y)dy \leq 0$ for all $x \in [a,b]$

Definition 3:

$F(x) \succ_{\mathrm{TSD}} G(x)$ if and only if

$F(x) \neq G(x)$ and $H_3(x) = \int_a^x H_2(y)dy \leq 0$ for all $x \in [a,b]$

Definition 4:

$\overline{F}(x) \succ_{\mathrm{SISD}} \overline{G}(x)$ if and only if

$\overline{F}(x) \neq \overline{G}(x)$ and $\overline{H}_2(x) = \int_x^b \overline{H}_1(y)dy \geq 0$ for all $x \in [a,b]$

where: $\overline{H}_1 = \overline{F}(x) - \overline{G}(x)$

Definition 5:

$$\overline{F}(x) \succ_{\mathrm{TISD1}} \overline{G}(x) \text{ if and only if}$$

$$\overline{F}(x) \neq \overline{G}(x) \text{ and } \overline{H}_3(x) = \int_x^b \overline{H}_2(y)dy \geq 0 \text{ for all } x \in [a,b]$$

Definition 6:

$$\overline{F}(x) \succ_{\mathrm{TISD2}} \overline{G}(x) \text{ if and only if}$$

$$\overline{F}(x) \neq \overline{G}(x) \text{ and } \widetilde{H}_3(x) = \int_a^x \overline{H}_2(y)dy \geq 0 \text{ for all } x \in [a,b]$$

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Włodzimierz Ogryczak

REFERENCE POINT METHOD WITH LEXICOGRAPHIC MIN-ORDERING OF INDIVIDUAL ACHIEVEMENTS*

Abstract

The reference point method (RPM) is a very effective technic for interactive analysis of the multiple criteria optimization problems. It provides the DM with a tool for an open analysis of efficient frontier either connected or disconnected. The interactive analysis is navigated by the commonly accepted control parameters expressing reference levels for the individual objective functions. The individual achievement functions quantify the DM's satisfaction from the individual outcomes with respect to the given reference levels. The final scalarizing function is built as the augmented max-min aggregation of individual achievements which means that the worst individual achievement is essentially maximized, but the optimization process is additionally regularized with the term representing the average achievement. The regularization by the average achievement is easily implementable, but it may disturb the basic max--min aggregation. In order to avoid inconsistencies caused by the regularization, the max-min solution may be regularized according to the lexicographic min-order, thus leading to the nucleolar RPM model. The nucleolar RPM implements a consequent max-min aggregation taking into account also the second-worst achievement, the third--worse, and so on, thus resulting in much better modeling of the reference levels concept. The lexicographic min-ordering regularization is more complicated in implementation due to the requirement of pointwise ordering of partial achievements. Nevertheless, by taking advantage of piecewise linear expression of the cumulated ordered achievements, the nucleolar RPM can be formulated as a standard lexicographic optimization. Actually, in the case of concave piecewise linear partial achievement functions (typically used in the RPM), the resulting formulation extends the original constraints and criteria with simple linear inequalities, thus allowing for a quite efficient implementation. It can be also approximated with the analytic form using the ordered weighted averages. The paper analyzes both the theoretical and practical issues of the nucleolar RPM.

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Keywords

Multicriteria optimization, reference point method (RPM), lex-min order.

INTRODUCTION

Typical multiple criteria optimization methods aggregate the individual outcomes with some scalarizing functions to generate a satisfactory efficient solution. The scalarizing functions may have various constructions and properties depending on the specific approach to preference modeling applied in several methods. Nevertheless, most scalarizing functions can be viewed as two-stage transformation of the original outcomes. First, the individual outcomes are rescaled to some uniform measures of achievements with respect to several criteria and preference parameters. Thus, the individual achievement functions are built to measure actual achievement of each outcome with respect to the corresponding preference parameters. In particular, in the reference point method (RPM) the strictly monotonic partial achievement functions are built to measure individual performance with respect to given reference levels. Similar constructions appear in fuzzy approaches where the membership functions for various fuzzy targets are such individual achievement measures scaled to the unit interval or in goal programming where scaled deviations from targets may be considered individual achievements.

Having all the outcomes transformed into a uniform scale of individual achievements they are aggregated at the second stage to form a unique scalarization. The aggregation usually measures the total (the average) or the worst individual achievement. While several technic and tools for better modeling of preferences with partial achievement functions are developed [3], the aggregation itself is much less studied. The RPM is based on the so-called augmented (or regularized) max-min aggregation. Thus, the worst individual achievement is essentially maximized, but the optimization process is additionally regularized with the term representing the average achievement. The max-min aggregation guarantees fair treatment of all individual achievements by implementing an approximation to the Rawlsian principle of justice.

The max-min aggregation is crucial for allowing the RPM to generate all efficient solutions even for nonconvex (and particularly discrete) problems. On the other hand, the regularization is necessary to guarantee that only efficient solution are generated. The regularization by the average achievement is easily implementable, but it may disturb the basic max-min model. Actually, the only consequent regularization of the max-min aggregation is the lexicographic max-min (nucleolar) solution concept where in addition to the worst achievement, the second worst achievement is also optimized (provided that the worst remains on the optimal level), the third worst is optimized (provided that the two worst remain optimal), and so on. Such a nucleolar regularization is the only max-min regularization satisfying the addition/deleting principle, thus making the corresponding nucleolar RPM not affected by any passive criteria. The recent progress in optimization methods of ordered averages allows one to implement the nucleolar RPM quite effectively. The paper analyzes both the theoretical and practical issues of the nucleolar RPM.

1. SCALARIZATIONS OF THE REFERENCE POINT METHOD

In this paper, without loss of generality, it is assumed that all the criteria are maximized (that is, for each outcome "more is better"). Hence, we consider the following multiple criteria optimization problem:

$$\max \{ \ (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})) \ : \ \mathbf{x} \in Q \ \} \tag{1}$$

where x denotes a vector of decision variables to be selected within the feasible set $Q \subset R^n$, and $\mathbf{f}(x) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ is a vector function that maps the feasible set Q into the criterion space R^m. Note that neither any specific form of the feasible set Q is assumed nor any special form of criteria $f_i(\mathbf{x})$ is required. We refer to the elements of the criterion space as outcome vectors. An outcome vector $\mathbf{y}$ is attainable if it expresses outcomes of a feasible solution, i.e. $\mathbf{y} = \mathbf{f}(\mathbf{x})$ for some $\mathbf{x} \in Q$.

Model (1) only specifies that we are interested in maximization of all objective functions f_i for $i \in I = \{1, 2, \ldots, m\}$. Thus, it allows only to identify (to eliminate) obviously inefficient solutions leading to dominated outcome vectors, while still leaving the entire efficient set to look for a satisfactory compromise solution. In order to make the multiple criteria model operational for the decision support process, one needs assume some solution concept well adjusted to the DM preferences. This can be achieved with the so-called quasi-satisficing approach to multiple criteria decision problems. The best formalization of the quasi-satisficing approach to multiple criteria optimization was proposed and developed mainly by Wierzbicki [20] as the reference point method. The RPM was later extended to permit additional information from the DM and, eventually, led to efficient implementations of the so-called aspiration/reservation based decision support (ARBDS) approach with many successful applications [5,21].

The RPM is an interactive technic. The basic concept of the interactive scheme is as follows. The DM specifies requirements in terms of reference levels, i.e. by introducing reference (target) values for several individual outcomes. Depending on the specified reference levels, a special scalarizing achievement function is built which may be directly interpreted as expressing utility to be maximized. Maximization of the scalarizing achievement function generates an efficient solution to the multiple criteria problem. The computed efficient solution is presented to the DM as the current solution in a form that allows comparison with the previous ones and modification of the reference levels if necessary.

While building the scalarizing achievement function the following properties of the preference model are assumed. First of all, for any individual outcome y_i more is preferred to less (maximization). To meet this requirement the function must be strictly increasing with respect to each outcome. Second, a solution with all individual outcomes y_i satisfying the corresponding reference levels is preferred to any solution with at least one individual outcome worse (smaller) than its reference level. That means, the scalarizing achievement function maximization must enforce reaching the reference levels prior to further improving of criteria. Thus, similar to the goal programming approaches, the reference levels are treated as the targets, but following the quasi-satisficing approach they are interpreted consistently with basic concepts of efficiency in the sense that the optimization is continued even when the target point has been reached already.

The generic scalarizing achievement function takes the following form [20]:

$$S(\mathbf{y}) = \min_{1 \leq i \leq m} \{s_i(y_i)\} + \frac{\varepsilon}{m} \sum_{i=1}^{m} s_i(y_i) \tag{2}$$

where ε is an arbitrary small positive number and $s_i : R \to R$, for $i = 1, 2, \dots, m$ are the partial achievement functions measuring actual achievement of the individual outcomes y_i with respect to the corresponding reference levels. Let a_i denote the partial achievement for the i-th outcome $(a_i = s_i(y_i))$ and $\mathbf{a} = (a_1, a_2, \dots, a_m)$ represent the achievement vector. The scalarizing achievement function (2) is, essentially, defined by the worst partial (individual) achievement, but additionally regularized with the sum of all partial achievements. The regularization term is introduced only to guarantee the solution efficiency in the case when the maximization of the main term (the worst partial achievement) results in a nonunique optimal solution. Due to combining two terms with arbitrarily small parameter ε, formula (2) is easily implementable and it provides a direct interpretation of the scalarizing achievement function as expressing utility. When accepting the loss of a direct utility interpretation, one may consider a limiting case with $\varepsilon \to 0_+$ which results in lexicographic order applied to two separate terms of function (2). That means, the regularization can be implemented with the second level lexicographic optimization [14]. Therefore, RPM may be also considered as the following lexicographic problem ([13] and references therein):

$$\operatorname{lex\,max} \{ \, (\min_{1 \le i \le m} a_i \,, \sum_{i=1}^{m} a_i) \; : \; a_i = s_i(f_i(\mathbf{x})) \; \forall \, i, \; \mathbf{x} \in Q \, \} \tag{3}$$

The following two properties of the lexicographic model (3) are crucial for the RPM methodology:

P1: The aggregation is strictly monotonic in the sense that increase of any partial achievement a_i leads to a preferred solution.

P2: For any given target value ϱ, the solution generating all partial achievements equal to ϱ $(a_i = \varrho \; \forall i)$ is preferred to any solution generating at least one partial achievement worse than ϱ.

Property P1 guarantees that while using strictly increasing partial achievement functions s_i, every generated solution is efficient. Property P2 guarantees that while using partial achievement function allocating the same value on achieving the reference level, the solution reaching all the reference levels is preferred to any solution failing achievement of at least one reference level.

Various functions s_i provide a wide modeling environment for measuring partial achievements [21,8]. To take advantages of properties P1 and P2 they need to be strictly increasing and to allocate the same value on reaching the reference level. The basic RPM model is based on a single vector of the reference levels, the aspiration vector $\mathbf{r}^a$. For the sake of computational simplicity, the piecewise linear functions s_i are usually employed. In the simplest models, they take a form of two segment piecewise linear functions:

$$s_i(y_i) = \begin{cases} \lambda_i^+(y_i - r_i^a), & \text{for } \; y_i \geq r_i^a \\ \lambda_i^-(y_i - r_i^a), & \text{for } \; y_i < r_i^a \end{cases} \tag{4}$$

where λ_i^+ and λ_i^- are positive scaling factors corresponding to under-achievements and overachievements, respectively, for the i-th outcome. Note that for any outcome reaching the corresponding aspiration level $y_i = r_i^a$ one gets $s_i(r_i^a) = 0$. Hence, when using the RPM (3) with partial achievement functions (4), the solution reaching all the aspiration levels is preferred to any solution failing achievement of at least one aspiration level. It is usually assumed that λ_i^+ and is much larger than λ_i^- . Actually, even linear functions:

$$s_i(y_i) = \lambda_i(y_i - r_i^a) \tag{5}$$

with positive scaling factors λ_i represent simplified (but still valid) partial achievement functions in the sense that while used in the lexicographic RPM scheme (3) it guarantees the property **P2**. Nevertheless, the differentiation of the scaling factor is important to enforce the preferences of achieving more aspiration levels rather than overstep the others, especially in the analytic RPM(2). Figure 1 depicts how differentiated scaling affects the isoline contours of the analytic scalarizing achievement function. Certainly, introducing lexicographic two-level partial achievements optimization would be a better way to model the aspiration properties [11], but also more complicated.

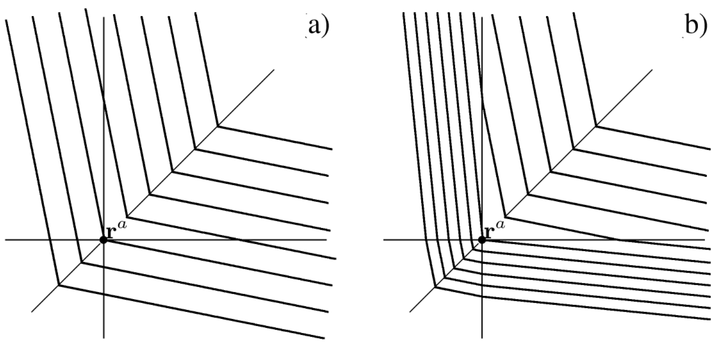

Fig. 1. Isoline contours for the analytic scalarizing achievement function (2): a) with partial achievements (5), (b) with partial achievements (4)

Real-life applications of the RPM methodology usually deal with more complex partial achievement functions defined with more than one reference point [21] which enriches the preference models and simplifies the interactive analysis. In particular, the models taking advantages of two reference vectors: vector of aspiration levels $\mathbf{r}^a$ and vector of reservation levels $\mathbf{r}^r$ [5] are used, thus allowing the DM to specify requirements by introducing acceptable and required values for several outcomes. The partial achievement function s_i can be interpreted then as a measure of the DM's satisfaction with the current value of outcome the i-th criterion. It is a strictly increasing function of outcome y_i with value $a_i = 1$ if $y_i = r_i^a$, and $a_i = 0$ for $y_i = r_i^r$. Thus, the partial achievement functions map the outcomes values onto a normalized scale of the DM's satisfaction. Various functions can be built meeting those requirements. We use the piece-wise linear partial achievement function introduced in an implementation of the ARBDS system for the multiple criteria transshipment problems with facility location [15]:

$$s_i(y_i) = \begin{cases} \gamma \quad (y_i - r_i^r)/(r_i^a - r_i^r), & \text{for} \quad y_i \le r_i^r \\ \quad (y_i - r_i^r)/(r_i^a - r_i^r), & \text{for} \quad r_i^r < y_i < r_i^a \\ \alpha \quad (y_i - r_i^a)/(r_i^a - r_i^r) + 1, & \text{for} \quad y_i \ge r_i^a \end{cases} \tag{6}$$

where α and γ are arbitrarily defined parameters satisfying $0 < \alpha < 1 < \gamma$. Parameter α represents additional increase of the DM's satisfaction over level 1 when a criterion generates outcomes better than the corresponding aspiration level. On the other hand, parameter $\gamma > 1$ represents dissatisfaction connected with outcomes worse than the reservation level.

For outcomes between the reservation and the aspiration levels, the partial achievement function s_i can be interpreted as a membership function μ_i for a fuzzy target. However, such a membership function remains constant with value 1 for all outcomes greater than the corresponding aspiration level and with value 0 for all outcomes below the reservation level (Figure 2).

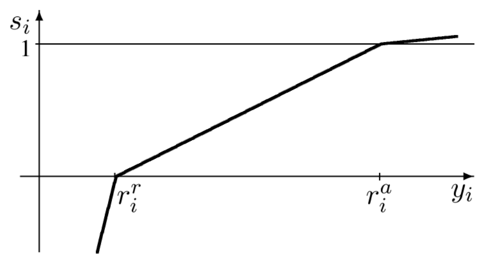

Fig. 2. ARBDS partial achievement function

Hence, the fuzzy membership function is neither strictly monotonic nor concave, thus not representing typical utility for a maximized outcome. The partial achievement function (6) can be viewed as an extension of the fuzzy membership function to a strictly monotonic and concave utility. One may also notice that the aggregation scheme used to build the scalarizing achievement function (2) from the partial ones may also be interpreted as some fuzzy aggregation operator [21]. In other words, maximization of the scalarizing achievement function (2) is consistent with the fuzzy methodology in the case of not attainable aspiration levels and satisfiable all reservation levels while modeling reasonable utility for any values of aspiration and reservation levels.

2. NUCLEOLAR RPM

The crucial properties of the RPM are related to the max-min aggregation of partial achievements while the regularization is only introduced to guarantee the aggregation monotonicity. Unfortunately, the distribution of achievements may make the max-min criterion partially passive when one specific achievement is relatively very small for all the solutions. Maximization of the worst achievement may then leave all other achievements unoptimized. In the lexicographic RPM defined by (3) the regularization term is then optimized on the second level, thus preventing one from selection of any inefficient solution. Nevertheless, the selection is then made according to linear aggregation of the regularization term instead of the max-min aggregation, thus destroying the preference model of the RPM. This can be illustrated with an example of a simple discrete problem of 7 alternative feasible solutions to be selected according to 6 criteria. Table 1 presents six partial achievements for all the solutions where the partial achievements have been defined according to the aspiration/reservation model (6), thus allocating 1 to outcomes reaching the corresponding aspiration level. Solution S7 is the only inefficient alternative. Solution S1 to S5 oversteps the aspiration levels (achievement values 1.2) for four of the first five criteria while failing to reach one of them and the aspiration level for the sixth criterion as well (achievement values 0.3). Solution S6 meets the aspiration levels (achievement values 1.0) for the first five criteria while failing to reach only the aspiration level for the sixth criterion (achievement values 0.3). One may easily notice that the sixth partial achievement (and the corresponding criterion) is constant for the seven alternatives under consideration. Hence, one may expect the same solution selected while taking into account this criterion or not. If focusing on only five first criteria, then the RPM (either lexicographic (3) or analytic (2)) obviously selects solution S6 as reaching all aspiration levels which results in the worst achievement value 1.0. However, while taking into account all six criteria all the solutions generate the same worst achievement value 0.3 and the final selection of the RPM depends on the total achievement (regularization term). Actually, either lexicographic RPM (3) or its analytic version (2) will select then one of solutions S1 to S5 as better than S6.

Table 1

Sample achievements with a passive criterion

Solution	a_1	a_2	a_3	a_4	a_5	a_6	$\min_{i=1,\ldots,6}$	$\sum_{i=1}^{6}$	$\min_{i=1,\ldots,5}$	$\sum_{i=1}^{5}$
S1	0.3	1.2	1.2	1.2	1.2	0.3	0.3	5.4	0.3	5.1
S2	1.2	0.3	1.2	1.2	1.2	0.3	0.3	5.4	0.3	5.1
S3	1.2	1.2	0.3	1.2	1.2	0.3	0.3	5.4	0.3	5.1
S4	1.2	1.2	1.2	0.3	1.2	0.3	0.3	5.4	0.3	5.1
S5	1.2	1.2	1.2	1.2	1.2	0.3	0.3	5.4	0.3	5.1
S6	1.0	1.0	1.0	1.0	1.0	0.3	0.3	5.3	1.0	5.0
S7	0.3	1.0	0.3	1.0	0.6	0.3	0.3	3.5	0.3	3.2

In order to avoid inconsistencies caused by the regularization, the max-min solution may be regularized according to the Rawlsian principle of justice. Formalization of this concept leads us to the lexicographic max-min ordered or nucleolar solution concept. The approach has been used for general linear programming multiple criteria problems [1,7] as well as for specialized problems related to (multiperiod) resource allocation [6]. In discrete optimization it has been considered for various problems including the location-allocation ones [10]. The lexicographic max-min approach can be mathematically formalized as follows. Within the space of achievement vectors we introduce map $\Theta = (\theta_1, \theta_2, \ldots, \theta_m)$ which orders the coordinates of achievements vectors in a nondecreasing order, i.e. $\Theta(a_1, a_2, \ldots, a_m) = (\theta_1(\mathbf{a}), \theta_2(\mathbf{a}), \ldots, \theta_m(\mathbf{a}))$ if there exists a permutation τ such that $\theta_i(\mathbf{a}) = a_{\tau(i)}$ for all i and $\theta_1(\mathbf{a}) \leq \theta_2(\mathbf{a}) \leq \ldots \leq \theta_m(\mathbf{a})$. The standard max-min aggregation depends on maximization of $\theta_1(\mathbf{a})$ and it ignores values of $\bar{a}_i$ for $i \geq 2$. In order to take into account all the achievement values, we look for a lexicographic maximum among the ordered achievement vectors.

Note that the lexicographic RPM model (3) can be expressed as the following problem:

$$\operatorname{lex}\max \{ (\theta_1(\mathbf{a}), \sum_{i=2}^{m} \theta_i(\mathbf{a})) \ : \ a_i = s_i(f_i(\mathbf{x})) \; \forall\, i, \; \mathbf{x} \in Q \}$$

thus, in the case of two criteria $(m = 2)$, representing exactly the lexicographic max-min aggregation. For larger number of criteria $(m > 2)$ model (3) only approximates the lexicographic max-min as all the lower priority objective terms are aggregated at the second priority level. One may consider the lexicographic max-min approach applied to the partial achievement functions (7) as a basis for a corresponding nucleolar RPM model:

$$\operatorname{lex}\max \{ (\theta_1(\mathbf{a}), \theta_2(\mathbf{a}), \ldots, \theta_m(\mathbf{a})) \ : \ a_i = s_i(f_i(\mathbf{x})) \; \forall\, i, \; \mathbf{x} \in Q \} \tag{7}$$

We will use the name nucleolar RPM to avoid any possible misunderstandings when referring to the lexicographic RPM. The nucleolar RPM implements a consequent max-min aggregation, thus resulting in much better modeling of the reference levels concept.

Table 2

Ordered achievements values

Solution	θ_1 (**a**)	θ_2 (**a**)	θ_3 (**a**)	θ_4 (**a**)	θ_5 (**a**)	θ_6 (**a**)
S1	0.3	0.3	1.2	1.2	1.2	1.2
S2	0.3	0.3	1.2	1.2	1.2	1.2
S3	0.3	0.3	1.2	1.2	1.2	1.2
S4	0.3	0.3	1.2	1.2	1.2	1.2
S5	0.3	0.3	1.2	1.2	1.2	1.2
S6	0.3	1.0	1.0	1.0	1.0	1.0
S7	0.3	0.3	0.3	0.6	1.0	1.0

One may easily notice that the nucleolar RPM is not affected by any adding or eliminating passive criterion. While applying the nucleolar RPM the ordered achievement are lexicographically minimized and therefore in our example solution S6 is selected for six criteria as it was selected for five criteria (Table 2). Actually, the lexicographic max-min is the only regularization of the max-min approach satisfying the reduction (addition/deleting) principle [2]. Namely, if the individual achievement of an outcome does not distinguish two solutions, then it does not affect the preference relation:

$$(a'_1, \ldots, a'_i, a^*, a'_{i+1}, \ldots, a'_q) \succeq (a''_1, \ldots, a''_i, a^*, a''_{i+1}, \ldots, a''_q) \quad \Leftrightarrow \quad a' \succeq a'' \tag{8}$$

Due to strictly monotonic individual achievement functions, the reduction principle is also satisfied in the original outcome space. Moreover, since the aggregation is impartial with respect to partial achievements, it depends only on distribution of achievements independently from their order. Hence, the nucleolar RPM works also properly if the max-min optimization becomes passive despite one cannot identify any passive original criterion. This can be illustrated with data from Table 3 which differ from those of Table 1 only due to permuted achievements of solution S7. This alternative is no longer dominated and the sixth criterion is no longer passive. Nevertheless, as the distributions of achievement values remain the same, the max-min optimization remains passive and the standard forms of the RPM select solution S1 to S5 according to regularization term. Similarly, the ordered values of achievements remain the same as in Table 2, and the nucleolar RPM still selects solution S6 as the best matching the aspiration levels.

Table 3

Sample achievements with passive max-min criterion

Solution	a_1	a_2	a_3	a_4	a_5	a_6
S1	0.3	1.2	1.2	1.2	1.2	0.3
S2	1.2	0.3	1.2	1.2	1.2	0.3
S3	1.2	1.2	0.3	1.2	1.2	0.3
S4	1.2	1.2	1.2	0.3	1.2	0.3
S5	1.2	1.2	1.2	1.2	0.3	0.3
S6	1.0	1.0	1.0	1.0	1.0	0.3
S7	0.3	0.3	0.3	1.0	0.6	1.0

The following assertions shows that the nucleolar RPM model (7) satisfies the basic requirements for the RPM approaches. Namely, model (7) guarantees the efficiency of solutions (Theorem 1) and it is possible to generate all efficient solutions using nucleolar RPM by appropriately choosing the reference vector (Theorem 2).

Theorem 1

For any strictly increasing partial achievement functions $s_i(y_i)$, *if* $\bar{\mathbf{x}}$ *is an optimal solution of the problem (7), then* $\bar{\mathbf{x}}$ *is also an efficient solution of the corresponding multicriteria problem (1).*

Proof

Suppose $\bar{\mathbf{x}}$ optimal to (7) is dominated by some $\mathbf{x}' \in Q$. Thus, due to strictly increasing partial achievement functions one gets $\bar{a}_i = s_i(f_i(\bar{\mathbf{x}})) \leq a'_i = s_i(f_i(\mathbf{x}')) \; \forall \, i$, with at least one inequality strict. Hence, $\Theta(\bar{\mathbf{a}}) <_{lex} \Theta(\mathbf{a}')$ which contradicts optimality $\bar{\mathbf{x}}$ to (7).

■

Theorem 2

For any $\bar{\mathbf{x}}$ efficient solution, if the reference level are defined as, $r_i = f_i(\bar{\mathbf{x}})$ and strictly increasing partial achievement functions s_i taking the same value at the reference levels $(s_i(r_i) = \varrho \; \forall \, i)$ are used, then $\bar{\mathbf{x}}$ is an optimal solution of the corresponding nucleolar RPM problem (7).

Proof

Note that $\bar{a}_i = s_i(f_i(\bar{\mathbf{x}})) = \varrho \; \forall \, i$. If there exist $a'_i = s_i(f_i(\mathbf{x}'))$ for $\mathbf{x}' \in Q$ such that $\Theta(\bar{\mathbf{a}}) <_{lex} \Theta(\mathbf{a}')$, then $\bar{a}_i =\leq a'_i \; \forall \, i$ with at least one inequality strict. This contradicts the efficiency of $\mathbf{x}$.

■

Note that all typical partial achievement functions, in particular piecewise linear functions of the form (4), (5), or (6) are strictly increasing and they assign the same value at the reference levels. Thus, Theorem 2 justifies the controllability of the nucleolar RPM in the sense that for any $\mathbf{x} \in Q$ efficient solution to multiple criteria problem (1) there exists the reference vector $\mathbf{r}^a$ such that $\mathbf{x}$ is an optimal solution of the corresponding nucleolar RPM problem (7) defined with this reference vector.

3. IMPLEMENTATION ISSUES

An important advantage of the RPM depends on its easy implementation as an expansion of the original multiple criteria model. Actually, even more complicated partial achievement functions of the form (6) are strictly increasing and concave (under the assumption that $0 < \alpha < 1 < \gamma$), thus allowing for implementation of the entire RPM model (2) by an LP expansion [15]. The ordered achievements optimized in the nucleolar RPM (7) are, in general, hard to implement due to the pointwise ordering. Let us consider cumulated achievements $\bar{\theta}_k(\mathbf{a}) = \sum_{i=1}^{k} \theta_i(\mathbf{a})$ expressing, respectively: the worst (smallest) achievement, the total of the two worst achievements, the total of the three worst achievements, etc. Within the lexicographic optimization a cumulation of criteria does not affect the optimal solution. Hence, the nucleolar RPM model (7) can be expressed in terms of the lexicographic maximization of quantities $\bar{\theta}_i(\mathbf{a})$:

$$\text{lex } \max \{ (\bar{\theta}_1(\mathbf{a}), \bar{\theta}_2(\mathbf{a}), \ldots, \bar{\theta}_m(\mathbf{a})) \; : \; a_i = s_i(f_i(\mathbf{x})) \; \forall \, i, \; \mathbf{x} \in Q \} \tag{9}$$

This simplifies dramatically the optimization problem since quantities $\bar{\theta}_k(\mathbf{a})$ can be optimized without use of any integer variables. First, let us notice that for any given vector $\mathbf{a}$, the cumulated ordered value $\bar{\theta}_k(\mathbf{a})$ can be found as the optimal value of the following LP problem:

$$\bar{\theta}_k(\mathbf{a}) = \min_{u_{ik}} \{ \sum_{i=1}^{m} a_i u_{ik} : \sum_{i=1}^{m} u_{ik} = k, \quad 0 \le u_{ik} \le 1 \quad \forall i \} \tag{10}$$

The above problem is an LP for a given outcome vector a while it becomes nonlinear for a being a vector of variables. This difficulty can be overcome by taking advantage of the LP dual to (10). Introducing dual variable t_k corresponding to the equation $\sum_{i=1}^{m} u_{ik} = k$ and variables d_{ik} correspon-ding to upper bounds on u_{ik} one gets the following LP dual of problem (10):

$$\bar{\theta}_k(\mathbf{a}) = \max_{t_k, d_{ik}} \{ k t_k - \sum_{i=1}^{m} d_{ik} : a_i \ge t_k - d_{ik}, \; d_{ik} \ge 0 \quad \forall \, i \} \tag{11}$$

Due the duality theory, for any given vector a the cumulated ordered coefficient $\bar{\theta}_k(\mathbf{a})$ can be found as the optimal value of the above LP problem. It follows from (11) that $\bar{\theta}_k(\mathbf{a}) = \max \{ k t_k - \sum_{i=1}^{m} (t_k - a_i)_+ \}$ where $(.)_+$ denotes the nonnegative part of a number and t_k is an auxiliary (unbounded) variable. The latter, with the necessary adaptation to the minimized outcomes in location problems, is equivalent to the computational formulation of the k-centrum model introduced by [17]. Hence, formula (11) provides an alternative proof of that formulation.

Taking advantages of both (9) and (11), the nucleolar RPM can be formulated as a standard lexicographic optimization. Moreover, in the case of concave piecewise linear partial achievement functions (as typically used in the RPM approaches), the resulting formulation extends the original constraints and criteria with linear inequalities. In particular, for strictly increasing and concave partial achievement functions (6), it can be expressed in the form:

$$
\begin{array}{lll}
\text{lex max} & (z_1, z_2, \ldots, z_m) & \\
\text{s.t.} & \mathbf{x} \in Q,\ y_i = f_i(\mathbf{x}) & \forall\, i \\
& z_k = kt_k - \sum_{i=1}^{m} d_{ik} & \forall\, k \\
& a_i \geq t_k - d_{ik},\ d_{ik} \geq 0 & \forall\, i,k \\
& a_i \leq \gamma(y_i - r_i^r)/(r_i^a - r_i^r) & \forall\, i \\
& a_i \leq (y_i - r_i^r)/(r_i^a - r_i^r) & \forall\, i \\
& a_i \leq \alpha(y_i - r_i^a)/(r_i^a - r_i^r) + 1 & \forall\, i
\end{array}
\tag{12}
$$

Thus, the nucleolar RPM can be effectively applied to various multiple criteria optimization problems including the discrete ones.

Model (12) provides us with an easily implementable sequential algorithm to generate efficient solutions according to the nucleolar RPM preference specification. However, it does not introduce any explicit scalarizing achievement function which could be directly interpreted as expressing utility to be maximized. In order to get such an analytical form (or rather approximation) of the nucleolar RPM one needs to replace the lexicographic (preemptive) optimization of the ordered achievements in (7) with its weighting approximation. Note that the weights are then assigned to the specific positions within the ordered achievements rather than to the partial achievements themselves, thus representing the so-called Ordered Weighted Averaging (OWA) aggregation. With the OWA aggregation one gets the following RPM model:

$$
\max \{ \sum_{i=1}^{m} v_i \theta_i(\mathbf{a}) \ :\ a_i = s_i(f_i(\mathbf{x}))\ \forall\, i,\ \mathbf{x} \in Q \} \tag{13}
$$

where $v_1 > v_2 > \ldots > v_m > 0$ are positive and strictly decreasing weights. When differences among weights tend to infinity, the OWA aggregation approximates the leximin ranking of the ordered outcome vectors [22]. That means, as the limiting case of (13), we get the nucleolar RPM model (7). Actually, the standard RPM model with the analytic scalarizing achievement function (2) can be expressed as the following OWA model:

$$
\max \{ (1 + \frac{\varepsilon}{m})\theta_1(\mathbf{a}) + \frac{\varepsilon}{m} \sum_{i=2}^{m} \theta_i(\mathbf{a}) \ :\ a_i = s_i(f_i(\mathbf{x}))\ \forall\, i,\ \mathbf{x} \in Q \}
$$

Hence, the standard RPM model exactly represents the analytic (utility) form of the OWA aggregation (13) with strictly decreasing weights in the case of $m = 2$ $(v_1 = 1 + \varepsilon/2 > v_2 = \varepsilon/2)$. For $m > 2$ it abandons the differences in weighting of the second worst achievement, the third worst one, etc $(v_2 = \ldots = v_m = \varepsilon/m)$.

The OWA aggregation is obviously a piecewise linear function since it remains linear within every area of the fixed order of arguments. Its optimization can be implemented by expressing in terms of the cumulated ordered achievements:

$$\max \{ \sum_{i=1}^{m} w_i \bar{\theta}_i(\mathbf{a}) \ : \ a_i = s_i(f_i(\mathbf{x})) \ \forall\, i;\ \mathbf{x} \in Q \}$$

where $w_i = v_i - v_{i+1}$ for $i = 1, \ldots, m-1$, and taking advantages of the LP expression (11) of $\bar{\theta}_i$ [16]. This leads to a single level computational model similar to (12).

$$\max \sum_{k=1}^{m} w_k z_k \quad \text{s.t.} \quad \begin{array}{ll} z_k = k t_k - \sum_{i=1}^{m} d_{ik} & \forall\, k \\ \mathbf{x} \in Q,\ y_i = f_i(\mathbf{x}) & \forall\, i \\ a_i \geq t_k - d_{ik},\ d_{ik} \geq 0 & \forall\, i, k \\ a_i \leq \gamma(y_i - r_i^r)/(r_i^a - r_i^r) & \forall\, i \\ a_i \leq (y_i - r_i^r)/(r_i^a - r_i^r) & \forall\, i \\ a_i \leq \alpha(y_i - r_i^a)/(r_i^a - r_i^r) + 1 & \forall\, i \end{array} \tag{14}$$

For some special sequences of the OWA weights v_i this solution concept can easily be defined without any need to order outcomes, thus the solution procedure may be quite simple. From the properties of the Gini's mean absolute difference [12] it follows that:

$$\sum_{i=1}^{m} \sum_{k=1}^{m} \min\{a_i, a_k\} = \sum_{i=1}^{m} a_i - \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} |a_i - a_k| = \sum_{i=1}^{m} [2(m-i)+1] \theta_i(\mathbf{a})$$

Hence, the OWA aggregation given by the decreasing sequence of weights v_i with a constant step $v_i - v_{i+1} = \Delta$ can be directly expressed as:

$$\sum_{i=1}^{m} v_i \theta_i(\mathbf{a}) = \bar{v}_1 \min_{1 \leq i \leq m} a_i + \frac{\Delta}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} \min\{a_i, a_k\} \tag{15}$$

where $\bar{v}_1 = v_1 - \Delta(2m-1)/2$. Note that formula (15) defines a piecewise linear concave function which guarantees its LP computability when maximized.

The following extension of the analytic RPM model (2)

$$\max \{ \min_{1\le i\le m} a_i + \frac{\varepsilon}{m^2} \sum_{i=1}^{m} \sum_{k=1}^{m} \min\{a_i, a_k\} \ : \ a_i = s_i(f_i(\mathbf{x}))\ \forall\, i,\ \mathbf{x} \in Q \} \quad (16)$$

due to (15), represents the OWA aggregation given by the decreasing sequence of weights with $v_1 = 1 + (2m-1)\varepsilon$ and the constant step $v_i - v_{i+1} = 2\varepsilon/m^2$. Certainly, such an analytic model is only rough approximation to the nucleolar RPM. Nevertheless, when applying (16) to our sample problem from Table 1, the solution S6 is selected. For strictly increasing and concave partial achievement functions (6) the model can be expressed as:

$$\begin{array}{lll} \max & \underline{a} + \frac{\varepsilon}{m^2} \sum_{i=1}^{m} \sum_{k=1}^{m} t_{ik} & \\ \text{s.t.} & \mathbf{x} \in Q,\ y_i = f_i(\mathbf{x}) & \forall\, i \\ & a_i \ge \underline{a} & \forall\, i \\ & a_i \ge t_{ik},\ a_k \ge t_{ik} & \forall\, i,k \\ & a_i \le \gamma(y_i - r_i^r)/(r_i^a - r_i^r) & \forall\, i \\ & a_i \le (y_i - r_i^r)/(r_i^a - r_i^r) & \forall\, i \\ & a_i \le \alpha(y_i - r_i^a)/(r_i^a - r_i^r) + 1 & \forall\, i \end{array} \quad (17)$$

4. ILLUSTRATIVE EXAMPLE

In order to illustrate the nucleolar RPM performances let us analyze the multicriteria problem of information system selection. We consider a billing system selection for a telecommunication company [19]. The decision is based on 7 criteria related to the system functionality, reliability, processing efficiency, investment costs, installation time, operational costs, and warranty period. All these attributes may be viewed as criteria, either maximized or minimized. Table 4 presents all the criteria with their measures units and optimization directions. There are also set the aspiration and reservation levels for each criterion.

Table 4

Criteria and their reference levels for the sample billing system selection

	f_1	f_2	f_3	f_4	f_5	f_6	f_7
	Functionality	Reliability	Efficiency	Investment cost	Installation time	Operational cost	Warranty period
Units	# modules	1-10	CAPS	mln PLN	months	mln PLN	years
Optimization	max	max	max	min	min	min	max
Reservation	4	8	50	2	12	1.25	0.5
Aspiration	10	10	200	0	6	0.5	2

Five candidate billing systems have been accepted for the final selection procedure. All they meet the minimal requirements defined by the reservation levels. Table 5 presents for all the systems (columns) their criteria values y_i and the corresponding partial achievement values a_i. The latter are computed according to the piece-wise linear formula (6) with $\alpha = 0.1$.

Table 5

Criteria values y_i and individual achievements a_i for five billing systems

i	System A		System B		System C		System D		System E	
	y_i	a_i	y_i	a_i	y_i	a_i	y_i	a_i	y_i	a_i
1	9	0.83	10	1.00	8	0.67	6	0.33	8	0.67
2	10	1.00	9	0.50	10	1.00	9	0.50	10	1.00
3	200	1.00	100	0.33	170	0.80	90	0.27	150	0.67
4	1	0.50	0.3	0.85	0.8	0.60	0.2	0.90	0.5	0.75
5	10	0.33	3	1.05	6	1.00	8	0.67	5	1.02
6	1	0.33	1	0.33	0.6	0.87	0.2	1.04	1	0.33
7	2	1.00	2	1.00	1	0.33	2	1.00	1.5	0.67

Table 6 presents for all the systems (columns) their partial achievement values ordered from the worst to the best $\theta_i(\mathbf{a})$.. Examining row $\theta_1(\mathbf{a})$ one may notice that except of system D all other systems have the same worst achievement value $\min_i a_i = 0.33$.. Selection among systems A, B, C, and E depends on the achievements aggregation used in the RPM approach. Comparing the second worst achievements (row $\theta_2(\mathbf{a})$) one can see that according to the nucleolar RPM (7) system E is the best selection guaranteeing at least 0:67 achievement levels for six criteria. These selection cannot be done if using the classical RPM with regularization based on the total achievements. Actually, according to row $\sum_i a_i$ either lexicographic RPM (3), or its analytic version (2) will select system C as better than all the others. However, according to row $\sum_i \sum_k \min\{a_i, a_k\}$ even an analytic model a rough approximation to the nucleolar RPM an analytic model (16) turns out to be strong enough to identify system E as the best selection.

Table 6

Ordered achievements for five billing systems

	A	B	C	D	E
θ_1 (**a**)	0.33	0.33	0.33	0.27	0.33
θ_2 (**a**)	0.33	0.33	0.60	0.33	0.67
θ_3 (**a**)	0.50	0.50	0.67	0.50	0.67
θ_4 (**a**)	0.83	0.85	0.80	0.67	0.67
θ_5 (**a**)	1.00	1.00	0.87	0.90	0.75
θ_6 (**a**)	1.00	1.00	1.00	1.00	1.00
θ_7 (**a**)	1.00	1.05	1.00	1.04	1.02
$\sum_i a_i$	4.99	5.06	5.27	4.71	5.11
$\sum_i \sum_k \min\{a_i, a_k\} a_i$	27.33	27.75	29.91	25.10	30.27

CONCLUSIONS

The reference point method is a very convenient technic for interactive analysis of the multiple criteria optimization problems. It provides the DM with a tool for an open analysis of the efficient frontier. The interactive analysis is navigated with the commonly accepted control parameters expressing reference levels for the individual objective functions. The partial achievement functions quantify the DM satisfaction from the individual outcomes with respect to the given reference levels. The final scalarizing function is built as the augmented max-min aggregation of partial achievements, which means that the worst individual achievement is essentially maximized, but the optimization process is additionally regularized with the term representing the average achievement. The regularization by the average achievement is easily implementable, but it may disturb the basic max-min aggregation. In order to avoid inconsistencies caused by the regularization, the max-min solution may be regularized according to the Rawlsian principle of justice leading to the nucleolar RPM model.

The nucleolar RPM implements a consequent max-min aggregation taking into account also the second worst achievement, the third worse, and so on, thus resulting in much better modeling of the reference levels concept. The nucleolar regularization is more complicated in implementation due to the requirement of pointwise ordering of partial achievements. Nevertheless, by taking advantages of piecewise linear expression of the cumulated ordered achievements, the nucleolar RPM can be formulated as a standard lexicographic optimization. Actually, in the case of concave piecewise linear partial achievement functions (typically used in the RPM), the resulting formulation extends the original constraints and criteria with simple linear inequalities, thus allowing for a quite efficient implementation. The nucleolar RPM can be also approximated with the analytic form using the ordered weighted averaging, thus introducing explicit scalarizing achievement function to be interpreted as utility.

The paper is focused on nucleolar refinement of the reference point method. Nevertheless, the same methodology can be easily applied to various multiple criteria approaches requiring some fair (equitable) aggregations. In particular, to the fuzzy goal programming models.

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REPRESENTING PARTIAL INFORMATION ON PREFERENCES WITH THE HELP OF LINEAR TRANSFORMATION OF OBJECTIVE SPACE

Abstract

We consider the case in which information about preferences in a multiobjective optimization problem is represented in the form of upper bounds on trade-off coefficients, defined for any pair of objective functions. We introduce some requirements for this preference system and derive a system of constraints which should be put on the upper bounds. We use a matrix which elements are numbers inverse to upper bounds on trade-offs as the linear operator for transforming the objective space. The presented constraint system allows for proving that if a given evaluation is weakly Pareto optimal in the transformed space, then in the initial space it is Pareto optimal and satisfies the corresponding bounds on trade-off coefficients.

Keywords

Multiple objective optimization, incomplete preference information, bounds on trade-off coefficients, linear transformation of objective space.

INTRODUCTION AND PROBLEM SETTING

Incomplete information about preferences is typical for practical multi-objective problems. Therefore much attention in multiobjective optimization is paid to methods of solving problems exploiting partial information on preferences. One of approaches to representing and treating such information is to put bounds on global trade-offs. This approach is elaborated in a series of works by Kaliszewski, Michalowski, and Zionts (see for example [1-3]) and is applied in the construction of interactive methods of solving multiobjective optimization problems [3].

In the present work we show that a linear constraint system should be put on the bounds mentioned above. Using these constraints we elaborate a new approach to setting bounds on trade-offs based on linear transformation of the objective space.

Below we give the problem setting and the definition of trade-off solutions known from the literature. In the first Section we introduce a strict preference relation which corresponds to bounding trade-offs. We require for this preference relation to be asymmetric and weakly transitive. From these requirements we derive a set of constraints which should satisfy the given bounds on trade-offs. In the second Section we present a new approach to constructing a preference relation based on the linear transformation of the objective space. Using the constraints mentioned above we establish a connection between the "classical" bounds on trade-offs and the new approach.

Let us consider the multiobjective optimization problem of the following form:

$$\max_{x \in X} f(x) \tag{1}$$

where X is the set of feasible solutions, $f(x) = (f_1(x), f_2(x), \ldots, f_k(x))$ is the vector objective, $f_i\colon X \to \mathbf{R}$, $i \in N_k$ are objective functions, $N_k = \{1,2,\ldots,k\}$; $k \geq 2$. To each feasible solution x we assign its (vector) evaluation $y = f(x)$.

Solving this problem means finding an element of X, which is the most preferred one from the point of view of Decision Maker (DM). When choosing the "most preferred" solution, we take into account only those vector evaluations which components should be "as large as possible". Therefore we go from the problem (1) to the problem of finding the most preferred evaluation:

$$\max_{y \in Y} y$$

where

$$Y = \{f(x) \colon x \in X\}$$

is the set of all evaluations, $Y \subset \mathbf{R}^k$.

Given Y, the set of Pareto optimal evaluations $P(Y)$ and the set of weakly Pareto optimal evaluations $W(Y)$ are defined as:

$$P(Y) = \{y \in Y \colon \nexists\, y' \in Y\ (y' \geq y \;\&\; y' \neq y)\}$$

$$W(Y) = \{y \in Y \colon \forall\, y' \in Y \setminus \{y\}\ \exists\, p \in N_k\ (y_p \geq y'_p)\}$$

A feasible solution x is called a Pareto optimal (weakly Pareto optimal) solution of the problem (1), if its evaluation is Pareto optimal (weakly Pareto optimal).

We use the notion of global trade-off coefficient introduced in Kaliszewski and Michalowski (1997).

For any $y^* \in \mathbf{R}^k$ and any $j \in N_k$, we define the set:

$$Z_j(y^*) = \{y \in \mathbf{R}^k:\ y_j < y_j^* \ \&\ \forall\ s \in N_k \setminus \{j\}\ (y_s \geq y_s^*)\}$$

Definition 1 [1]

Let $i, j \in N_k$, $i \neq j$. If $Z_j(y^*) \cap Y \neq \varnothing$, then the number

$$T_{ij}(y^*, Y) = \sup_{y \in Z_j(y^*) \cap Y} \frac{y_i - y_i^*}{y_j^* - y_j} \tag{2}$$

is called the (global) trade-off between i-th and j-th objective functions for the evaluation y^*.

If $Z_j(y^*) \cap Y = \varnothing$, then by definition, we assume $T_{ij}(y^*, Y) = -\infty$.

The trade-off $T_{ij}(y^*, Y)$ indicates by how much, at most, the evaluation y∗ can be improved in its i-th component with respect to its worsening in its j-th component when passing to any other evaluation from Y, under the condition that the remaining components do not become worse.

For any $i,j \in N_k$, $i \neq j$, let the number $\alpha\text{ij} > 0$, which represents the needed upper bound on the trade-off between i-th and j-th objective functions, be given.

Definition 2

Let $y^* \in Y$. The evaluation y∗ is called the trade-off evaluation of the problem (1), if it is Pareto optimal and the following inequalities hold:

$$T_{ij}(y^*, Y) \leq \alpha_{ij} \ \text{ for all } \ i,j \in N_k,\ i \neq j \tag{3}$$

Let $x \in X$. We call x the trade-off solution of the problem (1), if its evaluation is a trade-off evaluation.

The number α_{ij} can be interpreted as the minimal gain in the i-th objective function which in DM's opinion overweighs the unitary loss in j-th objective. If $T_{ij}(y^*, Y) > \alpha_{ij}$ for some $i,j \in N_k$, then there exists another evaluation y more preferable than y^* because when going from y^* to y, the DM gains (improving the i-th objective) relatively more than he/she looses (worsening the j-th objective).

1. TRADE-OFFS AND PREFERENCE RELATIONS

Let us introduce a strict preference relation on $\mathbf{R}^k$ such that its maximal elements among Pareto optimal evaluations (and only among them) are trade--off evaluations.

Definition 3

Let $y, y' \in \boldsymbol{R}^k$. *We say that vector* y ***t-dominates*** *vector* y' *and write*:

$$y \succ y'$$

if and only if there exist $s,p \in N_k$, $s \neq p$ *such that*:

$$y \in Z_p(y') \text{ and } \frac{y_s - y'_s}{y'_p - y_p} > \alpha_{sp}$$

This relation between evaluations y and y' means, that there exist $i,j \in N_k$, $i \neq j$ such that by passing from y to y' we get an improvement in the i-th component more than α_{ij} times as large as is the deterioration in the j-th component, at that all the other components do not deteriorate[1].

Note that the preference relation $\succ$ is $k(k$-1)-parametric because it depends on the numbers α_{ij}, $i,j \in N_k$, $i \neq j$. But we do not include these numbers in its designation, because changing values of the parameters is not assumed in this work.

The negation of relation ϕ is denoted by $\not\phi$.

We call our preference relation *t-dominance* where letter "t" is the abbreviation for "trade-off" because this relation is closely connected with the definition of trade-off evaluations. This connection is established by the following evident proposition.

Proposition 1

Let $y^* \in P(Y)$. *The evaluation* y^* *is a trade-off evaluation if and only if there does not exist another evaluation* $y \in Y$ *such that* $y \succ y^*$.

Indeed, the existence of such evaluation, and only it, ensures that (3) is violated.

Proposition 1 implies that the set of trade-off evaluations is the intersection of Pareto set with the set of maximal elements of preference relation $\succ$ on Y.

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[1] We use terms "better" and "worse" as synonyms for words "greater" and "smaller" according to the principle "better if more" for all components of the vector objective.

Consider an example showing one peculiarity of t-dominance preference relation.

Example 1

Let $k = 3$, $\alpha_{ij} = 3$ *for all* $i,j \in N_k$, $i \neq j$. *Let* $y=(2,6,4)$, $y'=(2,5,7)$, $y''=(3,2,4)$. *Then we have* $y-y'' = (-1,4,0)$, $y'-y'' = (-1,3,3)$, $\{y,y'\} \subseteq Z_1(y'')$. Assuming that all three criteria are of similar worth, y′ in comparison to y″ seems more advantageous than y in comparison to y″. But we have

$$\frac{y_2 - y_2''}{y_1'' - y_1} = 4 > \alpha_{21}, \quad \frac{y_2' - y_2''}{y_1'' - y_1'} = 3 \le \alpha_{21}, \quad \frac{y_3' - y_3''}{y_1'' - y_1'} = 3 \le \alpha_{31}$$

which means that $y \phi y''$ *and* $y' \not\phi y''$ *according to Definition 3.*

We see in this example that t-dominance preference relation does not have a cumulative effect when gain of passing from one solution to another is calculated. It takes into account only relation between gain and loss in a pair of objectives. As a result, y' does not dominate y'' and y does, while in sum y' has more gains in comparison with y.

The fact that evaluation y is t-dominated by evaluation y' means that in DM's opinion the evaluation y' is *better than* the evaluation y. It follows that t-dominance is a strict preference relation. There are some properties that strict preference relations are expected to satisfy. They have to be irreflexive and asymmetric. Recall that relation $\succ$ is:

– irreflexive, if $y \phi y$ for any $y \in \mathbf{R}^k$,
– asymmetric, if $y \phi y'$ implies $y' \phi y$ for any $y, y' \in \mathbf{R}^k$,
– transitive, if $y \phi y' \phi y''$ implies $y \phi y''$ for any $y, y', y'' \in \mathbf{R}^k$.

Note that transitivity is a desirable, but not necessary property of a strict preference relation.

It is evident that t-dominance preference relation is irreflexive since $y \notin Z_j(y)$ for any $y \in \mathbf{R}^k$.

The next proposition gives the necessary and sufficient condition for this relation to be asymmetric.

Proposition 2

Preference relation $\succ$ *is asymmetric if and only if the bounds on trade--offs satisfy the inequalities:*

$$1/\alpha_{ij} \le \alpha_{ji} \quad \text{for any } i,j \in N_k,\ i \neq j \tag{4}$$

Proof. Sufficiency

Let inequalities (4) hold. Suppose $y \succ y'$ and prove that $y' \not\succ y$. By definition, there exist $s, p \in N_k$, $s \neq p$ such that $y \in Z_p(y')$ and

$$\frac{y_s - y'_s}{y'_p - y_p} > \alpha_{sp} \tag{5}$$

There are two possible cases.

Case 1

There exists $q \in N_k$ such that $y' \in Z_q(y)$. From (5) we have $y'_s < y_s$, which implies $q = s$. Observe that from $y' \in Z_s(y)$ we have $y_i \leq y'_i$ for any $i \in N_k \backslash \{s\}$ and from $y \in Z_p(y')$ we have $y_i \geq y'_i$ for any $i \in N_k \backslash \{p\}$. It follows that:

$$y_i = y'_i \text{ for any } i \in N_k \backslash \{s,p\} \tag{6}$$

Combining (5) and (4) we get:

$$\frac{y'_p - y_p}{y_s - y'_s} < \frac{1}{\alpha_{sp}} \leq \alpha_{ps}$$

Taking into account (6) we conclude that there does not exist an $i \in N_k \backslash \{s\}$ satisfying

$$\frac{y'_i - y_i}{y_s - y'_s} > \alpha_{is}$$

Hence $y' \not\succ y$ in this case.

Case 2

There does not exist $q \in N_k$ such that $y' \in Z_q(y)$. Then $y' \not\succ y$ by definition.

The sufficiency has been proved.

Necessity

Suppose that (4) violates, i.e. for some s, $p \in N_k$, $s \neq p$, the inequality $1/\alpha_{sp} > \alpha_{ps}$ holds. Let $y \in \mathbf{R}^k$. Consider the vector $y' \in \mathbf{R}^k$ with elements defined as follows: $y'_i = y_i$, $i \in N_k \backslash \{s,p\}$, $y'_s = y_s - u$, $y'_p = y_p + v$, where u and v are positive numbers such that:

$$1/\alpha_{sp} > v/u > \alpha_{ps}$$

Then we have $y \in Z_p(y')$, $\dfrac{y_s - y'_s}{y'_p - y_p} > \alpha_{sp}$ and $y' \in Z_s(y)$, $\dfrac{y'_p - y_p}{y_s - y'_s} > \alpha_{ps}$

It follows that $y \succ y'$ and $y' \succ y$ which means that the preference relation $\succ$ is not asymmetric.

Now we will study the question of transitivity. It is evident that the t-dominance preference relation is not transitive due to the following technical reason. Relation y ϕ y″ requires that the inclusion $y \in Z_j(y'')$ hold for some $j \in N_k$. But it is evident that the relation $y \ \phi \ y' \ \phi \ y''$ in general case does not imply this inclusion, hence it does not imply $y \ \phi \ y''$.

What if we weaken the condition of transitivity to eliminate the technical problem described above? Namely, let's require that the implication $y \ \phi \ y' \ \phi \ y'' \Rightarrow y \ \phi \ y''$ hold only when $y \in Z_j(y'')$ for some $j \in N_k$. The following example shows that even in this case the implication does not hold.

Example 2

Let $i,j,s \in N_k$, $i \neq j \neq s \neq i$; v_i, v_j, v_s be positive numbers such that

$$\alpha_{ij} v_j < v_i < 2\alpha_{ij} v_j \tag{7}$$

$$\alpha_{sj} v_j < v_s < 2\alpha_{sj} v_j \tag{8}$$

Let $y \in \boldsymbol{R}^k$. Consider vectors y' and y'' with components defined as follows:

$$y_i' = y_i - v_i, \quad y_j' = y_j + v_j, \quad y_l' = y_l \text{ for all } l \in N_k \backslash \{i,j\}$$

$$y_s'' = y_s' - v_s, \quad y_j'' = y_j' + v_j, \quad y_l'' = y_l' \text{ for all } l \in N_k \backslash \{s,j\}$$

Observe that:

$$y_i'' = y_i - v_i, \quad y_s'' = y_s - v_s, \quad y_j'' = y_j + 2v_j, \quad y_l'' = y_l \text{ for all } l \in N_k \backslash \{i,s,j\}$$

Then we have $y \in Z_j(y')$, $y' \in Z_j(y'')$ and $y \in Z_j(y'')$.

Using (7) we obtain $\dfrac{y_i - y_i'}{y_j' - y_j} = \dfrac{v_i}{v_j} > \alpha_{ij}$ *which implies $y \ \phi \ y'$.*

Using (8) and obtain $\dfrac{y_s' - y_s''}{y_j'' - y_j'} = \dfrac{v_s}{v_j} > \alpha_{sj}$ *which implies $y' \ \phi \ y''$.*

From (7) and (8) we obtain $\dfrac{y_i - y_i''}{y_j'' - y_j} = \dfrac{v_i}{2v_j} < \alpha_{ij}$, $\dfrac{y_s - y_s''}{y_j'' - y_j} = \dfrac{v_s}{2v_j} < \alpha_{sj}$.

It follows that $y \ \phi \ y' \ \phi \ y''$, but $y \not\phi \ y''$ in spite of $y \in Z_j(y'')$.

The explanation of non-transitivity in this example is as follows. When passing from y'' to y' we improve the s-th objective at the expense of worsening the j-th objective; when passing from y' to y we improve the i-th objective at the expense of worsening the j-th objective. So when passing from y'' to y we improve the i-th and s-th criteria and get double worsening in the j-th objective. This double worsening is not formally compensated by the individual improvement of either the i-th or the s-th objective. While DM could be satisfied with these two improvements for the exchange of double worsening, this satisfaction is not taken into account by the preference relation because of the improvements are not cumulative.

To eliminate the described effect we weaken the condition of transitivity even more. We consider only those cases where y improves y'' in a single objective.

For any $y^* \in \mathbf{R}^k$ and any $i,j \in N_k$, $i \neq j$, we define the set:

$$Z_{ij}(y^*) = \{y \in \mathbf{R}^k : y_i \geq y_i^*,\ y_j < y_j^*,\ \forall\ s \in N_k \backslash \{j,j\}\ (y_s = y_s^*)\}$$

This is the set of vectors which improve y^* in the i-th objective and worsen y^* in the j-th objective while the values of other objectives do not change. Observe that $Z_{ij}(y^*) \subseteq Z_j(y^*)$.

Definition 4

The preference relation $\succ$ is called weakly transitive if for any $y,y',y'' \in \boldsymbol{R}^k$ the following implication holds:

$$(y \phi y' \phi y'' \text{ and } y \in Z_{ij}(y'') \text{ for some } i,j \in N_k) \Rightarrow y \phi y''$$

The following proposition shows that t-dominance preference relation satisfies this kind of weakened transitivity under some conditions imposed on trade-off bounds.

Proposition 3

The preference relation $\succ$ is weakly transitive if and only if the bounds on trade-offs satisfy the inequalities:

$$\alpha_{is}\alpha_{sj} \geq \alpha_{ij} \quad \text{for any } i,j,s \in N_k,\ i \neq s,\ j \neq s \tag{9}$$

The proof of Proposition 3 is given in Appendix.

Note that inequalities (4) are particular cases of (9) (when $i = j$).

2. LINEAR TRANSFORMATION AND TRADE-OFFS

We define the elements of a transformation matrix $\mathrm{B} = (\beta_{ij})_{k\times k} \in \mathbf{R}^{k\times k}$ by:

$$\beta_{ij} = \frac{1}{\alpha_{ji}},\ i,j \in N_k \tag{10}$$

The objective space is transformed by multiplying its vectors by the matrix. The transformed vector evaluations are compared componentwise:

$$\mathrm{B}y > \mathrm{B}y' \iff \sum_{j \in N_k} \beta_{ij} y_j > \sum_{j \in N_k} \beta_{ij} y'_j \quad \text{for all} \quad i \in N_k$$

A connection between such comparison and t-dominance relation is established by the following proposition.

Proposition 4

Let bounds on trade-offs satisfy inequalities (9). Then for any evaluations $y,y' \in \boldsymbol{R}^k$ the following implication holds:

$$y \phi y' \Rightarrow By > By'$$

Proof

Let $y \phi y'$. By Definition 3 there exist $s,p \in N_k$, $s \neq p$, such that $y_p < y'_p$,

$$y_l \geq y'_l \text{ for any } l \in N_k \backslash \{p\} \tag{11}$$

$$y_s - y'_s > \alpha_{sp}\left(y'_p - y_p\right) \tag{12}$$

To prove the proposition it is enough to show that $\sum_{j \in N_k} \beta_{ij}\left(y_j - y'_j\right) > 0$ for any $i \in N_k$.

Let $i \in N_k$. Observe that due to $\beta_{ii} = 1$ we have:

$$\sum_{j \in N_k} \beta_{ij}\left(y_j - y'_j\right) = y_i - y'_i + \sum_{j \in N_k \backslash \{i\}} \beta_{ij}\left(y_j - y'_j\right)$$

Consider three possible cases.

Case 1

$i=p$. Applying (10), (11), and (12) we obtain:

$$y_p - y'_p + \sum_{j \in N_k \backslash \{p\}} \beta_{pj}\left(y_j - y'_j\right) \geq y_p - y'_p + \beta_{ps}\left(y_s - y'_s\right) >$$

$$y_p - y'_p + \alpha_{sp}\beta_{ps}\left(y'_p - y_p\right) = 0$$

Case 2

$i = s$. From (9), (10), (11), and (12) we have:

$$y_s - y'_s + \sum_{j \in N_k \backslash \{s\}} \beta_{sj}\left(y_j - y'_j\right) \geq y_s - y'_s + \beta_{sp}\left(y_p - y'_p\right) >$$

$$\alpha_{sp}\left(y'_p - y_p\right) + \beta_{sp}\left(y_p - y'_p\right) = \left(\alpha_{sp} - \frac{1}{\alpha_{ps}}\right)\left(y'_p - y_p\right) \geq 0$$

Case 3

$i \notin \{s,p\}$. From (11) we have:

$$y_i - y'_i + \sum_{j \in N_k \setminus \{i\}} \beta_{ij} \left(y_j - y'_j \right) \geq \beta_{is} \left(y_s - y'_s \right) + \beta_{ip} \left(y_p - y'_p \right) =$$

$$\beta_{is} \left(y_s - y'_s \right) - \beta_{ip} \left(y'_p - y_p \right)$$

Applying (9), (10), and (12) we obtain:

$$\frac{\beta_{is} \left(y_s - y'_s \right)}{\beta_{ip} \left(y'_p - y_p \right)} > \frac{\alpha_{sp} \alpha_{pi}}{\alpha_{si}} \geq 1$$

It follows that $\beta_{is} \left(y_s - y'_s \right) - \beta_{ip} \left(y'_p - y_p \right) > 0$.

In each of three cases we obtained $\sum_{j \in N_k} \beta_{ij} \left(y_j - y'_j \right) > 0$.

■

The next proposition follows from the fact that all the elements of matrix B are positive.

Proposition 5

Let the evaluations $y, y' \in \boldsymbol{R}^k$ satisfy the inequalities $y \geq y'$, $y \neq y'$. Then $\mathrm{B}y > \mathrm{B}y'$.

By definition, put:

$$Y_{\mathrm{B}} = \{\mathrm{B}y\colon\ y \in Y\}$$

Corollary 1 [4]

If $\mathrm{B}y^* \in W(Y_B)$, then y^* is a trade-off evaluation of problem (1).

Proof

Let $\mathrm{B}y^* \in W(Y_{\mathrm{B}})$. Then there does not exist $y \in Y$ such that $\mathrm{B}y > \mathrm{B}y^*$. It follows from Proposition 5 that $y^* \in P(Y)$. It follows from Proposition 4 that there does not exist $y \in Y$ such that $y \phi y^*$. Then according to Proposition 1 y^* is a trade-off evaluation of problem (1).

■

According to Corollary 1, all weakly Pareto optimal solutions of the problem

$$\max_{x \in X} \mathrm{B} f(x) \tag{13}$$

are trade-off solutions of problem (1). But the converse is not true: not all the trade-off solutions of problem (1) can be found among weakly Pareto optimal solutions of problem (13) because the relations $y \phi y'$ and $By > By'$ are not equivalent. To illustrate the difference between them, we introduce an alternative interpretation of t-dominance preference relation.

By definition, let $\frac{a}{0} = +\infty$ for any $a > 0$. From (10) we obtain the following evident proposition which is actually a reformulation of Definition 3.

Proposition 6

Let $y, y' \in \boldsymbol{R}^k$. The evaluations y and y' satisfy the relation $y \succ y'$ if and only if there exist $i,j \in N_k$, $i \neq j$, such that:

$$y \in Z_i(y') \text{ and } \frac{y_i' - y_i}{y_j - y_j'} < \beta_{sp}$$

We see that t-dominance preference relation can be defined using numbers β_{ij} as parameters instead of α_{ij}. We propose the following interpretation of these numbers: β_{ij} is the maximum loss in the i-th objective which DM agrees to pay for unitary gain in the j-th objective under the condition that all the other objectives do not worsen. If during passing from y' to $y \in Z_i(y')$ the relation between loss and gain is less than β_{ij} for some $j \in N_k$, then y is more preferable than y' because DM loses (worsening the i-th objective) relatively less than the maximum he/she agreed to pay for the obtained gain (of improving the j-th objective).

Lemma 1

Let $y' \in \boldsymbol{R}^k$, $i \in N_k$, $y \in Z_i(y')$ be bounds on trade-offs satisfying (9). Then $By > By'$ if and only if

$$y_i' - y_i < \sum_{j \in N_k \setminus \{i\}} \beta_{ij} \left(y_j - y_j' \right) \tag{14}$$

Proof

The necessity is evident. Let (14) be valid and let's prove that an analogous inequality holds for any $s \in N_k \setminus \{i\}$. Observe that (9) implies:

$$\beta_{si} \beta_{ij} \leq \beta_{sj} \quad \text{for any} \quad i,j,s \in N_k,\ i \neq s,\ i \neq j \tag{15}$$

and $y \in Z_i(y')$ implies:

$$y_i < y_i' \tag{16}$$

$$y_l \geq y_l' \quad \text{for any} \quad l \in N_k \setminus \{i\} \tag{17}$$

Let $s \in N_k \backslash \{i\}$. Multiplying both sides of (14) by β_{si} and taking into account (15)-(17) we obtain:

$$\beta_{si}\left(y_i' - y_i\right) < \beta_{is}\beta_{si}\left(y_s - y_s'\right) + \sum_{j \in N_k \backslash \{i,s\}} \beta_{si}\beta_{ij}\left(y_j - y_j'\right) \le$$

$$\left(y_s - y_s'\right) + \sum_{j \in N_k \backslash \{i,s\}} \beta_{sj}\left(y_j - y_j'\right)$$

Thus, we get $\left(y_s' - y_s\right) < \sum_{j \in N_k \backslash \{s\}} \beta_{sj}\left(y_j - y_j'\right)$ for any $s \in N_k \backslash \{i\}$ which in combination with (14) yields $By > By'$.

■

Now compare two approaches to representing preferences in multi-objective problem (1): t-dominance preference relation and the relation of weak Pareto domination in the transformed objective space. Let $y \in Z_i(y')$. From Proposition 6 we have:

$$y \succ y' \;\Leftrightarrow\; y_i' - y_i < \beta_{ij}\left(y_j - y_j'\right) \text{ for some } j \in N_k \backslash \{i\}$$

From Lemma 1 we have:

$$\mathrm{B}y > \mathrm{B}y' \;\Leftrightarrow\; y_i' - y_i < \sum_{j \in N_k \backslash \{i\}} \beta_{ij}\left(y_j - y_j'\right)$$

In the first approach we check if the price of decreasing the i-th objective is small enough to agree to pay it for increasing one of the other objectives. In the second approach we "include in the bill" all the gains in other objectives and feel satisfied if we have paid by decreasing the i-th objective less than sum of the prices.

Let us return to Example 1 to see how the new approach functions in comparison with t-dominance relation. The following example is an extension of Example 1.

Example 3

Let $k = 3$, $\alpha_{ij} = 3$ for all $i,j \in N_k$, $i \ne j$. Let $y = (2,6,4)$, $y' = (2,5,7)$, $y'' = (3,2,4)$. Then we have $y - y'' = (-1,4,0)$, $y' - y'' = (-1,3,3)$, $\{y,y'\} \subseteq Z_1(y'')$

and $\dfrac{y_2 - y_2''}{y_1'' - y_1} = 4 > \alpha_{21}$, $\dfrac{y_2' - y_2''}{y_1'' - y_1'} = 3 \le \alpha_{21}$, $\dfrac{y_3' - y_3''}{y_1'' - y_1'} = 3 \le \alpha_{31}$

This means that $y \succ y''$ and $y' \not\succ y''$.

Using (10) we obtain β_{ij} = 1/3 for all $i,j \in N_k$, $i \neq j$ and calculate By = (5 1/3, 8, 6 2/3), By' = (6, 8, 9 1/3), By'' = (5, 4 1/3, 5 2/3). Thus, we have $By' \geq By > By''$.

In this example the new approach due to cumulativeness of calculating gains has revealed that y' is better than y'' and even “almost better” than y.

CONCLUSION

In this paper we presented two mutually connected results. In the first Section we showed that bounds on trade-offs should satisfy certain constraints (Propositions 2 and 3). These constrains ensure that the preference relation represented by these bounds meets some rational requirements. In the second Section we proposed a new approach to the representation of preferences based on linear transformation of the objective space. The connection between these two results is the following. To prove Proposition 4 which establishes a relation between linear transformation and trade-offs we use the constraints obtained in the first Section.

Appendix

Proof of Proposition 3. ***Sufficiency***

Let (9) hold. Suppose that the vectors $y, y', y'' \in \mathbf{R}^k$ satisfy the conditions:

$$y \phi y' \phi y'' \text{ and } y \in Z_{qr}(y'') \text{ for some } q, r \in N_k$$

Then by definition there exist numbers $i, j, s, p \in N_k$ such that:

$$y_i > y_i', \; y_j < y_j', \; y_l \geq y_l' \text{ for any } l \in N_k \backslash \{i, j\} \tag{18}$$

$$\frac{y_i - y_i'}{y_j' - y_j} > \alpha_{ij} \tag{19}$$

$$y_s' > y_s'', \; y_p' < y_p'', \; y_l' \geq y_l'' \text{ for any } l \in N_k \backslash \{s, p\} \tag{20}$$

$$\frac{y_s' - y_s''}{y_p'' - y_p'} > \alpha_{sp} \tag{21}$$

Observe that $y \in Z_{qr}(y'')$ implies:

$$y_q \geq y_q'', \; y_r < y_r'', \; y_l = y_l'' \text{ for any } l \in N_k \backslash \{r, q\} \tag{22}$$

To prove sufficiency it is enough to show that $y \phi y''$, i.e. that:

$$\frac{y_q - y''_q}{y''_r - y_r} > \alpha_{qr} \tag{23}$$

Observe that for any $l \in N_k \backslash \{j,p,r,q\}$ from (18), (20) and (22) we have $y_l \geq y'_l \geq y''_l$ and $y_l = y''_l$. It follows that:

$$y_l = y'_l = y''_l \quad \text{for any} \quad l \in N_k \backslash \{j,p,r,q\} \tag{24}$$

Since $y_i > y'_i$ and $i \neq j$, from (24) we have

$$i \in \{p,r,q\}. \tag{25}$$

Consider three possible cases.

Case 1

$r \neq s$, $r \neq p$. From (20) and (22) we have:

$$y_r < y''_r \leq y'_r \tag{26}$$

Using (18) we obtain:

$$r = j \tag{27}$$

Then taking into account $r \neq s$, (18), and (20) we have:

$$y_s \geq y'_s > y''_s \tag{28}$$

From (22) we get:

$$q = s \tag{29}$$

Since $i \neq j$, from (25) and (27) we have only two possibilities: $i = p$ or $i = q$. Consider these two subcases.

Case 1.1

$i = p$. Since $p \neq r$ and $p \neq s = q$ (see (20), (29)), from (22) we have $y_p = y''_p$. Using (26) we have:

$$\frac{y''_p - y'_p}{y''_r - y_r} \geq \frac{y_p - y'_p}{y'_r - y_r}$$

Taking into account (27) and applying (19) and we get:

$$\frac{y''_p - y'_p}{y''_r - y_r} \geq \frac{y_p - y'_p}{y'_r - y_r} > \alpha_{pr} \tag{30}$$

Taking into account (28), (29), and applying (21) we get:

$$\frac{y_q - y_q''}{y_p'' - y_p'} \geq \frac{y_q' - y_q''}{y_p'' - y_p'} > \alpha_{qp}$$

Combining the last inequality with (30) and applying (9) we obtain:

$$\frac{y_q - y_q''}{y_r'' - y_r} = \frac{y_q - y_q''}{y_p'' - y_p'} \frac{y_p'' - y_p'}{y_r'' - y_r} > \alpha_{qp}\alpha_{pr} \geq \alpha_{qr}$$

Case 1.2

$i = q$. From (28) and (29) we have $y_q' > y_q''$. From (26) we have $y_r' \geq y_r''$. Then:

$$\frac{y_q - y_q''}{y_r'' - y_r} > \frac{y_q - y_q'}{y_r' - y_r}$$

Taking into account $q = i$ and (27) we apply (19) and obtain:

$$\frac{y_q - y_q''}{y_r'' - y_r} > \frac{y_q - y_q'}{y_r' - y_r} > \alpha_{qr}$$

Case 2

$r = p$. Consider two possible subcases.

Case 2.1

$q \neq s$. Then taking into account $s \neq p = r$ we apply (20) and (22) to get:

$$y_s = y_s'' < y_s' \tag{31}$$

From (18) we obtain $j = s$. Thus, we have:

$$r = p \neq s = j \tag{32}$$

Then taking into account (18) and (22) we get:

$$y_p'' > y_p \geq y_p' \tag{33}$$

Since $q \neq r = p$, from (20) we have:

$$y_q' \geq y_q'' \tag{34}$$

From (25) and (32) we have $i \in \{p,q\}$. But i cannot be equal to p. Indeed, under assumption $i = p$ taking into account $j = s$ we apply (19) to have:

$$\frac{y_p - y_p'}{y_s' - y_s} > \alpha_{ps} \tag{35}$$

but on the other hand taking into account (31), (33) we have:

$$\frac{y'_s - y_s}{y_p - y'_p} > \frac{y'_s - y''_s}{y''_p - y'_p} \tag{36}$$

and applying (9), (21) we obtain:

$$\frac{y'_s - y_s}{y_p - y'_p} > \frac{y'_s - y''_s}{y''_p - y'_p} > \alpha_{sp} \geq \frac{1}{a_{ps}} \tag{37}$$

which contradicts to (35).

Hence $i = q$. Then taking into account $j = s$ (see (32)) and applying (19), (31), and (34) we obtain:

$$\frac{y_q - y''_q}{y'_s - y''_s} = \frac{y_q - y''_q}{y'_s - y_s} \geq \frac{y_q - y'_q}{y'_s - y_s} > \alpha_{qs}$$

Taking into account $r = p$ and applying consequently (33) and (21) we obtain:

$$\frac{y'_s - y''_s}{y''_r - y_r} \geq \frac{y'_s - y''_s}{y''_r - y'_r} > \alpha_{sr}$$

Combining the last two inequalities and applying (9) we obtain:

$$\frac{y_q - y''_q}{y''_r - y_r} = \frac{y_q - y''_q}{y'_s - y''_s}\frac{y'_s - y''_s}{y''_r - y'_r} > \alpha_{qs}\alpha_{sr} \geq \alpha_{qr}$$

Case 2.2

$q = s$. Consider three subcases.

a) $j \notin \{q,r\}$. Then from (18) we have $y_q \geq y'_q$ and $y_r \geq y'_r$. Taking into account $q = s$, $r = p$ and applying (21) we obtain:

$$\frac{y_q - y''_q}{y''_r - y_r} \geq \frac{y'_q - y''_q}{y''_r - y'_r} > \alpha_{qr}$$

b) $j = r$. Then taking into account $r = p$ and (25) we have $i = q$. From (19) we obtain:

$$\frac{y_q - y'_q}{y'_r - y_r} > \alpha_{qr}$$

Since $q = s$ and $r = p$, from (21) we have:

$$\frac{y'_q - y''_q}{y''_r - y'_r} > \alpha_{qr}$$

Using the last two inequalities we obtain:

$$\frac{y_q - y_q''}{y_r'' - y_r} = \frac{\left(y_q - y_q'\right) + \left(y_q' - y_q''\right)}{y_r'' - y_r} > \frac{\alpha_{qr}\left(y_r' - y_r\right) + \alpha_{qr}\left(y_r'' - y_r'\right)}{y_r'' - y_r} = \alpha_{qr}$$

c) $j = q$. Then taking into account $r = p$ and (25) we have $i = r$. From (9) and (19) we have:

$$\frac{y_r - y_r'}{y_q' - y_q} > \alpha_{rq} \quad \Rightarrow \quad \frac{y_q' - y_q}{y_r - y_r'} < \frac{1}{\alpha_{rq}} \le \alpha_{qr} \quad \Rightarrow \; y_q - y_q' > \alpha_{qr}(y_r' - y_r) \tag{38}$$

Since $q = s$ and $r = p$, from (21) we have:

$$\frac{y_q' - y_q''}{y_r'' - y_r'} > \alpha_{qr}$$

Combining this with (38) we obtain:

$$\frac{y_q - y_q''}{y_r'' - y_r} = \frac{\left(y_q - y_q'\right) + \left(y_q' - y_q''\right)}{y_r'' - y_r} > \frac{\alpha_{qr}\left(y_r' - y_r\right) + \alpha_{qr}\left(y_r'' - y_r'\right)}{y_r'' - y_r} = \alpha_{qr}$$

Case 3

$r = s$. From (20) and (22) we have:

$$y_r' > y_r'' > y_r \tag{39}$$

which due to (18) implies:

$$j = r = s \tag{40}$$

Consider two possible subcases.

Case 3.1

$q = p$. Taking into account (25) and (40) we have:

$$i = q = p \tag{41}$$

From (19), (40), and (41) we have:

$$\frac{y_q - y_q'}{y_r' - y_r} > \alpha_{qr} \tag{42}$$

From (9), (21), (40), and (41) we have:

$$\frac{y_r' - y_r''}{y_q'' - y_q'} > \alpha_{rq} \quad \Rightarrow \quad \frac{y_q'' - y_q'}{y_r' - y_r''} < \frac{1}{\alpha_{rq}} \le \alpha_{qr} \quad \Rightarrow \; y_q' - y_q'' > \alpha_{qr}(y_r'' - y_r')$$

Applying the last inequality and (42) we obtain:

$$\frac{y_q - y_q''}{y_r'' - y_r} = \frac{\left(y_q - y_q'\right) + \left(y_q' - y_q''\right)}{y_r'' - y_r} > \frac{\alpha_{qr}\left(y_r' - y_r\right) + \alpha_{qr}\left(y_r'' - y_r'\right)}{y_r'' - y_r} = \alpha_{qr}$$

Case 3.2

$q \neq p$. Then from (22) we have

$$y_p'' \geq y_p \tag{43}$$

and from (20) we have:

$$y_q' \geq y_q'' \tag{44}$$

Taking into account (25) and (40) we have $i \in \{p,q\}$. But i cannot be equal to p. Indeed, under assumption $i = p$ taking into account $j = s$ (see (40)) we apply (19) to have (35). But on the other hand, from (39), (40), and (43) we have (36), then applying (9) and (21) we obtain (37) which contradicts (35).

Hence $i = q$. Then using (39), (44), and applying (19) we obtain:

$$\frac{y_q - y_q''}{y_r'' - y_r} > \frac{y_q - y_q'}{y_r' - y_r} > \alpha_{qr}$$

In each of the cases considered, inequality (23) holds. Sufficiency has been proved.

Necessity

Suppose that (9) does not hold, i.e. there exist $i,j,s \in N_k$, $i \neq s, j \neq s$, such that:

$$\alpha_{is}\alpha_{sj} < \alpha_{ij} \tag{45}$$

Consider two possible cases.

Case 1

$i \neq j$. Let $y \in \mathbf{R}^k$. Take three positive numbers v_i, v_j and v_s satisfying the inequalities

$$\alpha_{is}\alpha_{sj}v_j < v_i < \alpha_{ij}v_j \tag{46}$$

$$\alpha_{is}\alpha_{sj}v_j < \alpha_{is}v_s < v_i \tag{47}$$

and put:

$$y' \in \mathbf{R}^k,\quad y_i' = y_i - v_i,\quad y_s' = y_s + v_s,\quad y_l' = y_l \text{ for all } l \in N_k \backslash \{i,s\}$$

$$y'' \in \mathbf{R}^k,\quad y_j'' = y_j' + v_j,\quad y_s'' = y_s' - v_s,\quad y_l'' = y_l' \text{ for all } l \in N_k \backslash \{j,s\}$$

Observe that:

$y_i'' = y_i' = y_i - v_i,\quad y_j'' = y_j' + v_j = y_j + v_j,\quad y_l'' = y_l$ for all $l \in N_k \backslash \{i,j\}$

Then we have:

$$y \in Z_{is}(y') \subseteq Z_s(y'),\quad y' \in Z_{sj}(y'') \subseteq Z_j(y''),\quad y \in Z_{ij}(y'')$$

Using (47) we obtain $\dfrac{y_i - y_i'}{y_s' - y_s} = \dfrac{v_i}{v_s} > \alpha_{is}$, which implies $y \phi y'$, and $\dfrac{y_s' - y_s''}{y_j'' - y_j'} = \dfrac{v_s}{v_j} > \alpha_{sj}$, which implies $y' \phi y''$. Using (46) we obtain $\dfrac{y_i - y_i''}{y_j'' - y_j} = \dfrac{v_i}{v_j} < \alpha_{ij}$, which implies $y \not\phi\ y''$. Thus, we have $y \phi y' \phi y''$, $y \in Z_{ij}(y'')$ and $y \not\phi y''$ which means that t-dominance preference relation is not weakly-transitive in Case 1.

Case 2

$i = j$. From (45) we have:

$$\alpha_{si} < \frac{1}{\alpha_{is}}$$

Take two positive numbers v_i and v_s satisfying the inequalities:

$$\alpha_{si} v_i < v_s < \frac{v_i}{\alpha_{is}}$$

Then we have:

$$\frac{v_s}{v_i} > \alpha_{si} \tag{48}$$

and

$$v_i - \alpha_{is} v_s > 0$$

It follows that there exist positive numbers v_i' and v_s' satisfying the inequalities:

$$v_i' > v_i,\ v_s' > v_s \tag{49}$$

and

$$\alpha_{is} v_s' < v_i' < \alpha_{is} v_s' + (v_i - \alpha_{is} v_s)$$

Then we have:

$$\frac{v_i'}{v_s'} > \alpha_{is} \tag{50}$$

and $v_i' - v_i < \alpha_{is}(v_s' - v_s)$ which implies:

Let $y \in \mathbf{R}^k$, then:

$$y' \in \mathbf{R}^k,\quad y'_i = y_i + v_i,\quad y'_s = y_s - v_s,\quad y'_l = y_l \text{ for all } l \in N_k \backslash \{i,s\}$$

$$y'' \in \mathbf{R}^k,\quad y''_i = y'_i - v'_i,\quad y''_s = y'_s + v'_s,\quad y''_l = y'_l \text{ for all } l \in N_k \backslash \{i,s\}$$

Observe that (49) implies:

$$y''_i = y_i + v_i - v'_i < y_i,\quad y''_s = y_s - v_s + v'_s > y_s,\quad y''_l = y_l \text{ for all } l \in N_k \backslash \{i,s\}$$

Then we have:

$$y \in Z_{si}(y') \subseteq Z_i(y'),\quad y' \in Z_{is}(y'') \subseteq Z_j(y''),\quad y \in Z_{is}(y'')$$

Using (48) we obtain $\dfrac{y_s - y'_s}{y'_i - y_i} = \dfrac{v_s}{v_i} > \alpha_{si}$ which implies $y \phi y'$. Using (50) we obtain $\dfrac{y'_i - y''_i}{y''_s - y'_s} = \dfrac{v'_i}{v'_s} > \alpha_{is}$ which implies $y' \phi y''$. Using (51) we obtain $\dfrac{y_i - y''_i}{y''_j - y_j} = \dfrac{v'_i - v_i}{v'_s - v_s} < \alpha_{is}$ which implies $y \not\phi y''$. Thus, we have $y \phi y' \phi y''$, $y \in Z_{is}(y'')$ and $y \not\phi y''$ which means that t-dominance preference relation is not weakly-transitive in Case 2.

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FUZZY MULTIOBJECTIVE METHODS IN MULTISTAGE DECISION PROBLEMS*

Abstract

In this paper we propose a new approach for solving dynamic multiobjective decision making problems. The decision variants are generated in a discrete multistage model by forward/backward procedure of finding the set of all maximal elements based on Bellman's principle of optimality. As the set of all maximal elements consists of a number of elements – decision variants, our problem is to find among them a compromise element based on decision maker´s preferences with respect to several decision criteria. The evaluation of the weights of the criteria is based on data given by pairwise comparison matrices using triangular fuzzy numbers. Extended arithmetic operations with fuzzy numbers for application of the generalized logarithmic least squares method are defined and six methods for ranking fuzzy numbers to compare fuzzy outcomes are proposed. A numerical example is presented to clarify the methodology.

Keywords

Multicriteria decision making, dynamic programming, multistage decision process, pairwise comparisons, fuzzy numbers, analytic hierarchy process (AHP).

INTRODUCTION

Most of decision making situations consist of sequential decision problems and have multiobjective character. One way of modeling such situations is multiobjective approach. The objectives can be described by means

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of decision criteria given by elements from a partially ordered set. Here we consider a finite process divided into periods and for the decision criteria we use a set of fuzzy values modeling uncertainty of human judgments and evaluations. The goals of the individual periods are appropriately associated with the overall goal of the system and we look for efficient elements – sequences of decisions of the individual periods maximizing the objectives in the sense of Pareto. Forward/backward procedures of finding the set of all maximal elements based on Bellman´s principle of optimality have been investigated in [9, 10, and 11], see also citations therein. In real multistage decision problems the set of all efficient elements is, however, too large to be used as a direct decision support in the decision making process. That is why specific methods are needed for finding a "compromise" solutions, i.e. for narrowing the set of efficient elements, ideally to generate a single element, satisfying additional requirements, and serving as a decision support in the decision process.

In this paper we propose a new method for finding such elements based on the analytical hierarchical process (AHP) which uses uncertain human preferences as input information. Instead of the classical eigenvector prioritization method, employed in the prioritization stage of the AHP, a fuzzy preference method, based on logarithmic least squares method, is applied. The resulting fuzzy AHP enhances the potential of the classical AHP for dealing with imprecise and uncertain human comparison judgments. It allows for multiple representations of uncertain human preferences by crisp, interval, and fuzzy judgments and makes it possible to find out a solution from incomplete sets of pair-wise comparisons.

When applying the classical AHP in decision making process one usually encounters two difficulties:

– evaluating pair-wise comparisons on a nine-point scale one does not deal with uncertainty,
– decision criteria are not independent as is usually required.

Here we deal with the first difficulty by proposing a new method which incorporates uncertainty by adopting pair-wise comparisons by triangular fuzzy numbers. The second difficulty taking into account interdependences between decision criteria is dealt with elsewhere, see [5].

The interface between hierarchies, multiple objectives, and fuzzy sets have been investigated by the author of the AHP, T.L. Saaty, as early as in 1978 in [6]. Later, van Laarhoven and V. Pedrycz extended the AHP to fuzzy pair-wise comparisons [8]. Here we propose a new and relatively simple method based on the original approaches from [1, 3] and extend this method for finding a compromise efficient variant generated by the multistage decision process [9-11]. Finally, we supply an example to demonstrate properties of the proposed method.

1. SEQUENTIAL DECISION PROBLEMS

In this paper we consider a dynamic process which consists of T periods, $t = 1,2,...,T$. At the beginning of each period the system is in one of a finite number of given states. When a decision is made, the system is transformed to another feasible state and the next period begins. The feasibility of states is measured by membership grades and the transformation of the system from one state to another is performed according to decisions based on multiple criteria − objectives. The criteria evaluations are given by fuzzy values. We shall use the following notation based on [11]:

1. X_t is the set of all *states* at the period t. Here we assume $X_t = X$ for all t and X is a finite set of given states.
2. $\widetilde{D}_t = \{\mu^t(d); d = (x, y), x \in X_t, y \in X_{t+1}\}$ is the fuzzy subset of $X \times X$, called the *fuzzy set of feasible decisions at the period t* (see Section 4 below). The feasibility of the decision $d = (x,y)$ transforming the system from state x to y is denoted by the *membership grade* $\mu^t(d)$, a real number from the unit interval [0,1]. If $\mu^t(d) = 0$, then $d = (x,y)$ is infeasible, i.e. it is impossible to transform the system from state x to y. On the other hand, if $\mu^t(d) = 1$, then transforming the system from x to y is fully feasible, i.e. possible. Given a feasibility level $\lambda \in [0,1]$, we define the set of all decisions at the period t with the feasibility at least λ (denoted by $[\widetilde{D}_t]_\lambda$), as follows.
3. $[\widetilde{D}_t]_\lambda = \{d = (x, y); \mu^t(d) \geq \lambda, x, y \in X\}$.
4. $\widetilde{c}_1^t(d)$, $\widetilde{c}_2^t(d)$,..., $\widetilde{c}_n^t(d)$ are fuzzy values (e.g. triangular fuzzy numbers) of n *decision criteria* $C_1, C_2, ..., C_n$, respectively, for decision $d = (x,y)$ at the period t (see Section 4 below).
5. $V = \{(x_1,x_2); (x_2,x_3);...; (x_T,x_{T+1})\}$ is the sequence of decisions $d_t = (x_t,x_{t+1})$, $x_t \in X$, $t = 1,2,...,T$. Here we call this sequence a *multistage decision variant* (*alternative*). If for each $d_t = (x_t,x_{t+1}) \in V$ it is true that $d_t = (x_t,x_{t+1}) \in [\widetilde{D}_t]_\lambda$, where $\lambda \in [0,1]$ is a given feasibility level, we call V the λ − *feasible multistage decision variant*.

2. ELICITING DECISION VARIANTS

For the case of partially ordered outcomes, e.g. evaluation of variants by fuzzy numbers, the concept of efficiency of decision variants as well as an algorithm for finding such variants has been presented in [10]. The essence of the algorithm is a forward/backward procedure of finding the set of all efficient variants based on Bellman's principle of optimality. This algorithm usually generates a large number of efficient variants which is difficult to use in a decision process. Assume that m efficient multistage decision variants have been generated by the algorithm [10], in particular, we have:

$$V_j = \{(x_1^j, x_2^j); (x_2^j, x_3^j); \ldots; (x_T^j, x_{T+1}^j)\} \tag{1}$$

where:

$x_t^j \in X$,
$j = 1,2,\ldots,m$,
$t = 1,2,\ldots,T$.

Here we propose a method for eliciting a "compromise" multistage decision variant chosen from the set of efficient variants. Moreover, in an environment with uncertainty, we take into consideration multistage decision variants which are also λ – feasible with a sufficiently high λ given in advance by decision maker. Our method is based on pair-wise comparisons of decision criteria and on the logarithmic least squares method for calculating the weights, i.e. the relative importance of the decision criteria. The result will make it possible to choose a single compromise variant corresponding to the decision maker's preferences in the decision process.

3. AHP AND PAIR-WISE COMPARISONS

In the Analytic Hierarchy Process (AHP) we consider a three-level hierarchical decision system: On the first level there is a decision goal G; on the second level, we have n independent decision criteria: C_1, C_2,...,C_n, such that $\sum_{i=1}^{n} w(C_i) = 1$, where $w(C_i) > 0$, $i = 1,2,\ldots,n$, $w(C_i)$ is a positive real number – weight, usually interpreted as a relative importance of the criterion C_i subject to the goal G. On the third level we have m variants of the decision outcomes. We take $V_1, V_2, \ldots, V_m$ such that, again, $\sum_{r=1}^{m} w(V_r, C_i) = 1$, where $w(V_r,C_i)$ is a nonnegative real number − an evaluation (weight) of V_r subject to the criterion C_i, $i = 1,2,\ldots,n$. This system is characterized by the *supermatrix* (see [7]):

$$\mathbf{W} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{W_{21}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{W_{32}} & \mathbf{I} \end{bmatrix}$$

a nonnegative matrix where $\mathbf{W_{21}}$ is the $n\times 1$ matrix (weighing vector of the criteria), i.e.:

$$\mathbf{W}_{21} = \begin{bmatrix} w(C_1) \\ \vdots \\ w(C_n) \end{bmatrix}$$

and $\mathbf{W_{32}}$ is the $m\times n$ matrix:

$$\mathbf{W}_{32} = \begin{bmatrix} w(C_1,V_1) & \cdots & w(C_n,V_1) \\ \vdots & \cdots & \vdots \\ w(C_1,V_m) & \cdots & w(C_n,V_m) \end{bmatrix}$$

The columns of this matrix represent evaluations of variants by the criteria. Moreover, $\mathbf{W}$ is a *column-stochastic matrix*, i.e. the sums of columns are equal to one. Then the limit matrix $\mathbf{W}^{\infty} = \lim_{k\to+\infty} \mathbf{W}^k$ (see [4]) exists and has the following form:

$$\mathbf{W}^{\infty} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{W_{32}W_{21}} & \mathbf{W_{32}} & \mathbf{I} \end{bmatrix} \tag{2}$$

Here $\mathbf{Z} = \mathbf{W_{32}W_{21}}$ is the $m\times 1$ matrix, i.e. the resulting *priority vector of weights of the variants*, $\mathbf{I}$ is the unit matrix. The variants can be ranked according to these priorities.

4. FUZZY NUMBERS AND FUZZY MATRICES

When applying the AHP in decision making, we usually encounter difficulties in evaluating pair-wise comparisons using the well known Saaty´s 5- (or 9-) point scale. In practice it is sometimes more convenient for the decision maker to express his/her evaluation in "words of natural language" by saying, e.g. "possibly 3", "approximately 4", or "about 5". Similarly, he/she could use evaluations of the form, e.g.: "A is possibly weakly preferable to B". Similarly, when evaluating individual decisions by some criterion, e.g. the level of inventory at a period t, we are often uncertain about values of the criterion. It is advantageous to express these evaluations by fuzzy sets of real numbers, in particular, by triangular fuzzy numbers (Figure 1). For the sake of convenience we now shortly refresh some basic concepts of the fuzzy set theory which we shall use in this paper.

A *fuzzy set* $\widetilde{A}$ *of* **R** is given by a *membership function* μ which is a mapping from the set of real numbers **R** into the unit interval [0,1], i.e. $\mu : \mathbf{R} \to [0,1]$. The *membership grade*, $\mu(x)$ of the element $x \in \mathbf{R}$, denotes the *possibility of occurrence (or realization) of* x, or, in other words, how strongly the element x belongs to the fuzzy set $\widetilde{A}$. The higher the value, the stronger the membership to $\widetilde{A}$, and vice versa. Full membership is denoted by the membership grade 1, full nonmembership by the grade 0. The evaluation of the membership grades of a fuzzy set may cause serious problems in practice; here we cannot go deeper into details – all what we will say here is that the membership grades of fuzzy sets may be estimated e.g. by experts. In order to distinguish fuzzy and nonfuzzy sets we shall denote the fuzzy sets, fuzzy vectors, and fuzzy matrices by a tilde above the symbol. For more information about fuzzy sets and related topics, see e.g. [12].

Let $\alpha \in [0,1]$ and the set $[\widetilde{A}]_\alpha = \{x \in \mathbf{R}; \mu(x) \geq \alpha\}$ is called an *alpha cut of* $\widetilde{A}$ (or α-cut). A *triangular fuzzy number* $\widetilde{a}$ is a fuzzy set of **R** defined by a triple of real numbers, i.e. $\widetilde{a} = (\underline{a}; a; \overline{a})$, where $\underline{a}$ is the *lower number*, a is the *middle number* and $\overline{a}$ is the *upper number*, $\underline{a} \leq a \leq \overline{a}$. The membership function $\mu(x)$ is continuous in $[\underline{a}, \overline{a}]$, increasing in $[\underline{a}, a]$, decreasing in $[a, \overline{a}]$, $\mu(a) = 1$ and $\mu(x) = 0$ for $x \notin [\underline{a}, \overline{a}]$. In what follows we shall use the most practical form of the membership function: a piecewise linear one.

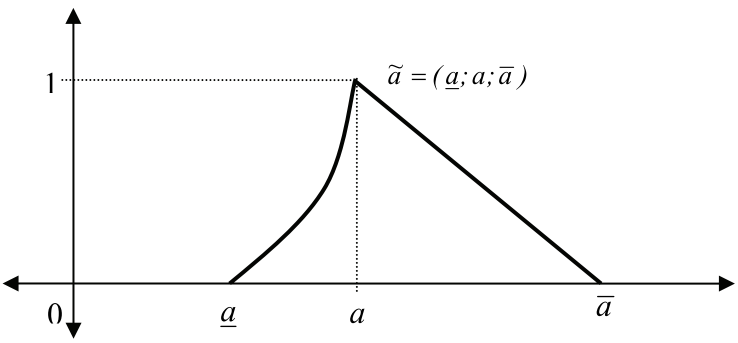

Fig. 1. A triangular fuzzy number

If $\underline{a} = a = \overline{a}$, then $\tilde{a}$ is said to be a *crisp number* (or a *nonfuzzy number*). Evidently, the set of all crisp numbers is isomorphic to the set of real numbers.

It is well known that the arithmetic operations +, –, *, and / on real numbers can be extended to fuzzy numbers by the Extension principle, see e.g. [12]. In the case of triangular fuzzy numbers $\tilde{a} = (\underline{a}; a; \overline{a})$ and $\tilde{b} = (\underline{b}; b; \overline{b})$, $\underline{a} > 0$, $\underline{b} > 0$, we obtain special formulae:

$$\tilde{a} \,\tilde{+}\, \tilde{b} = (\underline{a} + \underline{b}; a + b; \overline{a} + \overline{b})$$

$$\tilde{a} \,\tilde{-}\, \tilde{b} = (\underline{a} - \underline{b}; a - b; \overline{a} - \overline{b})$$

$$\tilde{a} \,\tilde{*}\, \tilde{b} = (\underline{a} * \underline{b}; a * b; \overline{a} * \overline{b})$$

$$\tilde{a} \,\tilde{/}\, \tilde{b} = (\underline{a} / \underline{b}; a / b; \overline{a} / \overline{b})$$

In operations of multiplication and division the form of the membership function of the result of the operation given by the Extension principle is nonlinear, even in the case when the operands are piecewise linear. In that case, piecewise linear membership functions in the above formulae give us good triangular approximations of the exact fuzzy numbers defined by the Extension principle. In what follows we shall use them for their simplicity and versatility.

If all elements in $m \times n$ matrix **A** are triangular fuzzy numbers, we call **A** the *triangular fuzzy matrix* and this matrix is composed of triples in the following way:

$$\widetilde{\mathbf{A}} = \begin{bmatrix} (\underline{a}_{11}; a_{11}; \overline{a}_{11}) & \cdots & (\underline{a}_{1n}; a_{1n}; \overline{a}_{1n}) \\ \vdots & \ddots & \vdots \\ (\underline{a}_{m1}; a_{m1}; \overline{a}_{m1}) & \cdots & (\underline{a}_{mn}; a_{mn}; \overline{a}_{mn}) \end{bmatrix}$$

Particularly, if $\widetilde{\mathbf{A}}$ is a triangular fuzzy matrix we say that it is *reciprocal*, if $\widetilde{a}_{ij} = (\underline{a}_{ij}; a_{ij}; \overline{a}_{ij})$ then $\widetilde{a}_{ji} = (\frac{1}{\overline{a}_{ij}}; \frac{1}{a_{ij}}; \frac{1}{\underline{a}_{ij}})$ for all $i,j = 1,2,...,n$.

Consequently, we have:

$$\widetilde{\mathbf{A}} = \begin{bmatrix} (1;1;1) & (\underline{a}_{12}; a_{12}; \overline{a}_{12}) & \cdots & (\underline{a}_{1n}; a_{1n}; \overline{a}_{1n}) \\ (\frac{1}{\overline{a}_{12}}; \frac{1}{a_{12}}; \frac{1}{\underline{a}_{12}}) & (1;1;1) & \cdots & (\underline{a}_{2n}; a_{2n}; \overline{a}_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\frac{1}{\overline{a}_{1n}}; \frac{1}{a_{1n}}; \frac{1}{\underline{a}_{1n}}) & (\frac{1}{\overline{a}_{2n}}; \frac{1}{a_{2n}}; \frac{1}{\underline{a}_{2n}}) & \cdots & (1;1;1) \end{bmatrix} \tag{3}$$

where $1 \le \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}$, $i,j = 1,2,...,n$. Without loss of generality we can assume that $1 \le a_{ij} \le a_{ik}$ whenever $i \le j \le k$.

5. RANKING FUZZY VARIANTS

Suppose we obtain m triangular fuzzy numbers, which we call here simply *fuzzy variants*:

$$\widetilde{z}_1 = (\underline{z}_1; z_1; \overline{z}_1),\ \widetilde{z}_2 = (\underline{z}_2; z_2; \overline{z}_2), \ldots, \widetilde{z}_m = (\underline{z}_m; z_m; \overline{z}_m) \tag{4}$$

Now, the problem is to rank them according to their "magnitude". The simplest way to do it is to rank fuzzy variants according to their middle value, which neglects the lower and upper parts of all fuzzy numbers. This is not a solution we ask.

A better way is the *center gravity method*. This method is based on computing the x-th coordinates x_i^g of the center of gravity of every "triangle" given by the corresponding membership functions $\tilde{z}_i$, $i = 1,2,...,m$. Evidently, the following holds:

$$x_i^g = \frac{\underline{z}_i + z_i + \overline{z}_i}{3} \tag{5}$$

By (5) the variants can be ordered from the best (with the largest value of (5)) to the worst (with the lowest value of (5)). Formula (5) incorporates in some sense the form of the triangular fuzzy number and that is why this method is more appropriate (Figure 2). Notice that in Figure 2, $a < b$, while $x_{\tilde{b}}^g < x_{\tilde{a}}^g$. More sophisticated methods for ranking fuzzy numbers exist, see e.g. [7]. For a comprehensive review of comparison methods see [3].

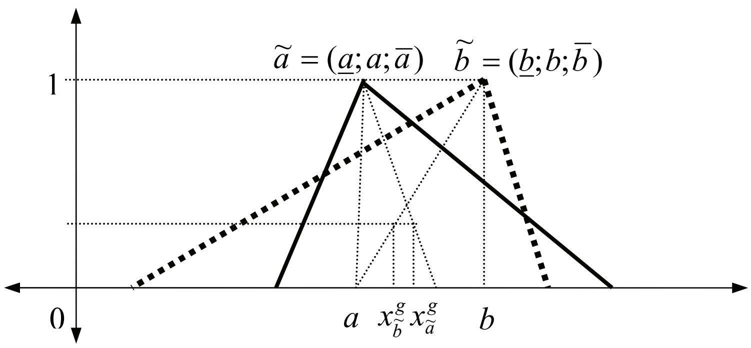

Fig. 2. Ranking fuzzy numbers

The advantage of the center gravity method is that the variants are ranked linearly (the ranking is complete) as each variant is represented by a real number. This representation may, however, become in some situations an unnecessary simplification of the situation. That is why other ranking methods have been proposed [2, 4]. Here we present five ranking methods based on the concept of dominance and aspiration level. The resulting order offered by the new rankings is, however, notcomplete, i.e. some variants remain noncomparable.

Definition 1

Let $\tilde{a}$ and $\tilde{b}$ be fuzzy numbers and $\delta,\gamma,\varepsilon \in [0,1]$ aspiration levels. We say that $\tilde{a}$ *is R-dominated by* $\tilde{b}$ (or $\tilde{b}$ *R-dominates* $\tilde{a}$) *on the aspiration level* δ if:

$$sup[\tilde{a}]_\alpha \le sup[\tilde{b}]_\alpha \text{, for all } \alpha \in [1-\delta,1]$$

We say that $\tilde{a}$ *is L-dominated by* $\tilde{b}$ (or $\tilde{b}$ *L-dominates* $\tilde{a}$) *on the aspiration level* γ if:

$$inf[\tilde{a}]_\alpha \le inf[\tilde{b}]_\alpha \text{, for all } \alpha \in [1\text{-}\gamma,1]$$

We say that $\tilde{a}$ *is LR-dominated by* $\tilde{b}$ (or $\tilde{b}$ *LR-dominates* $\tilde{a}$) *on the aspiration level* ε if $\tilde{a}$ is L-dominated and R-dominated by $\tilde{b}$ in the aspiration level ε. Here, L stands for “Left”, and R for “Right”.

Definition 2

Let $\tilde{a}$ and $\tilde{b}$ be fuzzy numbers and $\rho,\sigma \in [0,1]$ – aspiration levels. We say that $\tilde{a}$ *is P-dominated by* $\tilde{b}$ (or $\tilde{b}$ *P-dominates* $\tilde{a}$) *on the aspiration level* ρ if:

$$sup[\tilde{a}]_\alpha \le inf[\tilde{b}]_\alpha \text{, for all } \alpha \in [1\text{-}\rho,1]$$

We say that $\tilde{a}$ *is O-dominated by* $\tilde{b}$ (or $\tilde{b}$ *O-dominates* $\tilde{a}$) *on the aspiration level* σ if:

$$inf[\tilde{a}]_\alpha \le sup[\tilde{b}]_\alpha \text{, for all } \alpha \in [1\text{-}\sigma,1]$$

Here, P stands for “Pessimistic” and O – for “Optimistic”. We illustrate these concepts in the following example.

Example

In Figure 3, $\tilde{a}$ is LR-dominated by $\tilde{b}$ (or $\tilde{b}$ LR-dominates $\tilde{a}$) on the aspiration level β, $\beta \le \beta^*$. Moreover, for $\alpha \in [\alpha^*,1]$ $\tilde{a}$ is P-dominated by $\tilde{b}$ (or $\tilde{b}$ P-dominates $\tilde{a}$) on the aspiration level α, $\alpha \le \alpha^*$. At the same time, $\tilde{a}$ is O-dominated by $\tilde{b}$ (or $\tilde{b}$ O-dominates $\tilde{a}$) on the aspiration level $\delta \in [0,1]$.

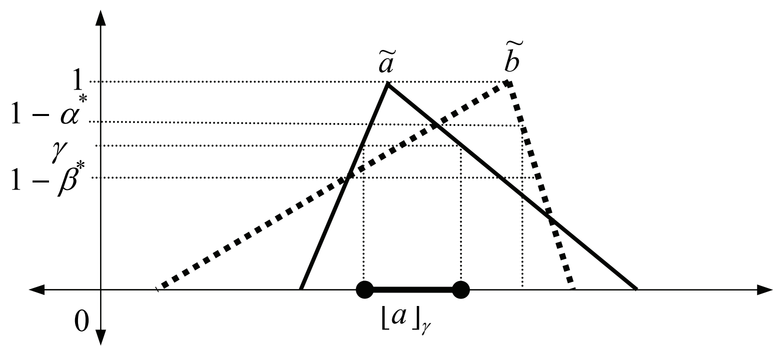

Fig. 3. Dominance of two fuzzy numbers

6. THE ALGORITHM

For each period $t = 1,2,...,T$ and decision $d = (x,y)$ we have n fuzzy values (triangular fuzzy numbers) $\tilde{c}_1^t(d), \tilde{c}_2^t(d),...,\tilde{c}_n^t(d)$ of decision criteria $C_1, C_2, ..., C_n$, respectively. The relative importance of the criteria is given by an $(n\times n)$ pair-wise comparison matrix $\widetilde{\mathbf{A}}^t$, a reciprocal fuzzy matrix whose elements are triangular fuzzy numbers $\tilde{a}_{ij}^t = (\underline{a}_{ij}^t; a_{ij}^t; \overline{a}_{ij}^t)$.

The proposed method for finding the "best" multistage decision variant (or for ranking all the variants) can be formulated in an algorithm in the following three steps:

1. Calculate the triangular fuzzy weights from the fuzzy pair-wise comparison matrices and from fuzzy triangular fuzzy numbers.
2. Calculate the aggregating triangular fuzzy evaluations of the multistage decision variants.
3. Find the "best" variant (or rank the variants) defined as triangular fuzzy numbers.

Below we explain in detail each step of this algorithm.

Step 1. Calculate the triangular fuzzy weights from the fuzzy pair-wise comparison matrix.

From now on we assume that the input data are uncertain and that they are given by triangular fuzzy numbers. Our purpose is to calculate the triangular fuzzy numbers – in this context we call them *fuzzy weights* – as evaluations of the relative importance of the criteria at each period.

Let a fuzzy pair-wise comparison matrix $\widetilde{\mathbf{A}}$ defined by (3) be given. We assume that there exists a fuzzy vectors of triangular fuzzy weights $\widetilde{w}_1, \widetilde{w}_2, ..., \widetilde{w}_n$, $\widetilde{w}_i = (\underline{w}_i\,; w_i\,; \overline{w}_i\,)$, $i = 1,2,...,n$ such that the pair-wise comparison matrix (3) is an estimation of the fuzzy matrix:

$$\widetilde{\mathbf{W}} = \begin{bmatrix} \frac{\widetilde{w}_1}{\widetilde{w}_1} & \frac{\widetilde{w}_1}{\widetilde{w}_2} & \cdots & \frac{\widetilde{w}_1}{\widetilde{w}_{nr}} \\ \frac{\widetilde{w}_2}{\widetilde{w}_1} & \frac{\widetilde{w}_2}{\widetilde{w}_2} & \cdots & \frac{\widetilde{w}_2}{\widetilde{w}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\widetilde{w}_n}{\widetilde{w}_1} & \frac{\widetilde{w}_n}{\widetilde{w}_2} & \cdots & \frac{\widetilde{w}_n}{\widetilde{w}_n} \end{bmatrix}$$

We shall find the fuzzy weights $\widetilde{w}_1, \widetilde{w}_2, ..., \widetilde{w}_n$ by minimizing the fuzzy functional:

$$\widetilde{H} = \sum_{i,j} \left(\widetilde{log} \frac{\widetilde{w}_i}{\widetilde{w}_j} - \widetilde{log}\, \widetilde{a}_{ij} \right)^2 \tag{6}$$

In (6), minimization of $\widetilde{H}$ is understood in the sense of solving the optimization problem:

$$\sum_{i,j} \left\{ \left(\log \frac{\underline{w}_i}{w_j} - \log \underline{a}_{ij} \right)^2 + \left(\log \frac{w_i}{w_j} - \log a_{ij} \right)^2 + \left(\log \frac{\overline{w}_i}{w_j} - \log \overline{a}_{ij} \right)^2 \right\} \longrightarrow \min \tag{7}$$

subject to:

$$\overline{w}_i \geq w_i \geq \underline{w}_i \geq 0\,, \; i = 1,2,...,n \tag{8}$$

It can be proven by standard methods of calculus that at each period $t = 1,2,...,T$ there exists a unique explicit solution of problem (7), (8) as follows:

$$\tilde{w}_k^t = (\underline{w}_k^t ; w_k^t ; \overline{w}_k^t) \text{ , } k = 1,2,...,n$$

where:

$$\underline{w}_k^t = \frac{\left(\prod_{j=1}^{n} \underline{a}_{kj}^t\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}^t\right)^{1/n}} \text{ , } w_k^t = \frac{\left(\prod_{j=1}^{n} a_{kj}^t\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}^t\right)^{1/n}} \text{ , } \overline{w}_k^t = \frac{\left(\prod_{j=1}^{n} \overline{a}_{kj}^t\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}^t\right)^{1/n}} \tag{9}$$

In [3], the method of calculating triangular fuzzy weights by (9) from the triangular fuzzy pair-wise comparison matrix (3) is called the *logarithmic least squares method*. This method can be applied both for calculating the triangular fuzzy weights of the criteria and for eliciting relative triangular fuzzy values of the criteria for the individual variants. Moreover, it can also be used for calculating feedback impacts of criteria on other criteria.

Step 2. Calculate the aggregating triangular fuzzy evaluations of the multistage variants.

Having calculated triangular fuzzy weights of the criteria for each period $t = 1,2,...,T$, we will calculate, for a decision $d = (x,y)$, its aggregation of evaluations as the weighting sum:

$$\tilde{S}^t(d) = \tilde{w}_1^t \tilde{\cdot} \tilde{c}_1^t(d) \tilde{+} \tilde{w}_2^t \tilde{\cdot} \tilde{c}_2^t(d) \tilde{+} \cdots \tilde{+} \tilde{w}_n^t \tilde{\cdot} \tilde{c}_n^t(d) \tag{10}$$

Here $\tilde{c}_i^t(d) = \left(\underline{c}_i^t(d); c_i^t(d); \overline{c}_i^t(d)\right)$ are fuzzy evaluations of the decisions. We also assume that the *normalization property* is satisfied, namely:

$$\sum_{i=1}^{n} c_i^t(d) = 1 \tag{11}$$

Otherwise, we normalize $\tilde{c}_i^t(d)$ by dividing its three components by $S = \sum_{j=1}^{n} c_j^t(d)$.

Now, let $V = \{d_1; d_2; ...; d_T\}$ be a λ – feasible multistage decision variant, where $d_1 = (x_1,x_2)$, $d_2 = (x_2,x_3)$, ..., $d_T = (x_T,x_{T+1})$, and $\lambda \in [0,1]$ is a given feasibility level.

We define a fuzzy evaluation $\widetilde{Z}(V)$ of the multistage decision variant V as:

$$\widetilde{Z}(V) = \mu^1(d_1)\widetilde{S}^1(d_1) \widetilde{+} \mu^2(d_2)\widetilde{S}^2(d_2) \widetilde{+} \cdots \widetilde{+} \mu^T(d_T)\widetilde{S}^T(d_T) \tag{12}$$

where $\mu^t(d_t)$ is the feasibility of the decision d_t. Notice that $\widetilde{Z}(V)$ given by (10), (11), and (12) is the resulting fuzzy evaluation of the multistage variant. Here, for addition, subtraction and multiplication of triangular fuzzy numbers we use the fuzzy operations defined earlier. That is why $\widetilde{Z}(V)$ is also a triangular fuzzy number, i.e. $\widetilde{Z}(V) = (\underline{z}; z; \overline{z})$. A group of multistage variants can be ranked according to these evaluations.

Step 3. Find the "best" variant, rank the variants.

In Step 2 we have found the fuzzy evaluations of the λ – feasible multistage variants described as triangular fuzzy numbers, i.e. by (11) we calculated the triangular fuzzy vector:

$$\widetilde{Z} = \left(\widetilde{Z}(V_1),...,\widetilde{Z}(V_m)\right) = \left((\underline{z}_1; z_1; \overline{z}_1),...,(\underline{z}_m; z_m; \overline{z}_m)\right)$$

The simplest method for ordering a set of triangular fuzzy numbers is the *center of gravity method*. This method is based on computing the x-th coordinates x_i^g of the center of gravity of every triangle given by the corresponding membership functions $\widetilde{z}_i$, $i = 1,2,...,m$. Evidently, it holds:

$$x_i^g = \frac{\underline{z}_i + z_i + \overline{z}_i}{3} \tag{13}$$

By (13) the variants can be ordered from the best (with the biggest value of (12)) to the worst (with the lowest value of (13)). Naturally, we can use more sophisticated methods for ranking fuzzy numbers [3].

7. EXAMPLE

In this section we analyze an example of decision making situation based on the example from [11]. The discrete process has 2 states, $X = \{x_1, x_2\}$, 3 periods, $t = 1,2,3$, and 2 fuzzy decision criteria C_1 and C_2 defined by the fuzzy functions $\widetilde{c}_1$ and $\widetilde{c}_2$, respectively. The evaluations of individual criteria by triangular fuzzy numbers and fuzzy state transformations are given in the following tables.

Table 1

Fuzzy data

$d = (x_i, x_j)$	$t = 1$	$t = 2$	$t = 3$
	$\mu^t(d_j)$ $\tilde{c}_1^t(d_j)$ $\tilde{c}_2^t(d_j)$	$\mu^t(d_j)$ $\tilde{c}_1^t(d_j)$ $\tilde{c}_2^t(d_j)$	$\mu^t(d_j)$ $\tilde{c}_1^t(d_j)$ $\tilde{c}_2^t(d_j)$
$d_1 = (x_1, x_1)$	0,8 (1;4;5) (1;2;3)	0,9 (4;5;8) (3;4;6)	0,8 (3;5;8) (2;3;4)
$d_2 = (x_1, x_2)$	0,8 (4;6;8) (2;2;2)	0,9 (2;4;5) (1;2;3)	0,8 (3;4;7) (6;7;8)
$d_3 = (x_2, x_1)$	0,9 (1;3;4) (5;7;8)	0,8 (3;6;8) (2;2;2)	0,9 (2;4;6) (1;2;3)
$d_4 = (x_2, x_2)$	1,0 (0;2;4) (3;5;6)	0,7 (2;5;9) (3;7;8)	0,7 (3;6;8) (6;7;8)

Here $\mu^1(d_1) = 0{,}8$, $\tilde{c}_1^1(d_1) = (1;4;5)$, $\tilde{c}_2^1(d_1) = (1;2;3)$, etc.

After normalization (11) we obtain the following table.

Table 2

Normalized fuzzy data

$d = (x_i, x_j)$	$t = 1$	$t = 2$	$t = 3$
	$\tilde{c}_1^t(d_j)$ $\tilde{c}_2^t(d_j)$	$\tilde{c}_1^t(d_j)$ $\tilde{c}_2^t(d_j)$	$\tilde{c}_1^t(d_j)$ $\tilde{c}_2^t(d_j)$
$d_1 = (x_1, x_1)$	(0,17; 0,67; 0,83) (0,17; 0,33; 0,50)	(0,44; 0,56; 0,89) (0,33; 0,44; 0,89)	(0,38; 0,63; 0,75) (0,25; 0,38; 0,50)
$d_2 = (x_1, x_2)$	(0,50; 0,75; 1,00) (0,25; 0,25; 0,25)	(0,33; 0,67; 0,83) (0,17; 0,33; 0,50)	(0,27; 0,36; 0,64) (0,55; 0,64; 0,73)
$d_3 = (x_2, x_1)$	(0,10; 0,30; 0,40) (0,50; 0,70; 0,80)	(0,38; 0,75;1,00) (0,25; 0,25; 0,25)	(0,33; 0,67; 1,00) (0,17; 0,33; 0,50)
$d_4 = (x_2, x_2)$	(0,00; 0,29; 0,57) (0,43; 0,71; 0,86)	(0,17; 0,42; 0,75) (0,25; 0,58; 0,67)	(0,23; 0,46; 0,62) (0,46; 0,54; 0,62)

The original multistage decision system is graphically depicted in Figure 4. The goal of this decision situation is to find the “best” multistage decision variant(s) from three preselected ones according to two criteria evaluated by triangular fuzzy numbers.

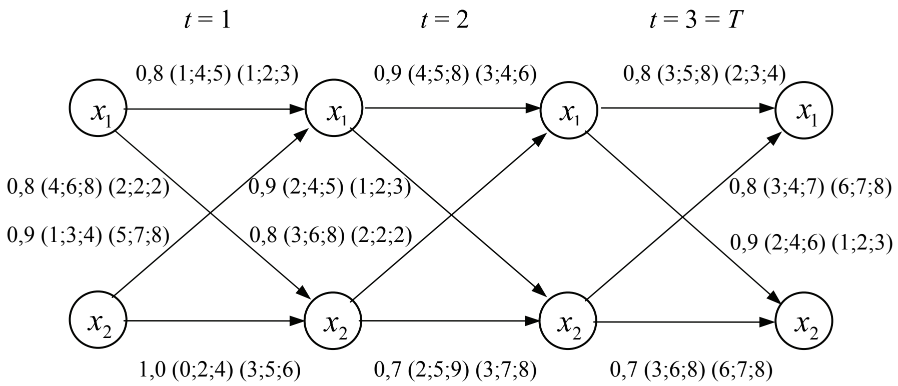

Fig. 4. The graph of the process

The weights of the criteria in the individual periods are given by the following 3 pair-wise comparison matrices:

$$A^1 = \begin{bmatrix} (1;1;1) & (2;3;4) \\ \left(\frac{1}{4};\frac{1}{3};\frac{1}{2}\right) & (1;1;1) \end{bmatrix},\ A^2 = \begin{bmatrix} (1;1;1) & (3;4;5) \\ \left(\frac{1}{5};\frac{1}{4};\frac{1}{3}\right) & (1;1;1) \end{bmatrix},\ A^3 = \begin{bmatrix} (1;1;1) & (4;5;6) \\ \left(\frac{1}{6};\frac{1}{5};\frac{1}{4}\right) & (1;1;1) \end{bmatrix}$$

By (9) we calculate the fuzzy weights of priorities of the criteria in the periods:

$$\widetilde{w}_1^1 = (0{,}67;0{,}75;0{,}80),\ \widetilde{w}_2^1 = (0{,}20;0{,}25;0{,}33)$$

$$\widetilde{w}_1^2 = (0{,}75;0{,}80;0{,}83),\ \widetilde{w}_2^2 = (0{,}17;0{,}20;0{,}25)$$

$$\widetilde{w}_1^3 = (0{,}80;0{,}83;0{,}86),\ \widetilde{w}_2^3 = (0{,}14;0{,}17;0{,}20)$$

In the example the set of 10 efficient realizations (i.e. multistage decision variants) has been generated using the Bellman Principle of optimality [9]. The ordered structure is based on the relation of LR-domination on the aspiration level $\varepsilon = 1$:

$V_1 = \{(x_1,x_2); (x_2,x_1);(x_1,x_1)\}$, $V_2 = \{(x_1,x_2); (x_2,x_2); (x_2,x_2)\}$,
$V_3 = \{(x_2,x_1); (x_1,x_2); (x_2,x_1)\}$,

$V_4 = \{(x_1,x_1); (x_1,x_1);(x_1,x_1)\}$, $V_5 = \{(x_1,x_2); (x_2,x_1); (x_1,x_2)\}$,
$V_6 = \{(x1,x2); (x2,x2); (x2,x1)\}$,

$V_7 = \{(x_2,x_1); (x_1,x_1);(x_1,x_1)\}$, $V_8 = \{(x_2,x_1); (x_1,x_1); (x_1,x_2)\}$,
$V_9 = \{(x_2,x_1); (x_1,x_2); (x_2,x_2)\}$,

$V_{10} = \{(x_2,x_2); (x_2,x_2);(x_2,x_2)\}$

In Figure 4 it is clear that we could generate 16 different variants. We limit ourselves, however, to the efficient realizations.

Using Table 2 we calculate evaluations of the variants according to the criteria from the formula (10), in particular:

$$\widetilde{S}^t(d) = \widetilde{w}_1^t \,\widetilde{\cdot}\, \widetilde{c}_1^t(d) \,\widetilde{+}\, \widetilde{w}_2^t \,\widetilde{\cdot}\, \widetilde{c}_2^t(d) = \left(\underline{S}^t(d); S^t(d); \overline{S}^t(d)\right)$$

The results are given in the following table:

Table 3

Normalized fuzzy data

$d = (x_i, x_j)$	$t = 1$	$t = 2$	$t = 3$
	$\left(\underline{S}^1(d); S^1(d); \overline{S}^1(d)\right)$	$\left(\underline{S}^2(d); S^2(d); \overline{S}^2(d)\right)$	$\left(\underline{S}^3(d); S^3(d); \overline{S}^3(d)\right)$
$d_1 = (x_1, x_1)$	(0,17; 0,67; 0,83)	(0,44; 0,56; 0,89)	(0,38; 0,63; 0,75)
$d_2 = (x_1, x_2)$	(0,50; 0,75; 1,00)	(0,33; 0,67; 0,83)	(0,27; 0,36; 0,64)
$d_3 = (x_2, x_1)$	(0,10; 0,30; 0,40)	(0,38; 0,75;1,00)	(0,33; 0,67; 1,00)
$d_4 = (x_2, x_2)$	(0,00; 0,29; 0,57)	(0,17; 0,42; 0,75)	(0,23; 0,46; 0,62)

Finally, applying formula (12), in particular:

$$\widetilde{Z}(V) = \mu^1(d)\widetilde{S}^1(d) \,\widetilde{+}\, \mu^2(d)\widetilde{S}^2(d) \,\widetilde{+}\, \mu^3(d)\widetilde{S}^3(d)$$

where *d* is a decision of the corresponding period, we obtain the resulting fuzzy values of the efficient variants and their centers of gravity (Table 4).

Table 4

Ranking of efficient variants based on their centers of gravity

Variants	x_i^g	Ranking
$\widetilde{Z}(V_1) = (0{,}83; 1{,}49; 2{,}02)$	1,50	1
$\widetilde{Z}(V_2) = (0{,}60; 1{,}15; 1{,}72)$	1,15	9
$\widetilde{Z}(V_3) = (0{,}76; 1{,}39; 2{,}21)$	1,41	3
$\widetilde{Z}(V_4) = (0{,}73; 1{,}41; 2{,}25)$	1,47	2

Table 4

Variants	x_i^g	ranking
$\widetilde{Z}$ (V_5) = (0,80; 1,35; 1,98)	1,38	6
$\widetilde{Z}$ (V_6) = (0,68; 1,37; 2,12)	1,39	5
$\widetilde{Z}$ (V_7) = (0,77; 1,31; 2,11)	1,40	4
$\widetilde{Z}$ (V_8) = (0,74; 1,17; 1,90)	1,27	7
$\widetilde{Z}$ (V_9) = (0,58; 1,23; 1,72)	1,18	8
$\widetilde{Z}$ (V_{10}) = (0,38; 1,04; 1,75)	1,06	10

The fuzzy values of the arbitrary chosen three variants (*V*1, *V*2, *V*3) are depicted in Figure 5.

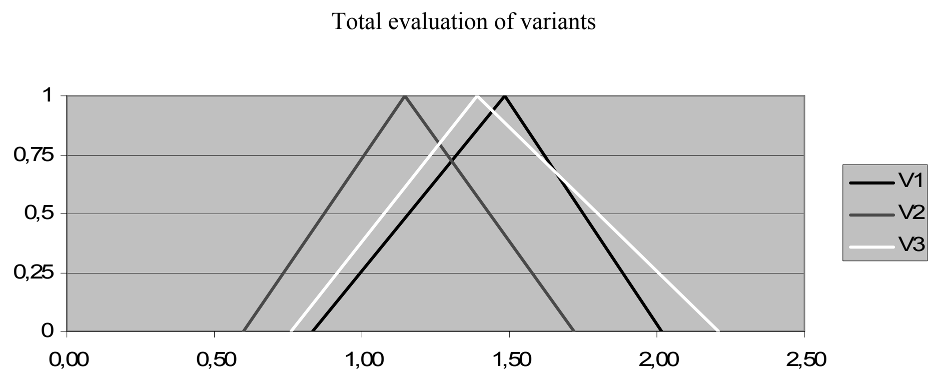

Fig. 5. Total evaluation of variants V1, V2, and V3.

In Figure 5, the rank of the variants is not clear, in particular, V_1 and V_3 are nearly "equally good", while V_2 is evidently the worst. We can confirm this observation using the center gravity ranking method. Applying the formula (13) we calculate x_1^g = 1,50, x_3^g = 1,41, x_2^g = 1,15 which confirms our first observation in Figure 5.

Consider now other rankings based on the concept of domination and on aspiration levels in Definitions 1 and 2. In Figure 5 it is evident that $\widetilde{Z}(V_3)$ is LR-dominated by $\widetilde{Z}(V_1)$ on the aspiration level $\beta = 0{,}25$. Moreover, $\widetilde{Z}(V_3)$ is P-dominated by $\widetilde{Z}(V_1)$ on the aspiration level $\alpha = 0{,}07$. At the same time, $\widetilde{Z}(V_3)$ is O-dominated by $\widetilde{Z}(V_1)$ on the aspiration level $\delta = 1{,}0$. With all rankings in mind, the multistage decision variant V_1 should be considered the best.

CONCLUSION

In this paper we proposed a new approach for solving the dynamic multiobjective decision making problems. The decision variants are generated in a discrete multistage model by forward/backward procedure of finding the set of all maximal elements based on Bellman's principle of optimality. As the set of all maximal elements consists of a number of elements – decision variants, our problem is to find among them a compromise variant based on decision maker´s preferences with respect to several decision criteria. The evaluation of the weights of the criteria is based on the data given by pair-wise comparison matrices using triangular fuzzy numbers. Extended arithmetic operations with fuzzy numbers for application of the generalized logarithmic least squares method are defined and six methods for ranking fuzzy numbers to compare fuzzy outcomes are proposed. A numerical example is presented to clarify the methodology.

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MULTIPLE CRITERIA VECTOR TESTING RESULTS EVALUATION MODEL

Abstract

In the paper one of the most important problems of modern testology which consists in quality definition of test task is discussed. Special attention is devoted to basic characteristic – the truth degree of test tasks answers. This characteristic is considered in pedagogical knowledge testing. A hierarchy of knowledge evaluation models is offered. The hierarchy allows to track features and conditions of the applications of various models. The examples of test tasks evaluation using one-parametrical knowledge evaluation model (binary, algebraic, or fuzzy) are presented. An idea and an example of using the fundamentally new vector (multiparametric) knowledge evaluation model are discussed. This model allows to evaluate the quality of answers to the test task with respect to several criteria simultaneously.

Keywords

Pedagogical testing, multiple criteria, knowledge evaluation model, knowledge control, truth degree, test tasks, automated knowledge testing.

INTRODUCTION

One of the most actual problems of modern pedagogy is the maintenance of quality of control materials (CM) [1]. The most important characteristic of CM that is considered in pedagogical knowledge testing is the truth degree of answers to test tasks.

The extension of the truth degree evaluation scale of test tasks answers allows to extend the analysis of a trainee's knowledge and to increase the reliability of pedagogical control. A hierarchy of knowledge evaluation models is presented in this article. A diagram of this is shown in Figure 1. The hierarchy allows to track features and conditions of the applications of various models and, based on that, to synthesize an essentially new vector knowledge evaluation model. Vector models allows to estimate the quality of answers to test tasks by several criteria simultaneously.

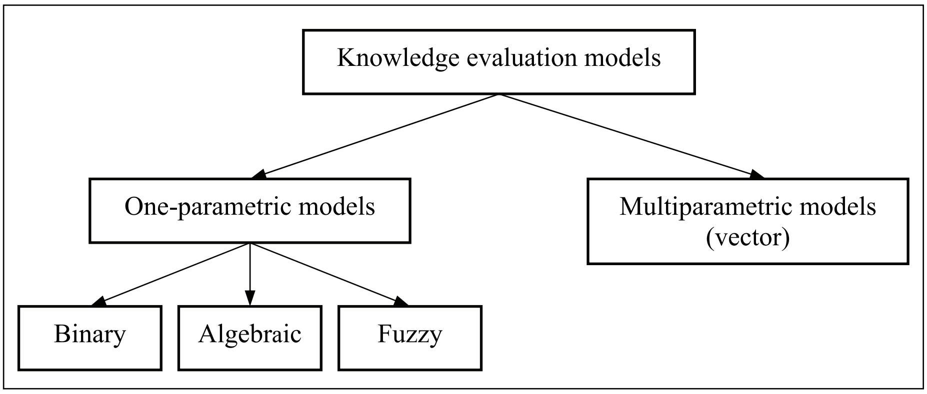

Fig. 1. Hierarchy of knowledge evaluation models

One of the most elementary one-parametrical model of truth degree evaluation (we shall name it ***binary***) is the popular approach to the truth degree evaluation of test tasks answers in terms of "true-false" [4]. Points 0 and 1 on a line (describing the truth degree of test task answers) correspond to this approach from the mathematical point of view (Figure 2).

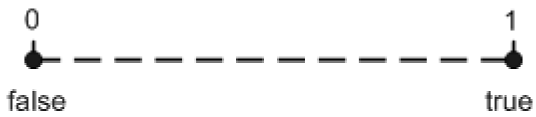

Fig. 2. Graphic representation of the truth degree for absolutely correct or absolutely wrong variant of the answer

The limitation of the acceptable truth degree of answer set to only two values allows to organize effective knowledge testing only in the case of knowledge revealing the concrete facts by the presentation to trainee of such types of questions, as: "What?", "Where?", "When?", "Who?", etc.

An example of answer evaluation for the elementary test task: "How much is 2×2 in decimal numeration?" using binary model is shown in Table 1.

Table 1

Variant of the answer	Truth degree
5	0
3	0
1	0
4	1

The second model with respect to its complexity, further named ***algebraic***, allows to evaluate the truth degree of variants of the test tasks answers by number whose values are taken from a range, for example, from the interval [0; 1] (Figure 3). The algebraic model gives an opportunity for a more flexible truth degree evaluation of variants of test tasks answers in comparison with the binary model, due to the wide use of incomplete, inaccurate, uncertain, etc. answers in pedagogical practice [4]. In the algebraic model the truth degree evaluation is represented by one of several points in the allowable values range. An example of the evaluation of the test task: "Choose the basic attributes of a parallelogram", using the algebraic model, is shown in Table 2.

Table 2

Variant of the answer	Truth degree
Opposite sides are equal	0,5
Opposite angles are equal	0,5
Opposite sides and angles are equal	1
Diagonals are equal	0

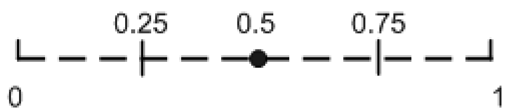

Fig. 3. Graphic representation of truth degree of the answer: "Opposite angles are equal"

One of the most complex one-parametrical models is the **fuzzy** model. The fuzzy model uses mathematical methods of fuzzy algebra for truth degree evaluation of answers to test tasks [2]. The quality of the answer is characterized by the membership function which is defined on the set of linguistic variable values “truth degree of answer” variants. The grade of membership of a variant of the answer to each category being estimated is defined by the number which values are taken from a range, for example, from the interval [0; 1]. In the fuzzy model the truth degree of an answer is characterized not by a unique point, but also by an aggregate of points which make up the membership function. The fuzzy model allows to estimate the truth degree of answers for test tasks in a more flexible manner as compared with the previous approaches to evaluation, in particular, in weakly-formalizable subjects (such as history, literature, etc.). An example of fuzzy evaluation of the truth degree of answers to the test task: “Where did Carlson live?” is shown in Table 3. In this example the symbol I designates a linguistic variable “the truth degree of the answer”, containing the values: I = [“true”, “not absolutely true”, „false”].

Table 3

Variant of the answer	Truth degree		
	true	not absolutely true	false
In the house	0,5	0,7	0,4
On the roof of a house	0,7	0,4	0,2
In a house in the attic	1	0	0
In a cellar	0	0	1

The membership function for the answer “On the roof of a house” is represented on Figure 4.

All models considered above allow to estimate the quality of CM by a unique parameter (in our case this parameter is the „truth degree of the answer to the test task”). From the position of mathematical analysis, such approach can be named the **scalar** approach.

However, it is clear that the quality of answers to test tasks can be characterized not only by the unique parameter „truth degree”, but can also be represented by the whole spectrum of parameters which can vary depending on the purpose of the pedagogical control, the character of discipline, the algorithm of testing, etc. An attempt to generalize scalar evaluation methods used in multivariate evaluation of qualities of answers to test tasks is discussed in this article. A formal model of evaluation of the qualities of answers to the test task by several criteria will be called the ***vector model***.

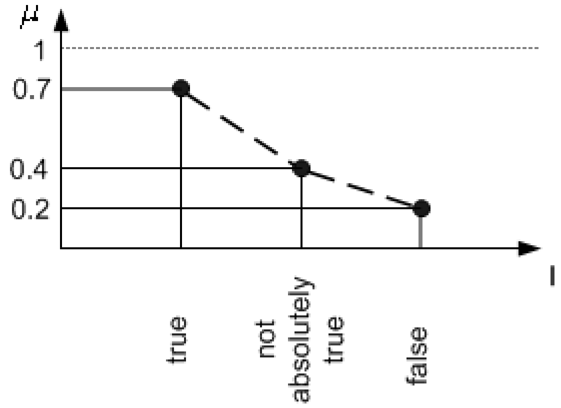

Fig. 4. Graphic representation of truth degree for the answer "On the roof of a house"

In the vector model the parameters playing the role of quality evaluation criteria are used as components of the vector basis which forms n-dimensional vector space (where *n* is the number of criteria used to evaluate the quality of answers to test tasks). Evaluation with respect to each criterion can be made using one of the abovelisted models (binary, algebraic, or fuzzy). The final estimation of the test task performed using the vector model will be determined by a point in n-dimensional space (for example, the point **A** in Figure 5 in the case of three-dimensional space of criteria). From the mathematical point of view, the final estimation can be exposed by each criterion separately or by the decision in a multiple criteria problem with various methods of deescalation, concession, method of the main criterion, method of an ideal point, etc. [3].

We shall consider the test task in the field "History of Russia", for example, "What role did the Communist Party of the Soviet Union play in the history of the development of Russia?" using the vector model. We shall assume that experts have estimated possible variants of answers, for example, by three criteria: truth (using binary model), completeness (using algebraic model), and originality (using fuzzy model, I = ["originally", "not very originally", "unoriginally"]). An example of the evaluation is shown in Table 4.

Table 4

Variant of the answer	Evaluation criteria				
	Truth	Completeness	Originality		
			Originally	Not very originally	Unoriginally
There are both positive and negative moments	1	0,5	0,5	0,3	0,9
Extremely positive role, democratic foundations put obstacles for the country	0	0,2	0,8	0,5	0,1
Extremely negative role, years of communism were the period of stagnation for the country	0	0,2	1	0	0
Domination of the Communist Party has not affected the development of Russia	0	0	0,3	0,1	0,5

A simplified illustration of the vector model by evaluation can be seen from Table 4 is shown in Figure 5. It is presented in a simplified manner as a coordinate on the axis "originality" where the fuzzy evaluation model is used and can be represented by a set of points which form a hyperplane. In general, a more complex case of three-dimensional space (which can be displayed as a hypercube) is considered.

As compared with the scalar quality evaluation models of answers to test tasks, the vector model presented above is more complex and full. It allows to expand the evaluation possibilities and to estimate the truth degree of test tasks more authentically. Thus, certainly, the labor input of expert evaluation of test tasks is increasing.

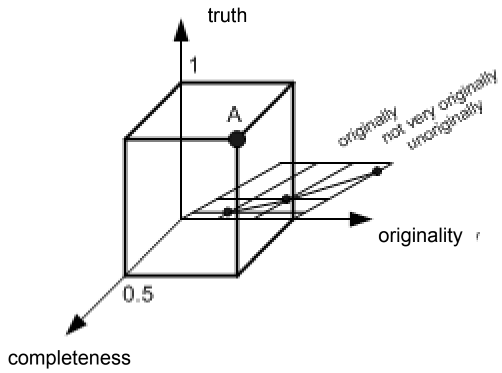

Fig. 5. A simplified illustration of the vector evaluation model

Nevertheless, the simplicity of test tasks duplication and their repeated use in knowledge control, and also the basic opportunity of application of a vector estimation of answers to test tasks in the scalar form (which can be demanded for knowledge evaluation with respect to one of the criteria) allows to consider this model to be a method for future use for realization in systems of automated knowledge testing.

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METRICS IN THE COMPROMISE HYPERSPHERE METHOD

Abstract

Compromise programming is one of the most often applied methods of multi-criteria optimization, both discrete and continuous. This paper deals with decision making in multicriteria linear programming problems. The approach presented here is based on finding a hypersphere (in the criteria space), which minimalizes the distance from the set of all nondominated extreme points. Next, we look for the nondominated extreme point closest to the hypersphere found previously. This point, called the best compromise nondominated solution, depends on the chosen metric. We consider the method of compromise hypersphere with different metrics and analyze their influence on the best compromise nondominated solution.

Keywords

Multicriteria linear programming, compromise programming.

INTRODUCTION

Compromise programming is one of the most often applied methods of multicriteria optimization, both discrete and continuous. Steuer and Choo [10] present an interactive weighted Tchebycheff procedure for multiple objective programming. A problem of weight choice in compromise programming is considered by Ballestro and Romero [3]. Other similar approaches are presented in the work of Carrizosa et al. [6] who consider the so-called AS norms in Ideal-Point methods. Ballestro [4] studies a problem of selection of a compromise programming metric and the risk aversion. For operative applications of compromise programming, the following works are worth mentioning: Opricovic and Tzeng [9], who discuss comparative analysis of compromise solution by the multicriteria decision making methods as well as Abdelaziz et al. [1], who discuss multiobjective programming technics with goal programming and compromise programming used to choose the portfolio which best satisfies the decision maker.

This paper deals with decision making in multicriteria linear programming problems. The concept of the method follows from the work of Gass and Roy [8]. The approach presented here is based on finding a hypersphere (in the criteria space) which minimizes the distance from the set of all nondominated extreme points. Next, we look for the nondominated extreme point which is closest to a hypersphere. This point, called the best compromise nondominated solution, depends on the chosen metric. We consider the method of compromise hypersphere with different metrics and analyze their influence on the best compromise nondominated solution.

The paper consists of three sections. Section 1 presents the general description of the compromise hypersphere for multiobjective linear programming. Section 2 describes methods of chosing distance functions. Section 3 contains an example. At the end, there are concluding remarks and further research.

1. DESCRIPTION OF THE COMPROMISE HYPERSPHERE METHOD

Let us consider the following multicriteria linear programme:

$$\text{VMax } \{\mathbf{Cx}: \mathbf{x} \in \text{X}\} \tag{1}$$

where:

$\text{X} = \{\mathbf{x} \in \mathcal{R}^N: \mathbf{Ax} \le \mathbf{b}, \mathbf{x} \ge 0\}$ or $\text{X} = \{\mathbf{x} \in \mathcal{R}^N: \mathbf{Ax} = \mathbf{b}, \mathbf{x} \ge \mathbf{0}\}$ – feasible region in decision space,

$\mathbf{x} \in \mathcal{R}^N$ – vector of decision variables,

$\mathbf{C} \in \mathcal{R}^{k \times N}$ – matrix of objective function coefficients,

$\mathbf{A} \in \mathcal{R}^{m \times N}$ – full row rank matrix of constraint coefficients,

$\mathbf{b} \in \mathcal{R}^m$ – right hand side vector.

We call $\text{y}^* \in \mathcal{R}^k$ a nondominated solution of (1) if:

$$\underset{\mathbf{x}^* \in \text{X}}{\exists} \mathbf{y}^* = \mathbf{Cx}^*$$

and

$$\sim \underset{\mathbf{x}' \in \text{X}}{\exists} \mathbf{Cx}^* \le \mathbf{Cx}' \wedge \mathbf{Cx}^* \ne \mathbf{Cx}'$$

The corresponding point $\mathbf{x}^* \in \mathcal{R}^n$ is called an efficient solution.

The aim of the method presented here is to rank the nondominated extreme points of the problem (1). The details of the method are presented below:

Step 1

Determination of the set of all nondominated extreme points (efficient solutions) of the problem (1). We will denote the nondominated extreme points as:

$$\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^n$$

Step 2

Solution of the programme:

$$\min_{\mathbf{y}^0, r_0} D\Big(r_0,\ \Big(d\big(\mathbf{y}^1, \mathbf{y}^0\big), \ldots, d\big(\mathbf{y}^n, \mathbf{y}^0\big)\Big)\Big) \tag{2}$$

where:

$\mathbf{y}^0 \in R^k$, $r_0 \in R$ denote the decision variables of (2),
d: $R^k \times R^k \rightarrow R$ denotes the distance between two vectors,
D: $R \times R^n \rightarrow R$ denotes the distance between one number and the set of n numbers (we will identify the set of n numbers as n dimensional vector).

We will denote the optimal solution of (2) as ${}^*y^0$, ${}^*r_0$ and the minimal value of the cost function as ${}^*\min(2)$.

Interpretation of problem (2). The problem is to find a hypersphere with the centre $\mathbf{y}^0 \in R^k$ and the radius $r_0 \in R$ such that its distance from the set $\{\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^n\}$ is minimal.

Step 3

Solution of the programme:

$$\min_{i=1,\ldots,k} \left| {}^*r_0 - d({}^*y^0, \mathbf{y}^i) \right| \tag{3}$$

We will denote the optimal solution of (3) as ${}^*i$, the optimal extreme point as ${}^*\mathbf{y}^i$, and the minimal value of the cost function as ${}^*\min(3)$.

Interpretation of problem (3). The problem is to find the extreme point which is closest to hypersphere found in Step 2.

Remark 1

We can find the set of all nondominated extreme points (efficient solutions) with the help of ADBASE [11].

Remark 2

The method of solving the programme (2) depends on the choice of D and d. In general, this programme is a complicated optimization problem (examples will be presented later).

Remark 3

Problem (3) is trivial, it suffices to compare n numbers which were used in Step 2.

2. MEASURE OF DISTANCE

As we have seen in Section 1, the compromise hypersphere method uses functions d and D as measures of distance. The following sections describe the methods of building these functions.

2.1 Choosing function *d*

We will use the well known family of metrics $l^p : R^k \times R^k \to R$ to measure the distance between two vectors:

$$l^p(\mathbf{y},\mathbf{z}) = \begin{cases} \left(\sum_{i=1}^{k} |y_i - z_i|^p \right)^{\frac{1}{p}}, & p \in [1,\infty) \\ \max\limits_{i=1,\dots,k} |y_i - z_i|, & p = \infty \end{cases}$$

where:

$\mathbf{y} = (y_1, \dots, y_k) \in R^k$, $\mathbf{z} = (z_1, \dots, z_k) \in R^k$.

Therefore, in the problem (2) we will use the function l^p as function d with parameter $p \in [1,\infty]$.

2.2. Choosing function *D*

We will use a modification of metrics l^q to measure the distance between a number and the set of n numbers (identified as an n dimensional vector). The modification function $l^q : R \times R^k \to R$ is defined as follows:

$$L^q(r,\mathbf{w}) = l^q((r,r,...,r),\mathbf{w})$$

where:

$r \in R$, $(r,r, \ldots, r) \in R^k$, $\mathbf{w} \in R^k$ and l^q is defined in Subsection 2.1.

Therefore, in the problem (2) we will use the function l^q as function D with parameter $q \in [1,\infty]$.

2.3. Problem $H(p, q)$

By using the function l^p as d and l^q as D in the problem (2) we obtain the following problem $H(p,q)$:

$$\min_{y^0, r_0} l^q(r_0,(l^p(\mathbf{y}^1,\mathbf{y}^0),...,l^p(\mathbf{y}^n,\mathbf{y}^0))) \qquad H(p,q)$$

The problem $H(2, \infty)$ is considered in the papers by Anthony et al. [2] and by Butler et al. [5]. However, Gass and Roy [8] consider an approximation of the problem $H(2, \infty)$ and it's quality can be found in Gass et al. [7].

3. EXAMPLE

Consider the following two-criteria problem (Figure 1):

$$\text{VMax } [x_1, x_2]$$

$$\begin{aligned} 3x_1 + 2x_2 &\leq 51 \\ x_1 + 2x_2 &\leq 21 \\ x_1 + 3x_2 &\leq 25 \\ x_1 + 4x_2 &\leq 30 \\ x_1 + 6x_2 &\leq 42 \\ x_1, x_2 &\geq 0 \end{aligned}$$

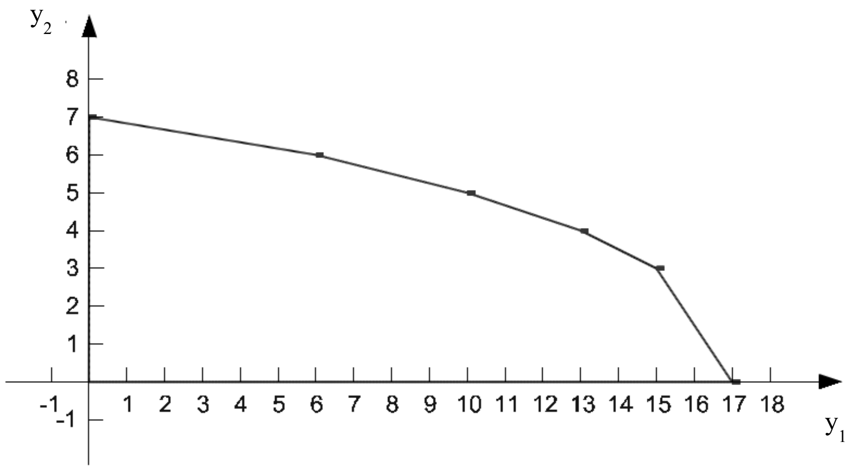

Fig. 1. Illustration of the example

We apply the hypersphere compromise method described in Section 1.

Step 1

We have the following six nondominated efficient extreme points:

$\mathbf{y}^1 = (0, 7)$, $\mathbf{y}^2 = (6, 6)$, $\mathbf{y}^3 = (10, 5)$, $\mathbf{y}^4 = (13, 4)$, $\mathbf{y}^5 = (15, 3)$, $\mathbf{y}^6 = (17, 0)$

Steps 2 and 3

In Step 2 we consider the problems $H(p, q)$ for $p = 1, 2, \infty$ and $q = 1, 2, \infty$. To solve these problems we use metaheuristic methods (genetic algorithms). The aim of the paper is not to analyze the numerical aspects of the optimization problems presented here, but to analyze the solutions obtained. Thus, we will not discuss numerical methods in details. The numerical analysis of the methods used will be the topic of future research.

In Table 1 there are optimal solutions of the problems (2) and (3); the optimal value of cost function of (2) is also presented. The symbols in the cells of Table 1 denote:

${}^*y^0$, ${}^*r_0$ – the centre and the radius of minimal hypersphere,

${}^*y^i$ – the optimal extreme point in problem 3,

${}^*\min(2)$ – the minimal value of the cost function in problem 2.

Table 1

Optimal solutions of the problems (2) and (3) and the optimal value of cost function of the problem (2)

$d = l^p$				
$p = 1$	$p = 2$	$p = \infty$		
(5, 0), 12 $\mathbf{y}^1, \mathbf{y}^4, \mathbf{y}^6$ 8	(0.8683, −15.0536), 22.0703 $\mathbf{y}^6$ 1.8139	(5.2700, −4.6828), 10.3157 $\mathbf{y}^2$ 6	q=1	$D=l^q$
(4.4, −0.3), 11.7 $\mathbf{y}^1$ 4.7749	(1.6262, −15.0776), 21.9415 $\mathbf{y}^3$ 0.8844	(7.2222, -3.2778), 8.7778 $\mathbf{y}^2, \mathbf{y}^3$ 2.6457	q=2	
(4.3333, −0.6667), 11.3333 $\mathbf{y}^3$ 3	(1.5906, −15.1667), 22.1007 $\mathbf{y}^1$ 0.4792	(7, −1), 8 $\mathbf{y}^5$ 2	q=∞	

Case analysis of $p = 2$ and $q = \infty$

We discuss the case of $p = 2$ and $q = \infty$ in detail. An example of minimal hypersphere with the centre ${}^*y^0 = (1.5906, -15.1667)$ and the radius ${}^*r_0 = 22.1007$ is presented in Figure 2.

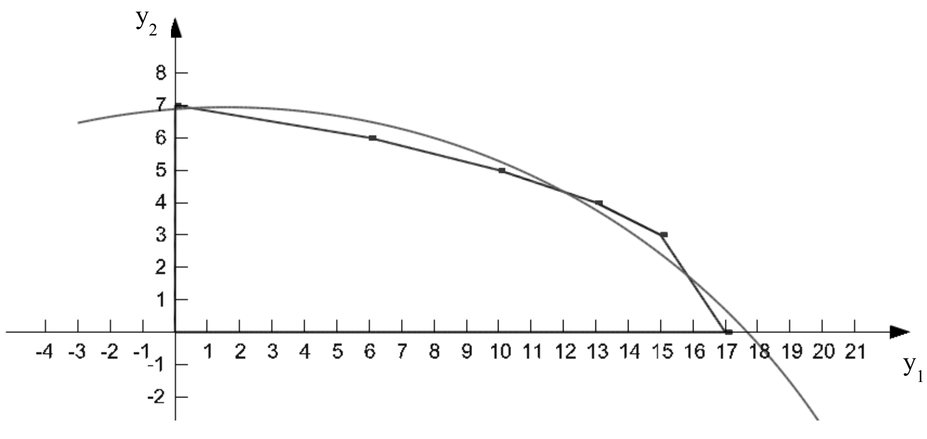

Fig. 2. Minimal hypersphere in the case of the problem $H(2, \infty)$

Table 2 presents distances between nondominated efficient extreme points and minimal hypersphere and a ranking of these points. In Table 2 an optimal solution of the problem (3) is also presented. The solution $\mathbf{y}^1$ is connected with the minimal distance which is equal to 0.1234.

Table 2

Distances between nondominated efficient extreme points and minimal hypersphere for $p = 2$ and $q = \infty$

	$\left|{}^*r_0 - d({}^*y^0, \mathbf{y}^i)\right|$	Ranking
$\mathbf{y}^1$	0.1234	1
$\mathbf{y}^2$	0.4792	4
$\mathbf{y}^3$	0.2506	3
$\mathbf{y}^4$	0.2051	2
$\mathbf{y}^5$	0.4792	4
$\mathbf{y}^6$	0.4792	4

Moreover, in the case analyzed here, the formula for the function D has the following form:

$$l^q\left({}^*r_0, \mathbf{w}\right) = l^\infty\left(\left({}^*r_0, {}^*r_0, \ldots, {}^*r_0\right), \mathbf{w}\right) = \max_{i=1,\ldots,6}\left|{}^*r_0 - w_i\right|$$

where:

$$\mathbf{w} = (d({}^*y^0, \mathbf{y}^1) d({}^*y^0, \mathbf{y}^2) d({}^*y^0, \mathbf{y}^3) d({}^*y^0, \mathbf{y}^4) d({}^*y^0, \mathbf{y}^5) d({}^*y^0, \mathbf{y}^6)) \in R^6$$

and

$$({}^*r_0, {}^*r_0, \ldots, {}^*r_0) = (22.1007, 22.1007, 22.1007, 22.1007, 22.1007, 22.1007) \in R^6$$

Using values presented in Table 2 we obtain:

$$l^q\left({}^*r_0, \mathbf{w}\right) = \max\{0.1234, 0.4792, 0.2506, 0.2051, 0.4792, 0.4792\} = 0.4792$$

Therefore, only vectors lying farthest from minimal hypersphere influence the value of cost function D. There are three $\mathbf{y}^2$, $\mathbf{y}^5$, $\mathbf{y}^6$ vectors with the maximum distance 0.4792 (see Table 2).

CONCLUDING REMARKS AND FURTHER RESEARCH

We have presented a method of decision supporting in problems of multicriteria linear programming. The method is based on finding a hypersphere which is closest to the set of the efficient extreme points. The method presented by Gass and Roy [8] has been developed using different methods of measuring the distance. We have presented an example with nine possible variants of hypersphere compromise programming. In the example we considered six nondominated extreme points. As we have shown (Table 1), each of the nondominated extreme points (depending on the assumed variant of measuring) turned out to be a optimal solution of the programmes $H(p, q)$. An extension of the presented method could be constructed by means of the augmented Tchebycheff metric [10].

The author suggests the following problems as the subject of further research:
- constructing a method of choosing functions d and D using interaction with decision maker,
- finding mathematical properties of the presented problem $H(p, q)$,
- numerical analysis of algorithms searching for optimal solutions of the problem $H(p, q)$.

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Tomasz Wachowicz

NEGOTIATION AND ARBITRATION SUPPORT WITH ANALYTIC HIERARCHICAL PROCESS

Abstract

While preparing to negotiations, the negotiating parties usually concentrate on the behavioral aspects of them. They consider various negotiations strategies, tactics, actions, and responses believed to be the key factors that allow them to achieve their goals. They pay not enough attention to the adequate definition of their objectives and to the consideration of how to balance shortages in one objective with excesses of the others when it is impossible to achieve the aspiration or reservation values on the individual criteria.

In the paper we consider PrOACT approach presented by Hammond, Luce, and Raiffa to structure the negotiation goals and to score and analyze the negotiation template. We also try to incorporate the AHP procedures in the process of offer evaluation that allow us to avoid simple assigning of the scores to the issues and resolutions. We believe this is important especially for the decision makers who are not skilled in the formal analysis and perceive the assigning as unclear and complicated. After the evaluation of the offers we focus on the search of the fair compromise by means of a well known game theory approach. Finally, we return to AHP which allows us to find a fair compromise in the situation where the negotiation strengths of the parties are not equal.

Keywords

Negotiation, negotiation analysis, multiple attribute decision analysis, AHP, game theory.

INTRODUCTION

Negotiations are usually perceived as conflicting processes, the solving or wining of which requires some intrinsic interpersonal skills of behavioral nature and well-trained abilities of using negotiation strategies and tactics. The vast majority of the literature on negotiation is devoted to the problem of how to act and behave during the negotiation process to achieve a satisfying agreement; it also gives some descriptive advice to the negotiators [1, 10, 9].

But since the 1980s a parallel approach to the negotiation problems has been developed, called negotiation analysis[1]. It derives from decision analysis, game theory, multiple objective programming, and other mathematical procedures and aims at giving the negotiating subjects an advice of prescriptive or normative nature [17, 12]. Nowadays these two approaches combined together allow for successful negotiation support [8]. The negotiation support, being mostly based on formal analysis, can be easily conducted in a semi-automatic way by means of the negotiation support systems, that is, expert software with implemented formal procedures and algorithms which give to the negotiators support in evaluating and comparing offers and concessions, making proposals, and conducting pre- and postnegotiation analysis. There are some successful applications of the NSS into solving real-world negotiation problems such as negotiating the reduction of the pollution emission to the atmosphere in Europe with RAINS system [5] or the problem of the Law of the Sea [15]. Presently, while e-business expands, the NSS are implemented as e-Negotiation Systems that allow negotiating via Internet, beyond the bounds of time and space, by people from very different parts of the World at a time which is the most convenient for each of them. They give not only negotiation support to the parties, but they also facilitate the communication between them and conduct the arbitration and mediation analysis [6]. They became very sophisticated tools which use may cause resistance or concern, especially to the negotiators unfamiliar with newest computer technology and formal analysis. Therefore, it is very important to equip NSS and eNS with formal models that are, on the one hand, efficient and, on the other hand, can be intuitively operated by decision makers (negotiator, arbitrator, mediator, facilitator). The problem occurring for negotiation analysts or NSS designers is: What combination of formal methods applied satisfies these two criteria simultaneously.

In the paper we will try to apply the simplest possible mathematical tools to the computer-based negotiation and arbitration support. All the procedures that require more advanced calculations and analysis will be programmed in a spreadsheet, to show that they can be easily copied into more sophisticated software. Finally, we will show an example of the use of the proposed methodology and software in the solution of a hypothetical negotiation case.

[1] The first works on formal modeling of negotiation by means of mathematical tools have been undertaken much earlier [14], but the discipline of negotiation analysis began to shape after 1982 [11].

1. NEGOTIATION STRUCTURE

We will consider two-party multi-issue integrative negotiations[2] conducted according to the negotiation template agreed upon by both parties in the prenegotiation phase. The negotiation template is a list of all issues that are to be negotiated with predefined full range of possible resolutions (options). The template is going to be analyzed and evaluated by the parties separately in order to build their own scoring systems that reflect their individual preferences. It will allow to compare the sequence of offers, analyze the concessions, and measure the quality of the agreement under discussion. There is no need to assume that the negotiation template remains stable during the entire negotiation process, but any changes made to it, such as adding issues or modifying options, will require additional calculations and rescoring of the template. The negotiation issues reflect the parties' objectives and can be both of qualitative and quantitative nature. Since we are going to introduce a separate scoring method for options and issues comparison, it does not matter whether the options are given as nominal, ordinal, interval, or ratio.

To construct a solid scoring system we will require the negotiators to follow the PrOACT [4, 12] approach for decision making. This approach consists of five elements that are the basis of successful decision making. They are: Problem, Objectives, Alternatives, Consequences, and Trade-offs. Each of these elements requires a thoughtful analysis and together they will result in proper definition and structuralization of the negotiation problem and in realization of the relations between the issues and options, and their importance.

1. **Problem.** The first element of the PrOACT approach requires the analysis of the actual negotiation problem. The work need to be undertaken by both parties together in the pre-negotiation phase. They need to recognize the background of the conflict and all its aspects to be solved to see what must be decided, which will lead to the definition of the negotiation subject.
2. **Objectives**[3] are the criteria used to evaluate the offer which reflect the negotiator's needs, hopes, and wishes. To assess the objective true necessity the negotiator should consider why it is important to her/him and what she/he means by it. This will lead to clarification of the negotiation issues. In the negotiations, objectives are defined separately by the parties and are included in the mutual negotiation template.

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[2] The subject of the negotiation does not matter in fact.
[3] In the negotiation theory "Objectives" are sometimes called "Interests".

3. **Alternatives** are the actions that can be taken to satisfy negotiators' needs. Thinking of the alternatives will result in identifying those options that constitute the range of all possible resolutions for each negotiation issue. The process of the generation of alternatives, similar to the generation of objectives, should be conducted jointly by the parties in the pre--negotiation discussion.
4. **Consequences** of all the alternatives approved in pre-negotiation phase should be recognized at that time. The evaluation of the subjective quality and value of each alternative for the negotiators is required. Many different types of analysis can be provided here such as conditional analysis, utility analysis, ordinal ranking, etc.
5. **Trade-offs.** Since at least some of the objectives are conflicting, it is usually not possible to end with the compromise that is overall the best for both parties. In such a situation it is necessary for the parties to sacrifice some of the objectives in favor of others. Each negotiator should realize the balance of options between the issues. This requires evaluating the importance of the objectives and then the importance of the options within the objectives.

Following the PrOACT approach will result in preparing a solid common negotiation template and the negotiators' individual scoring systems that are sufficient for negotiation support.

Finally, we assume the existence of a third party – an arbitrator or mediator (or an NSS/eNS playing this role) – who will facilitate the negotiation process by helping with scoring end evaluating the offers and suggesting a "fair" agreement. The third party has access to all the data describing the negotiators' structure of preferences required to find a compromise satisfying both of them.

Having described the negotiation situation, in the next section we present a set of analytic tools whose software implementation will allow for negotiation and arbitration support.

2. ANALYTIC TOOLS FOR NEGOTIATION AND ARBITRATION SUPPORT

2.1. Negotiation template evaluation

2.1.1. Additive scoring system

One crucial issue in the negotiation structure described above requires special consideration, namely, the problem of the evaluation of objectives and alternatives with respect to the negotiator's structure of preference. This evaluation is required for analyzing alternatives consequences and trade-offs. We suggest the application of the additive scoring system which is the simplest possible tool that have already been successfully applied in such eNSs as Inspire [6] or Negoisst [15]. It requires a simple qualitative between-issues and within-issues analysis consisting of two steps:

1. Distributing a certain amount of scoring points among all the issues established in the negotiation template.
2. Assigning scoring points allocated to the issue to all its resolution levels.

The scoring points allocated to each negotiation issue describe its importance. If the score of 40 is assigned to an issue, this issue is more important than two others which scores sum up to 30. Since the values which we assign to each issue come from a particular amount of scoring points, we can also explain the scores on a ratio scale calling the issue with the score of 20 twice as important as the issue with the score of 10.

The allocation of scoring points to resolution levels within issues follows a different rule. The level that best satisfies the issue receives the maximum possible score, while the level that least satisfies it, the score of 0. The other levels receive scores from the range $\langle 0; \max \rangle$, but the distributions do not have to be linear. A simple example of template evaluation is shown in Table 1. The table shows the analysis conducted by the employee that negotiates a new contract with the management.

Table 1

Scoring the negotiation template

Issue	Issue score	Resolution	Resolution score
Salary	50	3000 PLN	0
		4000 PLN	15
		5000 PLN	40
		6000 PLN	50
Holiday	30	20 days	0
		25 days	10
		30 days	30
Life-insurance	20	covered by employee	0
		covered by employer	20

Note: The issue scores sum up to 100.

One restriction has to be fulfilled to make the additive scoring system legitimate. Values associated with a given resolution of one issue cannot depend on the resolutions of other issues [11]. If the trade-offs of two issues depend on the levels of the third one then the new composite issue should be created that comprises these three issues and which values do not depend on the resolutions of the other issues.

2.1.2. AHP for template evaluation support

Even though the template evaluation with additive scoring systems seems to be rather easy, the process of assigning scores to issues and resolution levels can be a little vague and artificial, especially for the negotiators who had never followed such quantitative analysis before. Therefore we suggest applying the AHP procedure as a support tool for construction of a scoring system[4]. To use AHP methodology for negotiation support with the structure described in Section 1, we obviously need to satisfy the axioms that are the basis of this approach [13]. The reciprocal axiom is satisfied when we assume that our negotiators act rationally. The software support we suggest allows us to satisfy this axiom by using the ratio scale interpretation. If a negotiator rates one resolution level to be 3 times better than another one, the support system will

[4] The problem of usefulness of AHP for negotiation support has been discussed before in some papers, e.g. [7].

automatically interpret the latter to be 1/3 as good as the former one. The homogeneity axiom should be sufficient because the pre-negotiation phase analysis is conducted that results in the construction of negotiation template. The resolution levels established for each issue in negotiation template comprise the subset of all possible levels for this issue. Since the pre-negotiation phase allows to discuss the negotiation problem and find the preliminary negotiation set, the resolution levels proposed should not differ by more than one order of magnitude. The third axiom of judgment independence is similar to the one we had to satisfy when applying the additive scoring system described in Subsection 2.1.1. Having satisfied the three main AHP axioms we can assume that AHP procedure will result in appropriate judgments and can be incorporated in our notion of negotiation support.

We will use a nine-point verbal scale for comparison of the importance of issues and resolution levels, and the AHP rating approach for large numbers of alternatives [3]. In our negotiation case the procedure will consists of three steps:

1) application of the AHP procedure to pairwise comparisons of the issues,
2) application of the AHP procedure to pairwise comparisons of the resolution levels within each issue,
3) synthesis of results.

The characteristic AHP hierarchy for our negotiation problem described in Table 1 in terms of overall goal, criteria (issues), and resolution levels is shown in Figure 1.

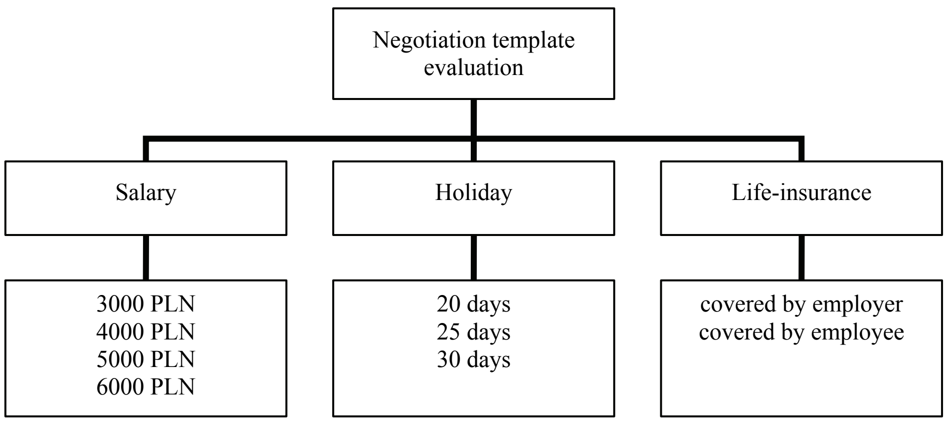


Fig. 1. AHP hierarchy for negotiation template evaluation

After completing Step 2 we will need to rescale the scores determined for the resolution levels of every issue. It is required to obtain an additive scoring system corresponding to the one proposed in section 2.1.1. We can apply the AHP rescaling formula proposed in Expert Choice [2]:

$$\tilde{s}_i^{rl} = s_i^{rl} \frac{s_i}{s_i^{\max}} \tag{1}$$

where:

$\tilde{s}_i^{rl}$ – is the rescaled score of the resolution level *rl* of issue *i*,

s_i^{rl} – is the original score of the resolution level *rl* of issue *i* obtained in Step 2,

s_i – is the original score of issue *i* obtained in Step 1,

$s_i^{\max}$ – is the maximum original score of any resolution level of issue *i*.

But this rescaling method leads us to the scoring system with the scores of different interpretation from the ones in Table 1. Since we had applied the AHP for within-issue analysis (Step 2), we obtained the reservation level scores of ratio scale interpretation. Hence, the score assigned to the worst resolution level is not 0, but a positive value, which describes how many times it is worse than the best one for this issue. To avoid this side effect we can apply a different rescaling method, but we suggest retaining this ratio scale interpretation, since it can be useful in comparing complete alternatives and in considering how much one alternative is better than another one.

2.2. Negotiation template analysis

Having completed the between- and within-issues quantitative analysis with the tools proposed in subsection above, the negotiators obtain the scoring system of the negotiation template that can be used in the negotiation phase for analyzing consequences and trade-offs (which we call template analysis). The negotiator knowing the scores of each resolution level within each issue may now compare two offers and judge which one is better based on the total sum of scores that they receive from the negotiation template. This is the negotiation support aspect we wanted to achieve. For instance, based on the template described in Table 1 the negotiator knows that the difference between agreeing for 25 days of holidays or 30 days of holidays is of the same importance as the difference between agreeing for life-insurance coverage by the employer or by the employee himself, herself. And further, she/he will agree for a salary reduction from 6000 PLN to 5000 PLN, but will request 25 days holidays instead of 20 days to balance the difference in scores. Finally, she/he can judge the offer of 4000 PLN, 25 days of holidays, and life-insurance coverage by the employee to be worse than the offer of 3000 PLN, 25 days holidays, and life-insurance coverage by the employer, since the former produces the total payoff of 25 and the latter, the payoff of 30. The negotiation support system can easily support this simple calculation.

But there is another aspect of the template analysis that can be successfully supported with the tools proposed previously. It is an arbitration process that focuses on the search of a mutually accepted and satisfying agreement.

2.2.1. Game theory approach for arbitration support

The simplest way to support the arbitration proceduree s is to incorporate the game theory approach [12, 16]. When randomization is assumed (which is acceptable for integrative negotiations), the approach focuses on finding the set of extreme-efficient contracts in order to derive from it the single alternative as the equitable or fair one. We recommend that three conceptions of the symmetric analysis be considered, which are most frequently applied for solving such two-person conflict:

1. **Maximizing the sum**.

   We seek an alternative a^e that produces the maximum sum of the payoffs of both negotiators:

$$s\left(a^e\right) = \max\left\{s_A(a^e) + s_{\mathbf{B}}(a^e) : a^e \in A^e\right\} \tag{2}$$

   where:

   $s\left(a^e\right)$ is a total payoff for the alternative a^e,
   $s_A(a^e)$ is a payoff the negotiator *A* receives for the alternative a^e,
   ($s_B(a^e)$ is defined similarly for the negotiator *B*),
   A^e is the set of extreme-efficient contracts.

2. **Maximizing the minimum**.
   This approach was originally proposed by von Neumann and Morgenstern for solving two-person non-cooperative games. It identifies a fair alternative a^e that maximizes the minimum payoff of either negotiator A or negotiator B and which global score is given as:
   $$s(a^e) = \max\{\min\{s_A(a^e), s_{\mathbf{B}}(a^e)\} : a^e \in A^e\} \qquad (3)$$
   We will apply this approach in a modified form. Since we accept the reservation levels which the parties can derive from their BATNA, we will maximize the minimum of the proportion of potential [12] (for details see Section 3).
3. **Maximizing the product**.
   Basing on the concept of the Nash solution of the game we seek an alternative a^e that maximizes the payoffs product of both negotiators:
   $$s(a^e) = \max\{s_A(a^e) \cdot s_{\mathbf{B}}(a^e) : a^e \in A^e\} \qquad (4)$$

Since these three notions can be easily explained to and interpreted by the negotiators, we will use them simultaneously in our arbitration analysis.

2.2.2. AHP for arbitration support

The game theory approach is commonly applied for seeking a fair or equitable compromise, but it does not take into consideration the negotiation strength of the parties. When we face a problem with disproportional negotiation strength, we need to analyze how the weaker negotiator is impacted by the acceptance of a compromise which gives a much better payoff to his/her partner than to himself/hereself. This is a very complicated psychological problem widely discussed in many papers on the psychology of conflict and is not the subject of this paper. But we will propose a procedure to determine the best compromise in the situation in which we are able to describe the negotiation strength quantitatively. We will incorporate the AHP approach for many players [3]. This approach will follow the analysis we conducted before in Subsection 2.1.2 for negotiation template evaluation and will require assigning to the negotiation parties weights which reflect their negotiation strength. The weights have to sum up to 1. If there are more than two negotiating parties we can recommend the AHP procedure with verbal judgments to find appropriate weights. If there are only two parties, not more than a simple calculation is required to solve the equation:

$$ax + x = 1 \tag{5}$$

where:

a – describes how many times the negotiation strength of one negotiator is greater than that of the other, and must be subjectively assumed by NSS/sNS or the arbitrator,

ax, x – is the weight reflecting the negotiation strength of the parties.

This step will lead us to the weights of the ratio scale interpretation required for AHP analysis. Before the analysis we need to add another level of hierarchy to the current AHP hierarchy structure. This level will reflect the parties' negotiation strength. The AHP synthesis procedure will then use the weights of resolution levels, the weights of issues, and the weights of the parties. The offer that receives higher priority should be recommended as the fair compromise.

The hierarchy structure of the problem of the application of AHP for determination of negotiation compromise is shown in Figure 2.

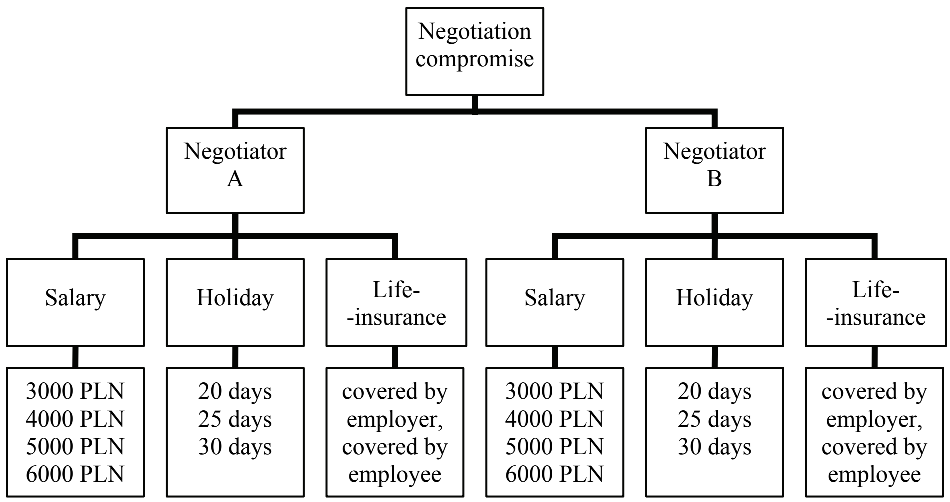


Fig. 2. AHP hierarchy for determination of negotiation compromise

3. EXAMPLE

We will now apply the above idea to the negotiation and arbitration support for a hypothetical example of bilateral negotiation. Let us explore the negotiation problem as presented in Figure 1. Next, we will consider three issues: the first of four resolution levels, the second of three, and the last of two. In order to construct an additive scoring system we use the AHP pairwise comparison to evaluate, first, the ranking of the issues and then the ranking of the resolution levels for each party, separately. After completing this comparison we will check the consistency of the evaluation based on a consistency ratio [3]. The process of ranking construction can be simply programmed in a spreadsheet (we will use MS Excel) based only on standard formulas without necessity of incorporating macros or VB scripts. An adequately programmed spreadsheet for ranking construction is shown in Figure 3.

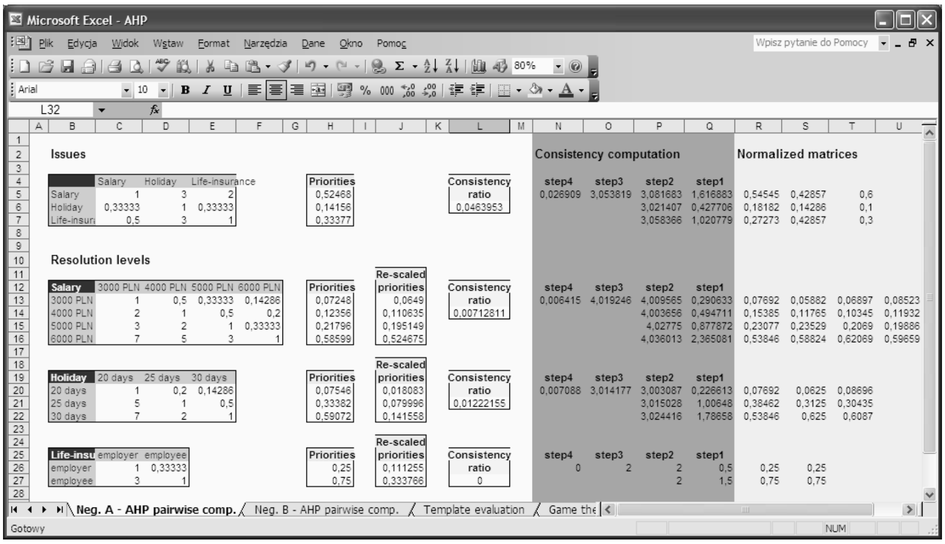
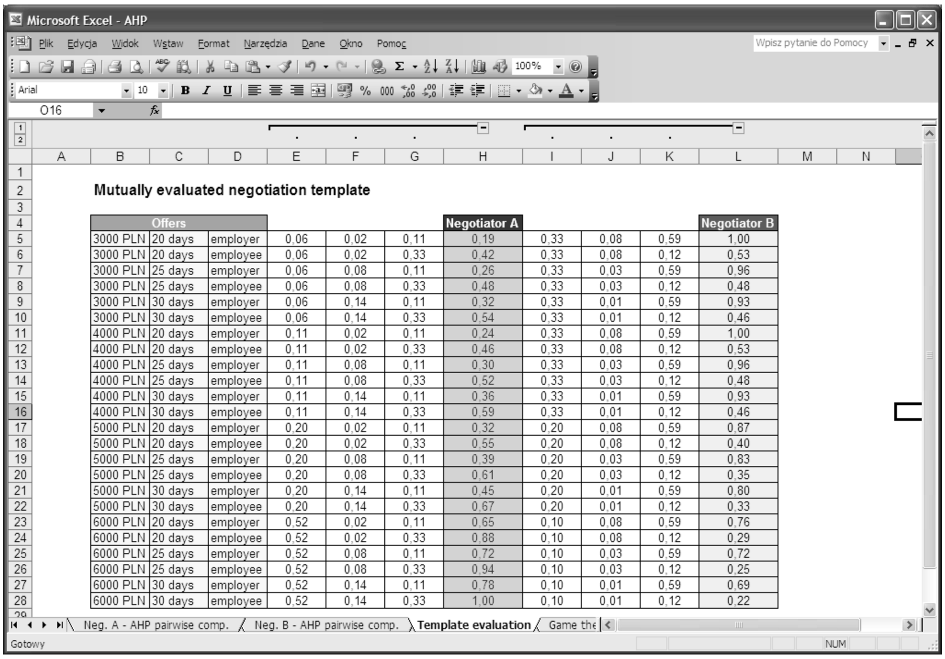


Fig. 3. Spreadsheet for issues and resolution levels ranking evaluation

Another sheet is similarly programmed for the negotiator B. The rankings of acceptable consistency determined for both parties allow for construction of mutually evaluated negotiation template (see Figure 4).



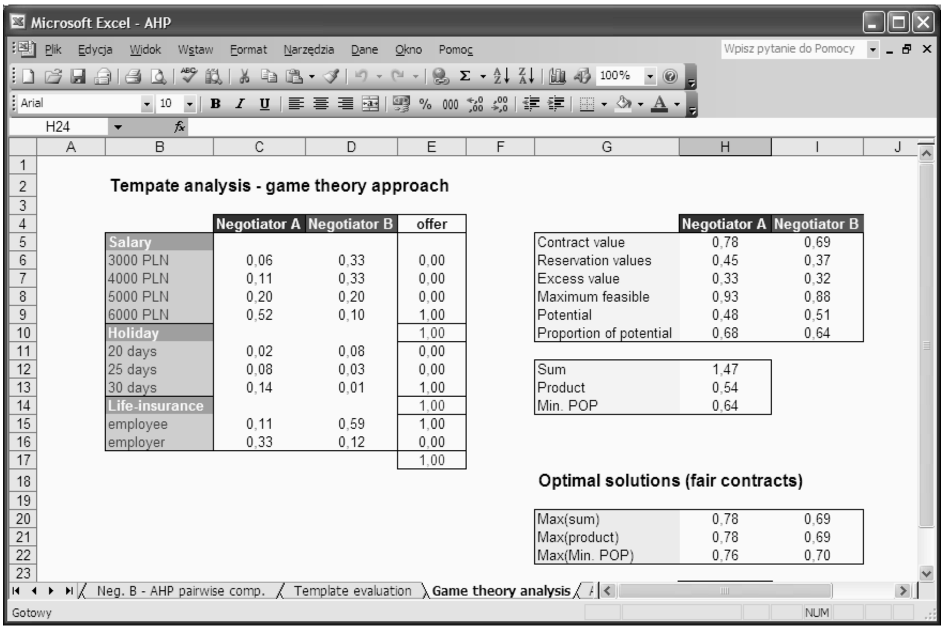
Mutually evaluated negotiation template

Offers						Negotiator A				Negotiator B
3000 PLN	20 days	employer	0,06	0,02	0,11	0,19	0,33	0,08	0,59	1,00
3000 PLN	20 days	employee	0,06	0,02	0,33	0,42	0,33	0,08	0,12	0,53
3000 PLN	25 days	employer	0,06	0,08	0,11	0,26	0,33	0,03	0,59	0,96
3000 PLN	25 days	employee	0,06	0,08	0,33	0,48	0,33	0,03	0,12	0,48
3000 PLN	30 days	employer	0,06	0,14	0,11	0,32	0,33	0,01	0,59	0,93
3000 PLN	30 days	employee	0,06	0,14	0,33	0,54	0,33	0,01	0,12	0,46
4000 PLN	20 days	employer	0,11	0,02	0,11	0,24	0,33	0,08	0,59	1,00
4000 PLN	20 days	employee	0,11	0,02	0,33	0,46	0,33	0,08	0,12	0,53
4000 PLN	25 days	employer	0,11	0,08	0,11	0,30	0,33	0,03	0,59	0,96
4000 PLN	25 days	employee	0,11	0,08	0,33	0,52	0,33	0,03	0,12	0,48
4000 PLN	30 days	employer	0,11	0,14	0,11	0,36	0,33	0,01	0,59	0,93
4000 PLN	30 days	employee	0,11	0,14	0,33	0,59	0,33	0,01	0,12	0,46
5000 PLN	20 days	employer	0,20	0,02	0,11	0,32	0,20	0,08	0,59	0,87
5000 PLN	20 days	employee	0,20	0,02	0,33	0,55	0,20	0,08	0,12	0,40
5000 PLN	25 days	employer	0,20	0,08	0,11	0,39	0,20	0,03	0,59	0,83
5000 PLN	25 days	employee	0,20	0,08	0,33	0,61	0,20	0,03	0,12	0,35
5000 PLN	30 days	employer	0,20	0,14	0,11	0,45	0,20	0,01	0,59	0,80
5000 PLN	30 days	employee	0,20	0,14	0,33	0,67	0,20	0,01	0,12	0,33
6000 PLN	20 days	employer	0,52	0,02	0,11	0,65	0,10	0,08	0,59	0,76
6000 PLN	20 days	employee	0,52	0,02	0,33	0,88	0,10	0,08	0,12	0,29
6000 PLN	25 days	employer	0,52	0,08	0,11	0,72	0,10	0,03	0,59	0,72
6000 PLN	25 days	employee	0,52	0,08	0,33	0,94	0,10	0,03	0,12	0,25
6000 PLN	30 days	employer	0,52	0,14	0,11	0,78	0,10	0,01	0,59	0,69
6000 PLN	30 days	employee	0,52	0,14	0,33	1,00	0,10	0,01	0,12	0,22

Fig. 4. Negotiation template evaluated by both parties

This negotiation template can be used directly by parties for negotiation support. The parties can analyze negotiation offers and consider the concessions made by the partner using a trade-off analysis. For instance, the offer of the negotiator B of {4000 PLN; 20 days, employer} gives to the negotiator A the score (0,24 scoring points) which is better than the one of {3000 PLN; 20 days, employer} by 0,19 points. Therefore it will be perceived by the negotiator A as a concession, although it did not require a true concession from the negotiator A (both offers have the same score of 1 for her/him). And, further, if the negotiator B suggests the salary reduction from 5000 PLN to 4000 PLN while leaving 30 days of holidays and life-insurance coverage by the employee, which for the negotiator A is a move from the offer of 0,45 to one of only 0,36 then the negotiator A will ask to leave the salary at the level of 5000 PLN, but to lower the request of 30 days of holidays to 25 days instead (an offer which ensures the score of 0,39). Many other trade-off analyses can be conducted similarly.

A further template analysis of arbitration support requires somewhat more advanced tools. First, we apply the game theory approach. To find a fair compromise the spreadsheet has to be prepared by listing all the resolution levels for all the issues considered. Then, binary cells corresponding to every resolution level must be identified that will indicate the level chosen for a compromise. The values of these cells must sum up to 1 within each issue. Adequately prepared spreadsheet cells are presented in Figure 5 in the range B4:E17.



Tempate analysis - game theory approach

	Negotiator A	Negotiator B	offer
Salary			
3000 PLN	0,06	0,33	0,00
4000 PLN	0,11	0,33	0,00
5000 PLN	0,20	0,20	0,00
6000 PLN	0,52	0,10	1,00
Holiday			1,00
20 days	0,02	0,08	0,00
25 days	0,08	0,03	0,00
30 days	0,14	0,01	1,00
Life-insurance			1,00
employee	0,11	0,59	1,00
employer	0,33	0,12	0,00
			1,00

	Negotiator A	Negotiator B
Contract value	0,78	0,69
Reservation values	0,45	0,37
Excess value	0,33	0,32
Maximum feasible	0,93	0,88
Potential	0,48	0,51
Proportion of potential	0,68	0,64

Sum	1,47
Product	0,54
Min. POP	0,64

Optimal solutions (fair contracts)

Max(sum)	0,78	0,69
Max(product)	0,78	0,69
Max(Min. POP)	0,76	0,70

Fig. 5. Arbitration support based on game theory

In the next step we identify (and store in cells H5:I5) the scores which each party receives for a contract described with binary cells that are the sum of products of column C and E (for the negotiator A) and D and E (for the negotiator A). These values will be used to calculate the total sum (H12) and the total product (H13) of offers that we are going to maximize according to the approach presented in Subsection 2.2.1. We also allow for introducing reservation values that come from parties' BATNA, but they can be set to 0 if not known. We are now looking for the maximum of minimum of the proportion of potential, which is (individually) the excess (the difference between the contract and the reservation values) divided by the potential (the difference between the maximum feasible value that can be achieved for the partner's reservation value and his/her own reservation value).

To find a fair compromise we need to run the Solver thrice. First, we maximize the cell H12 (the sum) allowing Solver to manipulate the variables from column E with the following constraints:

– the sum of the values from column E must be 1 within the groups of issues: SUM(E6:E9) = 1, SUM(E11:E13) = 1, SUM(E15:E16) = 1
– the values from column E must be non-negative (we can also wish them to be integers, but it is not necessary, since we assumed that randomization is possible),
– the excesses (H7:I7) must be non-negative.

Next, we receive the optimal solution, which is the offer {6000 PLN, 30 days, employee} that gives the score of 0,78 points to the negotiator A and of 0,69 points to the negotiator B. The Solver gives the same recommendation if we maximize the product (cell H13). When we maximize the minimum proportion of potential (cell H14) the negotiators receive the scores 0,76 and 0,70, respectively, but for the randomized offer. The randomized offer is shown in Figure 6.

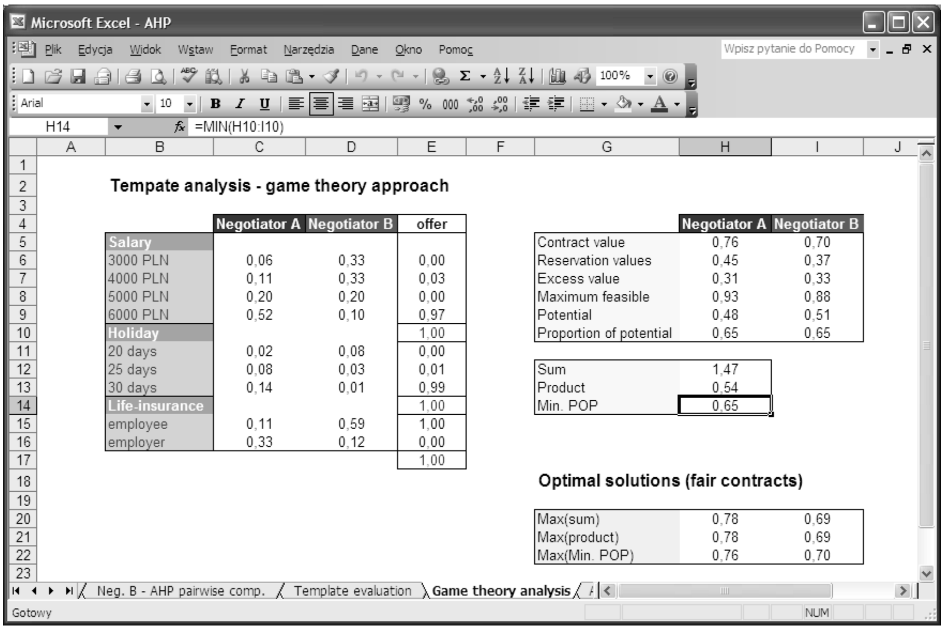
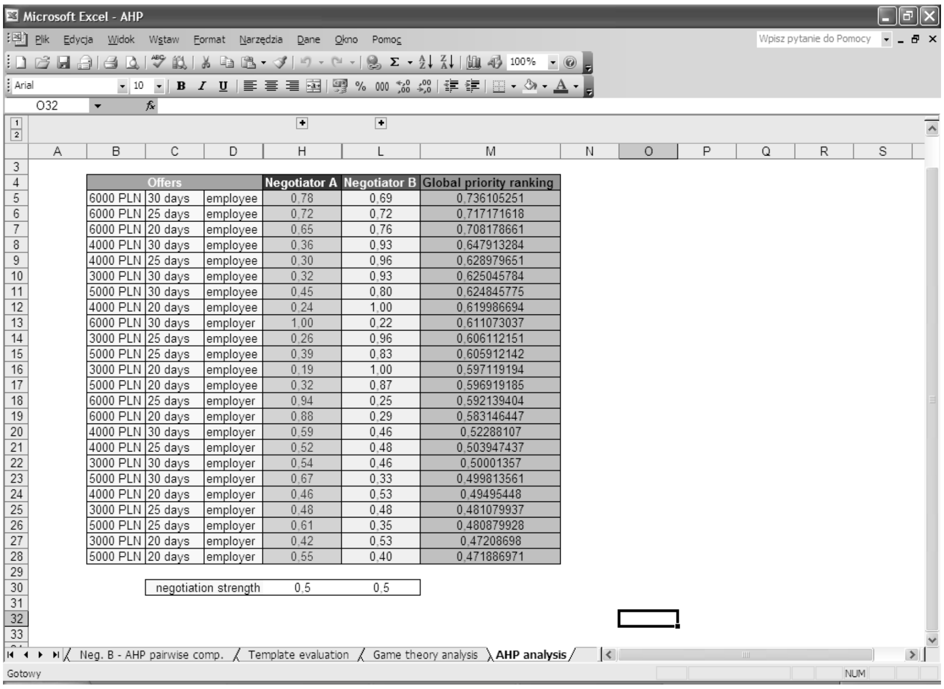


Fig. 6. Randomized compromise generated for maximizing the minimum proportion of potential

The arbitration support with the AHP approach is far easier. We need to assume (or calculate) the negotiation strength of the parties first and then to compute the global score for every feasible offer, which is a weighted sum of the individual scores multiplied by the strength weights. Thus, we obtain the global ranking where the offer with the highest score should be recommended as the fair one (see Figure 7). In our example, using weights reflecting equal negotiation strength of each party, we obtain the same recommendation for the fair compromise as in the case of maximizing the sum and maximizing the product in game theory based approach.



Offers			Negotiator A	Negotiator B	Global priority ranking
6000 PLN	30 days	employee	0,78	0,69	0,736105251
6000 PLN	25 days	employee	0,72	0,72	0,717171618
6000 PLN	20 days	employee	0,65	0,76	0,708178661
4000 PLN	30 days	employee	0,36	0,93	0,647913284
4000 PLN	25 days	employee	0,30	0,96	0,628979651
3000 PLN	30 days	employee	0,32	0,93	0,625045784
5000 PLN	30 days	employee	0,45	0,80	0,624845775
4000 PLN	20 days	employee	0,24	1,00	0,619986694
6000 PLN	30 days	employer	1,00	0,22	0,611073037
3000 PLN	25 days	employee	0,26	0,96	0,606112151
5000 PLN	25 days	employee	0,39	0,83	0,605912142
3000 PLN	20 days	employee	0,19	1,00	0,597119194
5000 PLN	20 days	employee	0,32	0,87	0,596919185
6000 PLN	25 days	employer	0,94	0,25	0,592139404
6000 PLN	20 days	employer	0,88	0,29	0,583146447
4000 PLN	30 days	employer	0,59	0,46	0,52288107
4000 PLN	25 days	employer	0,52	0,48	0,503947437
3000 PLN	30 days	employer	0,54	0,46	0,50001357
5000 PLN	30 days	employer	0,67	0,33	0,499813561
4000 PLN	20 days	employer	0,46	0,53	0,49495448
3000 PLN	25 days	employer	0,48	0,48	0,481079937
5000 PLN	25 days	employer	0,61	0,35	0,480879928
3000 PLN	20 days	employer	0,42	0,53	0,47208698
5000 PLN	20 days	employer	0,55	0,40	0,471886971
	negotiation strength		0,5	0,5	

Fig. 7. AHP based arbitration support

SUMMARY

In the paper we have suggested simple negotiation and arbitration support. This support is based on an additive scoring system that have already been applied in real-word negotiation support systems, but we combine it with AHP methodology to find the process of the negotiation template evaluation easier and more intuitive. We believe that this approach is an alternative to the simple assigning of scores, which can be perceived by as too abstract. Furthermore, we apply the commonly known game theory approach for seeking a fair compromise within such evaluated negotiation template and show that AHP can still be used for a similar analysis. Finally, we show that all the computations can be done in a simple spreadsheet, and consequently, they can be performed by a negotiation and arbitration support system which simplifies the negotiators' task even more. The spreadsheet, as presented in the paper, requires some preparation work, but can be easily automated with a VB programme which uses dialog boxes. Writing such programme is the next step of our research; it will show how easy it is to construct support tools that allow us to make the negotiation process clearer, faster, and fairer.

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DECISION MAKING PROBLEM WITH TWO INCOMPARABLE CRITERIA – GAME THEORY SOLUTION

Abstract

In this paper the solution concept of decision making problem with two incomparable criteria is presented. The incomparability consists in having no premises to aggregate the assessments of the decision variants or to aggregate the relations of partial preferences, which are associated with criteria considered – it is possible to show the best variant with respect of the particular criterion, but there is no information which could enable the definition of relations between the criteria considered.

Construction of the model of multicriteria decision making problem is the starting point of consideration. The problem is presented in literature as a two-person zero-sum game; it is noticed, however, that the multicriteria decision making problem is not antagonistic. Our premise is the construction of model on the ground of nonantagonistic game theory. The proposal is that the criteria play the game with themselves. It is not necessary, but the player as a person can be identified with decision maker who considers the problem from the point of view of one criterion. The set of strategy is defined by the set of decision variants – for each player. The payoffs are defined by assessments of the decision variants if the players use the same strategy (choose the same decision variants). Otherwise, they reach much worse result. Also they know that they must choose the same variant. The characteristic feature of the game is the problem of coordination between equilibria – each equilibrium is equivalent to one decision variant. Thus, the problem is: Which equilibrium should be chosen? Harsanyi and Selten's "A general theory of equilibrium selection in games" and their concept of risk domination is applied to answer this question.

The consequences of the proposal's acceptance are that the solution of the primary bicriterial decision making problem is the decision variant which belongs to a nondominated set of decision variants and for which the sum of the assessments of the decision variants is maximal (the assessments are subtracted for the minimized criteria and added for the maximized ones).

Keywords

Coordination game, incomparable criteria.

INTRODUCTION

The solution of a multiple criteria decision problem usually consists in scalarizing the question [2], i.e. in reducing it to the decision problem with one synthetic criterion or in building a model aggregating the assessments of the decision variants or aggregating the partial preferences [7].

However, it is difficult to obtain information which would allow for the aggregation of the assessments of decision variants or preferences. Therefore, the decision maker must face the problem of making decisions in precarious conditions.

The question that will be considered here is connected with building a multiple criteria decision model on theoretical basis; it is also associated with Harsanyi and Selten's paper [3], in particular, with the concept of equilibrium selection in games presented in it. A multicriteria decision making problem can be presented in the context of game theory as a two-person zero sum game, i.e. as an antagonistic game [4, 5]. However, it has been noticed that such extreme conflict is hardly ever characteristic for multiple criteria problems [1, p. 534], and that is why it is suggested that a model of a multiple criteria problem be constructed, in the form of a multi-person non-zero sum game [8, 9].

The main purpose of this paper is to describe the solution of a multiple criteria decision problem in a precarious situation due to the lack of clear premises that would allow a comparison of criteria.

CONSTRUCTION OF THE MODEL

If X is a set of acceptable decision variants, assessed according to a finite set of criteria $F = \{f_1, f_2\}$ then the assessment of the decision variant $x \in X$ according to its j-th criterion is the value $f_j(x)$. Let the decision maker consider each problem separately, from the point of view of every accepted criterion, while there are no premises allowing any kind of aggregation of assessments or preferences. Moreover, the decision maker wants the assessment of each decision variant to be the highest possible (the criteria are maximized).

The problem can be represented in the form of two-person non-cooperating non-zero sum game, with all the information, played between the criteria considered. The set of acceptable strategies is defined by the set X, and the payoff of the j-th player is defined by the value $f_j(x)$, but only in the situation when the other player also chooses the option x. If this is not the case, the player obtains a considerably lower score. The result of the game can be represented as a function $F(x_1, x_2)$ in the following way (assuming that both criteria are maximized):

$$F(x_1, x_2) = \begin{cases} (f_1(x_1), f_2(x_2)) & \text{gdy} \quad x_1 = x_2 \\ (M, M) & \text{gdy} \quad x_1 \neq x_2 \end{cases} \tag{1}$$

where:

x_1 is the decision variant, $x_1 \in X$, chosen by the player representing the first criterion,
x_2 is the decision variant, $x_2 \in X$, chosen by the player representing the second criterion,
M stands for any value fulfilling the following conditions:

$$M << f_j(x_i), \text{ for } j = 1,2, i = 1,2 \tag{2}$$

In the game defined here we come across the problem of coordination of the players' actions performed in order to reach the equilibrium[1]. The players-criteria choose their strategy (decision variant) independently, but the result of this choice depends solely on the other player's choice. It follows from the defined payoff function (1) in the case of both players choosing the same strategy that they will achieve much higher payoffs than if they choose different strategies. Therefore, the players are keen on choosing the same decision variant. To sum up, in the game defined here there are as many equilibria in the strategies set as there are elements in the set X. It should also be emphasized that these equilibria can be dominated by another equilibrium[2] only, while each equilibrium in the game represents one decision variant. In other words, the small number M is arbitrary in the sense that situations in the game considered here, where players-criteria choose the same strategy (variant), are equilibria in the game.

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[1] In this paper the equilibrium is understood as Nash equilibrium.
[2] This game is not of the Prisoner's Dillema type.

In this solution of the problem one can suggest choosing the equilibrium which is dominating due to the players' payoffs. It means that one should choose the decision variant corresponding to this equilibrium. Next, in a nontrivial situation, when the equilibrium dominating due to the payoffs does not exists, the equilibrium is chosen (from the non-dominated ones) according to the risk (risk dominating).

The analysis of the game begins with the comparison of pairs of equilibria. Let us consider the payoff matrix A representing the situation when the players have to deal with two equilibria:

$$A = \begin{bmatrix} (f_1(x_1), f_2(x_1)) & (M, M) \\ (M, M) & (f_1(x_2), f_2(x_2)) \end{bmatrix} \tag{3}$$

where:

$f_j(x_1)$ is the payoff of the j-th player when all the players choose the strategy (the decision variant) x_1,
$f_j(x_2)$ is the payoff of the j-th player when all the players choose the strategy (the decision variant) x_2.

The players maximize their expected payoffs. The expected payoff of the first player, due to using the first strategy $E_1(x_1)$, equals:

$$E_1(x_1) = f_1(x_1) \cdot q_1 + M \cdot (1 - q_1) \tag{4}$$

where:

q_1 is the subjective probability (according to the first player's assessment) that the other player will use the strategy x_1
the expected payoff of the first player, due to the use of the second strategy $E_1(x_2)$ – equals:

$$E_1(x_2) = f_1(x_2) \cdot (1 - q_1) + M \cdot q \tag{5}$$

For the second player the expected winnings equal, respectively:

$$E_2(x_1) = f_2(x_1) \cdot (1 - p_2) + M \cdot p_2 \tag{6}$$

$$E_2(x_2) = f_2(x_2) \cdot p_2 + M \cdot (1 - p_2) \tag{7}$$

where:

p_2 is the subjective probability that the other player will use the second strategy.

If the following condition is fulfilled:

$$E_1(x_1) > E_1(x_2) \tag{8}$$

the first player will choose the first strategy. If, in turn, the condition:

$$E_2(x_1) < E_2(x_2) \tag{9}$$

is fulfilled the other player will choose the second strategy.

In the situation when players choose different strategies the equilibrium will not be achieved and both will receive lower payoffs than they would if one of the players “gave up”. If the players have the same information about the situation they should use strategies implying the equilibrium, which choice can be based on stronger premises.

The substitution of the expressions (4) and (5) in the condition (8) yields, after consecutive transformations:

$$f_1(x_1)\cdot q_1 + M\cdot(1-q_1) > f_1(x_2)\cdot(1-q_1) + M\cdot q_1 \tag{10}$$

$$q_1\cdot[f_1(x_1) - M + f_1(x_2) - M] > f_1(x_2) - M \tag{11}$$

$$q_1 > \frac{f_1(x_2) - M}{[f_1(x_1) + f_1(x_2) - 2\cdot M]} = q_0 \tag{12}$$

while it is assumed that the expression in the denominator is positive.

Analogously, the substitution of the expressions (6) and (7) in the condition (9) yields consecutively:

$$f_2(x_1)\cdot(1-p_2) + M\cdot p_2 < f_2(x_2)\cdot p_2 + M\cdot(1-p_2) \tag{13}$$

$$p_2\cdot[f_2(x_2) - M + f_2(x_1) - M] > f_2(x_1) - M \tag{14}$$

$$p_2 > \frac{f_2(x_1) - M}{[f_2(x_1) + f_2(x_2) - 2\cdot M]} = p_0 \tag{15}$$

while it is also assumed that the value of the expression in the denominator is positive.

The value q_0 is the limit value of the subjective probability that the other player will choose the first strategy, while the value p_0 is the limit value of the subjective probability that the first player will choose the second strategy. One can then acknowledge that if the condition [3, p. 216]:

$$q_0 < p_0 \tag{16}$$

is fulfilled there are stronger premises for all the players to choose the first equilibrium (implying the choice of the variant x_1). It is more likely that the first player will choose the first strategy than that the other will choose the second one. This reasoning presents the idea of dominance associated with the risk (risk-dominance) of the first equilibrium dominating the other – the first equilibrium risk-dominates the second.

The substitution of expressions (12) and (15) in the condition (16) yields, after consecutive transformations:

$$\frac{f_1(x_2)-M}{f_1(x_1)+f_1(x_2)-2\cdot M}<\frac{f_2(x_1)-M}{f_2(x_1)+f_2(x_2)-2\cdot M} \tag{17}$$

$$\frac{f_1(x_2)\cdot f_2(x_2)-M\cdot f_1(x_2)-M\cdot f_2(x_2)}{f_1(x_1)\cdot f_2(x_1)-M\cdot f_1(x_1)-M\cdot f_2(x_1)}<1 \tag{18}$$

if the expression in the denominator is positive and, if this expression is negative:

$$\frac{f_1(x_2)\cdot f_2(x_2)-M\cdot f_1(x_2)-M\cdot f_2(x_2)}{f_1(x_1)\cdot f_2(x_1)-M\cdot f_1(x_1)-M\cdot f_2(x_1)}>1 \tag{19}$$

In the model we assumed that M is any arbitrary number satisfying the condition (2). Therefore, when the assessments of the decision variants are any finite real numbers, one can assume that $M\to-\infty$ and the condition (2) is always fulfilled. In this case the assumption that the denominator in the expressions (12) and (15) is positive is always fulfilled. Thus:

$$\lim_{M\to-\infty}\frac{f_1(x_2)\cdot f_2(x_2)-M\cdot f_1(x_2)-M\cdot f_2(x_2)}{f_1(x_1)\cdot f_2(x_1)-M\cdot f_1(x_1)-M\cdot f_2(x_1)}\le 1 \tag{20}$$

or

$$\lim_{M\to-\infty}\frac{f_1(x_2)\cdot f_2(x_2)-M\cdot f_1(x_2)-M\cdot f_2(x_2)}{f_1(x_1)\cdot f_2(x_1)-M\cdot f_1(x_1)-M\cdot f_2(x_1)}\ge 1 \tag{21}$$

if the expression in the denominator is negative. Therefore:

$$\frac{f_1(x_2)+f_2(x_2)}{f_1(x_1)+f_2(x_1)}\le 1 \tag{22}$$

or

$$\frac{f_1(x_2)+f_2(x_2)}{f_1(x_1)+f_2(x_1)}\ge 1 \tag{23}$$

Taking the above reasoning into consideration one can state that in the situation when the condition:

$$f_1(x_2) + f_2(x_2) \le f_1(x_1) + f_2(x_1) \quad (24)$$

[3]

is fulfilled, both players will choose the first strategy (the first strategy risk-dominates the other), therefore according to the idea presented here the first variant is better.

CONCLUSIONS

To sum up, one should emphasize that the idea suggested here can be used in the situation when there are neither premises which could allow for the comparison of the assessments of the variants with respect to the criteria nor partial preferences associated with the criteria considered. Moreover, the solution of the problem is obtained by addition (for the maximized criteria) and subtraction (for the minimized) of the assessments of the decision variants. However, due to the apparent incompatibility of the units and the scale in which these assessments are expressed, one should bear in mind the assumptions of usability as well as, above all, the rule (24) which aim is to indicate the risk-dominating decision variant. Pursuing our reasoning one can state that in the case when the third and the next variants are considered, the best variant is the one for which the sum of the assessments (in the case of maximized criteria) is the highest. This reasoning implies the statement that in the model suggested the relation of risk-dominance is transitive.

One can also notice that no conflict between risk-dominance and payoff-dominance exists in this model. Therefore, it is true that if one equilibrium dominates the other due to the payoffs, it also risk-dominates the other[4].

In the case when two equilibria in the set of pure strategies exist for which both sides of the expression (24) have equal value one should regard their respective variants as equivalent[5].

This reasoning and the assumptions of the usability of the method result in the fact that the question of normalization of the decision variants assessments in the model suggested becomes a serious problem, since

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[3] One can reach the same conclusions starting from the reasoning presented in [6, p. 42].

[4] In the general case risk-dominance is not transitive; also, conflict between risk-dominance and payoff-dominance can occur.

[5] In the general case one should also consider the equilibria in the set of mixed strategies, but in the model suggested here the interpretation of mixed equilibrium is ambiguous as it would imply at least two decision variants.

the choice of the normalization method can have a considerable influence on the choice of the best variant. Therefore, taking into account also the fact that the normalization method can be justified, one can assume that the method suggested here can be used in the situation when there are no reasons for normalization or when the normalization of the assessments is acceptable, but one can investigate the influence of the normalization on the solution. One should also emphasize the sensitivity of the condition (24) introduced to the change of the scale caused by the assessments normalization.

To conclude, one can state that when the decision is taken with regard to two incomparable maximized criteria she/he can choose the decision variant from those which are not dominated due to the assessments and for which the sum of the assessments of the decision variants is the highest. This results from the construction of the model of the problem as a coordination game, where an arbitrary small number (payoff) M implies that the players-criteria should choose the same strategy (variant). In this game, the problem of selection of one equilibrium occurs. The selected equilibrium should indicate the best decision variant in the situation under consideration. The procedure of selection is based on the notion of risk dominance presented by Harsanyi and Selten [3].

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