MULTIPLE CRITERIA DECISION MAKING

ANNUAL

VOL. 10 (2015)

THE UNIVERSITY OF ECONOMICS IN KATOWICE 2015

Printed by

EXPOL P. Rybiński, J. Dąbek Spółka Jawna, ul. Brzeska 4, 87-800 Włocławek

ISSN 2084-1531

Edition: 100 copies

Original version of the MCDM is the paper version

Publisher of The University of Economics in Katowice
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Vol. 10 2015

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USING MULTI-OBJECTIVE AFFINITY MODEL FOR MINING THE RULES OF REVISITS WITHIN 72 HOURS FOR EMERGENCY DEPARTMENT PATIENTS

Abstract

When patients return to the emergency department (ED) within 72 hours after their previous ED discharge, it is generally assumed that their initial evaluation or treatment had been somehow inadequate. Mining data related to unplanned ED revisits is one method to determine whether this problem can be overcome, and to generate useful guidelines in this regard. In this study, we use the receiver operating characteristic (ROC) curve to compare the data mining model by affinity set to other well known approaches. Some scholars have validated the affinity model for its simplicity and power in handling information systems especially when showing binary consequences. In experimental results, SVM showed the best performance, with the affinity model following only slightly behind. This study demonstrated that when patients visit the ED with normotensive status or smooth breath patterns, or when the physician-patient ratio is moderate, the frequency with which patients revisit the ED is significantly higher.

Keywords: Revisit, Emergency Department (ED), Data Mining, Affinity Set, Multi-objective.

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1 Introduction

Emergency physicians are expected to diagnose diseases accurately and efficiently. However, in fast-paced situations, time limitations and dynamic changes in the number of patients awaiting treatment lead to the inevitable risk of diagnostic error, by the simple fact that seemingly insignificant symptoms can be overlooked (Aaland, Smith, 1996; Brooksa, Holroydb, Riley, 2004; Kohn, Corrigan, Donaldson, 1999; Leape et al., 1991). Ignorance of such details could lead to a higher frequency of patients revisiting emergency rooms. Because emergency departments (EDs) are required to assume ever greater responsibilities, public interest in the quality of service they provide is increasing (Furnival, Woodward, Schunk, 1996; Hanlon, Pickette, 1979). Unscheduled revisits to EDs are known as audits of emergency care quality. Unscheduled revisits are commonly defined as patients presenting for the same chief complaint within 72 hours of discharge from the ED. A rate of less than 1% has been proposed as acceptable quality care (Wu et al., 2008). Unscheduled revisits are a reflection of ED performance, and the underlying causes must be investigated. A number of doctors have proposed traditional statistical methods to deal with this issue. Pierce et al. (1990) began an investigation into this important issue in 1990, followed by Hu (1992), Gordon et al. (1998). Recently, Wu et al. (2008) used the categorical analysis of patient revisits to the emergency department, in which age, sex, final discharge, reason for revisit, and the symptoms of most common complaints were calculated from 34714 records. Nuñez et al. (2006), studied 250 cases and 250 controls from the ED. The measured outcomes were unscheduled returns, post-ED destination, and patient dissatisfaction. They concluded that unscheduled returns were associated with medical errors in prognosis, treatment, follow-up care, and information. Marcantonio et al. (1999) performed a matched case control study among patients who had been admitted to an academic hospital in a Medicare managed care plan. The patients were aged 65 years or older and had been readmitted to the hospital as emergency cases within 30 days of discharge. They suggested that interventions, such as improved discharge education programs, could reduce unplanned readmission. However, most of the above studies applied traditional categorical analysis to the statistics, and tended to agree that revisits are generally illness-related. Further studies are needed to identify the most common and the most serious contributing factors related to revisits, to determine whether improvements can be made.

Early in 2004, Freitas (2004) reviewed the basics of multi-objective optimization for data mining, and suggested these optimization techniques are appropriate in data mining. Recently in 2012, Corne et al. (2012) proposed similar ideas for integrating multi-objective programing in supporting vector machines

(SVMs) (Cortes, Vapnik, 1995), decision trees (Abu-Hanna, Keizer, 2003), neural networks (Zbikowski, Hunt, ed., 1996) etc. These previous efforts validate the feasibility of using multi-objective optimization for mining big data. However, there are still limited multi-objective applications devoted to this area in addition to the popular evolutionary/soft methods (Freitas, 2008).

In this study we eschewed traditional statistical analysis, and employed a number of popular data mining techniques (Aguilar-Ruiz, Costa, Divina, 2004; Berman, 2002; Grupe, Owrang, 1995) to analyze collected clinical data of EDs rather than evolutionary/soft approaches. We adopted neural networks (Zbikowski, Hunt, 1996), rough sets (Rosetta) (Pawlak, 1991), SVM, decision trees, association rules (Delgado et al., 2001) and logistic regressions (Collett, 2003; Delen, Walker, Kadam, 2005). All of them are applied to uncover the relationship between causes and consequences of ED revisits. The affinity models has been validated/tested by a number of scholars (Alanazi, Abdullah, Larbani, 2013; Chen et al., 2009; Esfandiaria et al., 2014; Larbani, Chen, 2009; Michnik, Michnik, Pietuch, 2008; Paoin, 2011; Wu et al., 2009) in the areas of medicine and finance. In this study, a multi-objective affinity model was originally proposed to construct the *k*-core, presenting a number of advantages over the other data mining models evaluated in this study.

This paper is organized as follows: Section 2 introduces the basic concepts and definitions of affinity sets, and proposes the basic data-mining model of affinity. Section 3 reviews the popular data mining models and summarizes their advantages and disadvantages. Section 4 presents the multi-objective affinity model of data mining. Section 5 takes the actual samples of revisiting patients from Kaohsiung Medical University Hospital of Taiwan, to validate the data mining concept using our multi-objective affinity model, to identify the key factors in the high frequency of patient revisits. In addition, we compare the performance of multi-objective affinity model and other popular data mining models, according to the receiver operating characteristic (ROC) curve (Zweig, Campbell, 1993). Finally, in Section 6, we present our conclusions and recommendations based on the data mining results.

2 Preparation for Study

First, we review the basic concepts and definitions of affinity, as well as its potential use in data mining (Chen et al. 2009; Larbani, Chen, 2009; Michnik, Michnik, Pietuch, 2008; Wu et al., 2009). Interestingly, the word of affinity is popularly used in the chemical/medical/social field with various definitions. In chemical physics, chemical affinity is the electronic property by which dissimilar chemical species are capable of forming chemical compounds (Matejtshuk, 1997). In medicine, affinity is mentioned with various biomedical definitions, such as affinity membranes for the removal of endotoxins (Wei et al., 2002) and the immune system (Achenbach et al., 2004). A number of scholars have applied the biometric concept to soft computing where they used the affinity function to develop artificial immune systems (Hunt, Cooke, 1995). In social sciences, scholars give affinity a different meaning: affinity is characterized by high levels of intimacy and sharing, usually in similar groups, also known as affinity groups (Cattell, 2001; Ve-McConnell, 1999). Marketing managers believe that people are more likely to buy brands that affinity groups like. In this manner, they are able to track consumer behaviour according to the social interaction of affinity (Zinkhan, 2002).

Based on the various definitions of affinity given above, we concluded that no formal framework or theory dealing with affinity as a unified concept have been developed, and few researchers have discovered that the basic idea of affinity could be used to provide models valuable in information sciences. Fuzzy set theory is among the best tools for representing vague and imprecise concepts (Zadeh, 1965); however, a type of membership function is necessary in fuzzy sets. In this paper, we use the well known concept of closeness or distance between any two objects in topology to represent affinity and develop a data mining model. Due to its general nature, this new relationship theory, affinity set theory, is able to describe the degree of similarity between objects, and represent general relationships between objects, such as closeness, belongingness, equivalence, which enable decision makers to use this simple concept for modeling. The affinity set theory has been recently introduced in (Larbani, Chen, 2009). For further details we refer the reader to (Larbani, Chen, 2009).

2.1 Basic Definitions

We introduce the definition of an affinity set.

Definition 2.1
An affinity set consists of any two object (real or abstract) that create affinity.

Definition 2.2
Let e be a subject and A an affinity set. Let W be a subset of X⊆U. The affinity between e and A is represented by the function:

$$aff_{\mathrm{A}}^{e}\,(\,.\,)\colon \mathrm{W} \to [0,1] \tag{1}$$

$$w \to aff_{\mathrm{A}}^{e}(w)$$

The value $aff_{\mathrm{A}}^{e}(w)$ expresses the degree of affinity between subject e and affinity set A with respect to variable w. When $aff_{\mathrm{A}}^{e}(w) = 1$ this means that the affinity degree of e with affinity set A is at the maximal level with respect to variable w; but $aff_A^e(w) = 1$ does not mean that e belongs to A, unless the affinity measure $aff_A^e(w)$ is the degree of belongingness. When $aff_A^e(w) = 0$ this means that e has no affinity with A with respect to variable w. When $0 < aff_A^e(w) < 1$, this means that e has partial affinity with A with respect to w. Here we emphasize the fact that the notion of affinity is more general than the notion of membership or belongingness: the latter is just a particular case of the former.

Definition 2.3
The universal set, denoted by U, is the affinity set representing the fundamental principle of existence. We have:

$$aff_U^e(\,.\,): U \rightarrow [0,1] \qquad (2)$$
$$w \rightarrow aff_A^e(w)$$

and $aff_U^e(w) = 1$, for all existing objects with respect to w.

In other words the affinity set defined by the affinity "existence" has complete affinity with all previously existing objects, that exist in the present, and that will exist in the future. In general, in real-world situations, a traditional referential set S, such that for objects e not in S, $aff_A^e(w) = 0$ for all $w \in$ W, can be determined. In order to make the notion of affinity set operational and for practical reasons, in the remainder of the paper, instead of dealing with the universal set U, we only discuss affinity sets defined on a traditional referential set S. Thus, in the remainder of the paper when we refer to an affinity set, we assume that sets S and W are given.

Definition 2.4
Let A be an affinity set. Then the function defining A is:

$$F_A(.,.): S \times W \rightarrow [0,1] \qquad (3)$$
$$(e, w) \rightarrow F_A(e, w) = aff_A^e(w)$$

An element in real-life situations often belongs to a set for some variables and does not for other variables. Such behavior can be represented using the notion of an affinity set. The behavior of affinity set A over time can also be investigated through its function $F_A(.,.)$.

Interpretation 2.1

i) For a fixed element e in S, the function (3) which defines affinity set A reduces to the fuzzy set describing the variation of the degree of affinity of the element e over variable w.

ii) For a fixed w, the function (3) reduces to a fuzzy set defined on S that describes the affinity between elements S and affinity set A with respect to variable w. Roughly speaking, it describes the shape or "content" of affinity set A with respect to w.

iii) In addition to **i)** and **ii)**, we cannot say or check that an affinity set is a special fuzzy set, unless we can prove that any affinity set A is contained in a fuzzy set B, and vice versa.

Definition 2.5

Let A be an affinity set and $k \in [0,1]$. We say that an element e is in the t-k-Core of affinity set A with respect to w, denoted by w-k-Core(A), if $aff_{\mathrm{A}}^{e}(w) \geq k$, that is:

$$w - k - Core(\mathrm{A}) = \{e \mid aff_{\mathrm{A}}^{e}(w) \geq k\} \tag{4}$$

when k = 1, w-k-Core(A) is called simply the core of A with respect to w, denoted by w-Core(A). In addition, w-k-Core(A) ≡ k-Core(A(w)).

Definition 2.6

A life range is defined as the continuous or discrete mapping from the behavior of an element e of S to an affinity set A with respect to w: an illustration of the continuous case is given in Figure 1 below. However, a discrete case for v is also possible.

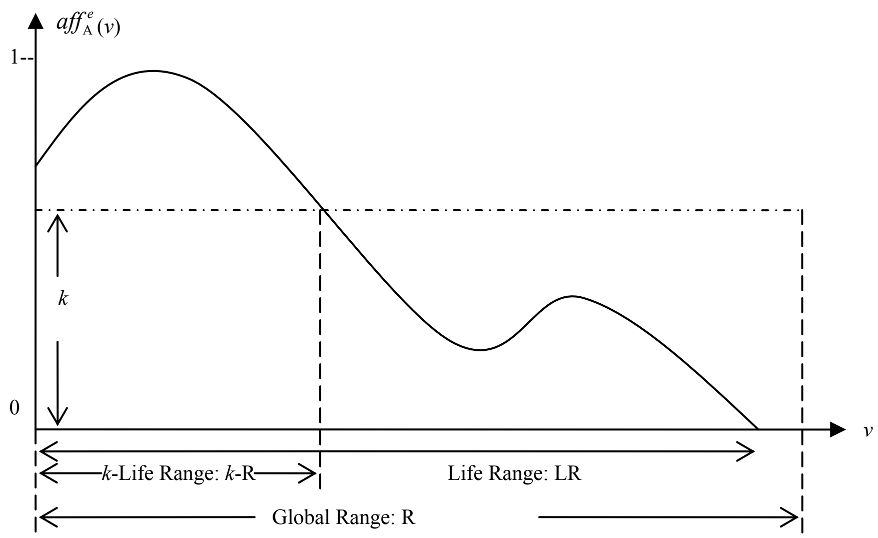

Figure 1. Illustration of the affinity between an element e and an affinity set A over a Global Range R (Continuous Case of v)

Here k-life range is the variable set: $\{v|$ for all $v \subseteq w$ such that $aff_{\mathrm{A}}^{e}(v) \geq k\}$; similarly, life range is the variable set $\{v|$ for all $v \subseteq w$ such that $aff_{\mathrm{A}}^{e}(v) \geq 0\}$.

The intersection and union operations on affinity sets are defined as follows.

Definition 2.7
The intersection of affinity sets A and B with respect to variable w, denoted by $\mathrm{A} \cap \mathrm{B}$, is defined by the function $\mathrm{F}_{\mathrm{A} \cap \mathrm{B}}(e, w) = aff_{\mathrm{A} \cap \mathrm{B}}^{e}(w) = \mathrm{Min}\{aff_{\mathrm{A}}^{e}(w), aff_{\mathrm{B}}^{e}(w)\}$, for all e in S. If A and B are considered over W, then $\mathrm{A} \cap \mathrm{B}$ is defined by the function:

$$\mathrm{F}_{\mathrm{A} \cap \mathrm{B}}(e, w) = aff_{\mathrm{A} \cap \mathrm{B}}^{e}(w) = \mathrm{Min}\{aff_{\mathrm{A}}^{e}(w), aff_{\mathrm{B}}^{e}(w)\}, \text{ for all } e \text{ in S and all } w \in \mathrm{W}.$$

Definition 2.8
The union of A and B with respect to variable w, denoted by $\mathrm{A} \cup \mathrm{B}$, is defined by the function $\mathrm{F}_{\mathrm{A} \cup \mathrm{B}}(e, w) = aff_{\mathrm{A} \cup \mathrm{B}}^{e}(w) = \mathrm{Max}\{aff_{\mathrm{A}}^{e}(w), aff_{\mathrm{B}}^{e}(w)\}$, for all e in S. If A and B are considered over W, then $\mathrm{A} \cup \mathrm{B}$ is defined by the function $\mathrm{F}_{\mathrm{A} \cup \mathrm{B}}(e, w) =$
$= \mathrm{Max}\{aff_{\mathrm{A}}^{e}(w), aff_{\mathrm{B}}^{e}(w)\}$, for all e in S and all $w \in \mathrm{W}$.

2.2 Affinity Data Mining

A static data mining model is proposed by using the basic theory of affinity.

Definition 2.9. Let V be a referential set endowed with distance $d(x, y)$, i.e. (V, d) is a metric space (Chen, 2009). Let X be a subset of V. The affinity set A in X is given by:

$$\mathrm{A} = (d', \mathrm{B}, \mathrm{X})$$

where d' is defined by:

$$d' : \mathrm{X} \to [0,1]$$
$$e \to d'(e, \mathrm{B}) = 1 - \alpha\, d(e, \mathrm{B})$$

where d is the affinity, the set B is called the core of the affinity set A, $d(e, \mathrm{B})$ is defined by:

$$d(e,\mathrm{B}) = \min_{z \in B} d(e, z)$$

Note that there is a difference between $d(e, \mathrm{B})$ and $d(x, y)$, although the same notation "d" is used. Indeed, $d(e, \mathrm{B})$ is the distance between an element e of X and the subset B of X, while $d(x, y)$ is the distance between two elements x and y of X. Note, that these two notions are different. Let $\alpha = \dfrac{1}{\max\limits_{(x,y) \in \mathrm{X} \times \mathrm{X}} d(x, y)}$, that is, α is the inverse of the maximal distance between elements of X.

Procedure 2.1

1) Define the affinity set A, determine the referential set V and define the metric space (V, d).
2) Determine the set X.
3) Choose a subset B of X which is a candidate for being the core of the affinity set A.
4) Use the affinity d' defined by:

$$d' : \mathrm{V} \rightarrow [0,1]$$
$$e \rightarrow d'(e, \mathrm{B}) = 1 - \alpha d(e, \mathrm{B})$$

to compute the k-core (A) when, once the value of k is given. Now we present an example illustrating how this idea works.

Example 2.1. Data Mining

Table 1: Sample Data of Patients

Sample	x_1 (Fever)	x_2 (Vomiting)	y (Death)
P_1	0	1	1
P_2	1	0	1
P_3	1	0	0
P_4	0	1	1
P_5	1	0	0

Here we assume that doctors have observed two symptoms for one new disease: one is "Fever", the other is "Vomiting", and they possibly lead to the death of patients. We collect the data of five patients, as in Table 1, using binary values to indicate whether these symptoms exist or not in each case. The input variables are "Fever" and "Vomiting". The output variable is "Death". For example, for the first patient P_1 it is observed that he/she is vomiting and finally he/she dies; for the second patient P_2 it is observed that he/she has fever and finally he/she dies,…, etc. Therefore, what meaningful conclusions can be derived from these cases by the affinity model? First, we denote a rule by a triple $r = (x_1, x_2, y)$, then use Procedure 2.1:

1) Define the metric space (V, d). Define the referential set V as the set of all guesses/rules that can be used to identify the disease. Distance d is the failure (inaccurate prediction) rate of a rule (a distance concept), defined as the failure frequency of rule; d' is used to present the hit rate of the rule and $d' = 1 - \alpha d$. The hit rate is defined as the frequency of accurate prediction divided by the number of samples observed. According to Definition 2.13, d' is used to measure the degree of affinity of rules.

2) Determine the referential set X. The referential set X $=\{r_i,\ i=\overline{1,m}\}$, is a subset of V, the set of all possible rules/guesses completing the vector space to three dimensions. All the attributes are binary as shown in Table 1, i.e., $r =$ $= (x_1, x_2, y)\in X$, $x_1\in\{0,1\}$, $x_2\in\{0,1\}$ and $y\in\{0,1\}$. Because we use binary values here for attributes, only eight combinations/guesses can be generated with respect to three discrete attributes. Each rule $r_i \in X$, $i=\overline{1,8}$ competes for better affinity with respect to affinity set A, which is the set of rules capable of predicting the consequence of disease at the fixed time.

3) Choose subset B of X as the core of affinity set A. We choose B as the set containing the rules with the maximal hit rate.

4) Use affinity d' as defined:

$$d' : X\to[0,1]$$
$$e \to d'(e, B) = 1-\alpha d(e, B)$$

Finally, compute the hit rate (degree of affinity) of each rule in X, and select k for the k-Core (A). Because guesses/rules are limited to eight combinations, by simultaneously considering three attributes, we summarize the degree of affinity for each rule (r_i) as follows:

r_1 : if $x_1 = 1$ and $x_2 = 1$, then $y = 1$, miss rate = 5/5 , hit rate (affinity degree) = 1 – 5/5 = 0
r_2 : if $x_1 = 1$ and $x_2 = 1$, then $y = 0$, miss rate = 5/5, hit rate (affinity degree) = 1 – 5/5 = 0
r_3 : if $x_1 = 1$ and $x_2 = 0$, then $y = 1$, miss rate = 4/5, hit rate (affinity degree) = 1 – 4/5 = 1/5
r_4 : if $x_1 = 1$ and $x_2 = 0$, then $y = 0$, miss rate = 3/5, hit rate (affinity degree) = 1 – 3/5 = 2/5
r_5 : if $x_1 = 0$ and $x_2 = 1$, then $y = 1$, miss rate = 3/5, hit rate (affinity degree) = 1 – 3/5 = 2/5
r_6 : if $x_1 = 0$ and $x_2 = 1$, then $y = 0$, miss rate = 5/5, hit rate (affinity degree) = 1 – 5/5 = 0
r_7 : if $x_1 = 0$ and $x_2 = 0$, then $y = 1$, miss rate = 5/5, hit rate (affinity degree) = 1 – 5/5 = 0
r_8 : if $x_1 = 0$ and $x_2 = 0$, then $y = 0$, miss rate = 5/5, hit rate (affinity degree) = 1 – 5/5 = 0

After computation, we obtain the 0.2-core(A) = $\{r_3, r_4, r_5\}$; if $k = 0.4$, then the 0.4-core(A) = $\{r_4, r_5\}$. If a rule/guess, for instance, $r = (x_1, x_2, y)$ (or r_i) is capable of hitting the observed samples with a higher frequency (i.e., lower frequency of missing), then $r = (x_1, x_2, y)$ or r_i, has a greater degree of affinity with A, or rule r_i is useful/valuable to explain the behavior of the samples collected/observed. Thus, if we set $k = 0.4$, we can easily determine the 0.4-core(A) by two rules: Rule 4 tells that the $x_1 = 1$ (Fever) is not fatal, but Rule 5 warns the doctors that the $x_2 = 1$ (Vomiting) caused by this new disease could kill a patient. Of course, as the sample size increases, and as the variety of these qualitative attributes increases, using such simple thinking can approximate any affinity set A.

Readers may be confused about the difference between our affinity data-mining model and the model of association rules (Brossette et al., 1998); however, these two models are significantly different because: (a) a model of association rules uses the support and confidence of conditional probability to mine useful rules, but an affinity model uses the subjectively defined closeness occurrence frequency of rules; (b) an affinity model assumes that, for instance, $r = (x_1, x_2, y)$ is a vector in a metric/vector space, but the model of association rules does not make this assumption, and, more importantly, (c) it is possible to use various definitions in an affinity model in order to measure the degree of affinity. In this manner, it is not only possible, but easy to define the closeness between any two rules, or the distance from a rule to a specified group/set for further use without statistical restrictions.

3 Popular Data Mining Models

In this section, we present a brief review of several data mining models popularly used in medicine. These models include neural network (NN), rough set (Rosetta), support vector machine (SVM), decision tree (DT), association rule (AR) and logistic regression (LR). The LR model is popularly used in traditional statistical analysis in medicine (Delen, Walker, Kadam, 2005; Lavarc, 1999).

The amount of data collected and stored in medical databases has dramatically increased, due to advancements in automated data collection, and traditional data analysis techniques are no longer adequate for this volume of data (Brossette et al., 1998; Burke et al., 1997). For this reason, a number of non-traditional techniques have been developed to represent these values. For example, Delen et al. (2005) used artificial neural networks (ANN), decision trees (DT) and logistic regression (LR) to predict the survivability of breast cancer, concluding that ANN and DT both performed better than LR. Chang and Chen (2009) also used DT in combination with NN for skin diseases with prediction accuracy as high as 92.62%, which also outperformed LR. The rough set is another powerful model in this field (Pawlak, 1991). Wilk et al. (2005) described a rough methodology used for identifying the most relevant clinical features and for generating decision rules based on selected attributes from a medical data set with missing values. These rules could help (ER) medical personnel in the triage (initial assessment) of children with abdominal pain. Hirono and Tsumoto (2005) introduced a rough representation of a region of interest (ROI) in medical images. The main advantage of this method was its ability to represent inconsistencies between the knowledge-driven shape and image-driven shape of an ROI. As for the SVM, Meyfroidt et al. (2009) proposed a general overview of machine learning techniques, with a more detailed discussion of a number of these techniques to encouraging doctors to use them. They also provided guidance for applications and directions of research for SVMs. When using SVM to predict the depth of infiltration in endometrial carcinoma based on transvaginal sonography

(Spackman, 1991), SVMs were more effective than logistic regression. Bazzani et al. (2001) used an SVM classifier to distinguish false signals from microcalcifications in digital mammograms. The SVM classifier performed slightly better than a classifier implemented using an ANN. Van Gestel et al. (2004) compared least squares SVMs with DT, Naive Bayes, and LR for the classification of 20 benchmark datasets. They reported that SVMs exceeded the other methods in most of the datasets and were not significantly worse in the remaining datasets.

Decision trees (DTs) and association rules (ARs) are other valuable tools in medical data mining. For example, Mugambi et al. (2004), addressed this issue using a novel hybrid multivariate decision tree comprising polynomial, fuzzy and decision tree structures. As for the association rules method, Delgado et al. (2001), introduced a new fuzzy approach to association rules among quantitative values in relational databases. These fuzzy association rules were more informative than rules related to precise values. They also introduced a new means to measure accuracy, and claimed that their work was more understandable and appropriate than typical systems. Kuo and Shih (2007) applied an ant colony system (ACS) to a large database of health insurance to derive association rules, and showed that the newly proposed method was able to provide more condensed rules than an *a priori* method. Computation time was also reduced. In addition, the LR model is commonly used in medicine; for example, Spackman (1991), Tu (1996) and Doig et al. (1993) all used LR models in their studies. However, the performance of LR was inferior to that of NN models.

To summarize, the above data mining models made considerable contributions to overcoming the problems associated with data mining. We simply compare the aforementioned models as in Table 2 for their advantages and disadvantages.

Table 2: Comparison of Data Mining Models

Characteristics/Models	SVM	NN	DT	LR	AR
Advantages	The prediction power is very strong	The graphical construction of model is clear	It is easy to use and explain	It is easy to use and explain	It is easy to catch the relationship between causes and consequences
Disadvantages	It is difficult to describe the clear rules between causes and consequences	It is difficult to describe the clear rules between causes and consequences	It is difficult to group and cluster when data are huge	The explanatory power is weak if the data do not follow the statistical assumptions	The explanatory power is weak if the data do not follow the statistical assumptions

Next, we compare the performance of the affinity model with that of the aforementioned models. The challenge for all of the data mining models is in the fact that the sample size was not large (only 645 units), and no statistical distribution was pre-assumed for the data.

4 Multi-objective Affinity Model for Data Mining

In this study, Step 4 in Procedure 2.1 was extended to consider multi-objectives of affinity. In Procedure 2.1, it was logical and reasonable for the decision maker to select the value of k first; for example, Michnik et al. (2008) proposed a similar idea using the iterated algorithm to find the final k-core(A). However, it was not easy to operate in this manner for most actual cases, and selecting the value of k at the beginning is a particularly difficult task for inexperienced decision makers. Early in 2006, Wu et al. (2009) used a multi-objective affinity classification system comparing ant colony optimization (ACO) in the classification of delayed diagnostics, and concluded that the multi-objective affinity set classification system was superior to the ACO system. Their fitness function of two objectives: z_1, z_2 is as follows (Wu et al., 2009):

$$f(z_1, z_2) = w_1 \times (N - z_1) + w_2 \times z_2 \tag{5}$$

where:

z_1 − number of rules in a subset, $z_1 < N$;
z_2 − prediction accuracy of rules in a subset;
N − maximal number of rules in a subset predetermined by the decision maker;
w − weight of objective predetermined by the decision maker.

In the above paper, Wu et al. (2009) used the weighing objective function (5) to rank the appropriate subset of rules by setting $w_1 = w_2 = 0.5$. Because z_1 and z_2 were not in the same scale, the performance of z_1 could be over-emphasized. In addition, Chen et al. (2009) used multi-objective ideas rather than selecting the value of k, and separated the data set into a training set and validation set, proposing two criteria to select the final k-core(A): one was that each rule had to include at least two causes (x), the other was that the rule base had to be able to catch the validation set 100% of the time. Thus, $\mathrm{M}_{\mathrm{A}}^{e}(w^0) \geq 0.247$ or $k = 0.247$ were finally achieved.

The study of Wu et al. (2009) did not demonstrate the potential power of multi-objective affinity classification system, which inspired us to compare the multi-objective affinity model with many traditional data mining methods. Furthermore, our fitness function for ranking the subset of rules was based on affinity, on which values ranged from 0 to 1 (normalized). This study extended and modified the research of Chen et al. (2009) and Wu et al. (2009) to a multi-objective problem (Steuer, 1986). We assumed that a decision maker is unable to select the value of k in the beginning, but has multiple goals to form the k-core (A). For example, he/she may want to minimize the size of the k-core(A), i.e., the number of rules is decreased, but desires the prediction accuracy of the k-core(A) to remain high. In such situations, there are conflicts between two goals, in attempting to minimize the number of rules while maximizing the prediction accuracy of the rule base. Each rule set presents a possible feasible solution, and each rule set plays the role of set B in Procedure 2.1. In this case, B is evaluated by its objective of minimizing the number of rules and simultaneously maximizing the prediction accuracy. In Section 3, these two objectives are clearly defined according to their affinities. To achieve this, the affinity d in Step 4 of Procedure 2.1 is newly defined by integrating the affinities of the aforementioned two objectives.

The following is used to illustrate our new multi-objective approach to computing d' in step 4 of Procedure 2.1. First, an initial rule set C of the best 100 rules with highest affinities is prepared by Procedure 2.1. Here, too, we use the idea of Example 2.1. If rule r_i is found in the training set once, then its corresponding affinity degree is one divided by the size of the training set; if rule r_i is found in the training set twice, then its corresponding affinity degree is two divided by the size of training set, and so on. The degree of affinity for a rule in the training set is used as the prediction reference for the validation set, which is denoted by aff_{r_i} in the following. It is logical to say that if a rule is frequently found in the training set, then it has a higher degree of prediction power for the validation set and should be kept in C. Second, assume set B is randomly generated and B⊆C. B is chosen to approximate the final core of affinity set A. If the size of B, i.e., the number of rules in B, is $norm$(B), then our first affinity d_1' is defined as follows:

$$d_1' = \min_{r_i \in \mathrm{B}}\left[\frac{aff_{r_i}}{norm(\mathrm{B})}\right] \tag{6}$$

Third, we assume that the decision maker expects the number of rules in the final core to be small, but he hopes that it will contain at least fifteen rules. When the number of rules is more than fifteen, his satisfaction is reduced. Thus, we can simply define the second affinity d_2' as follows:

$$d_2' = \frac{15}{norm(\mathrm{B})} \tag{7}$$

Here $norm$(B) is the size of B and 15 ≤ $norm$(B) ≤ 30 is assumed in this study. Thus, the new d' is defined as the well-known weighted function in multi-objective programming theory (Steuer, 1986):

$$d' = w_1 d_1' + w_2 d_2' \tag{8}$$

where $w_1 + w_2 = 1$ and $w_1, w_2 \geq 0$. The weights of (w_1, w_2) are selected subjectively at the beginning. According to the new definitions above and Procedure 2.1, the iteration steps of this study are as follows:

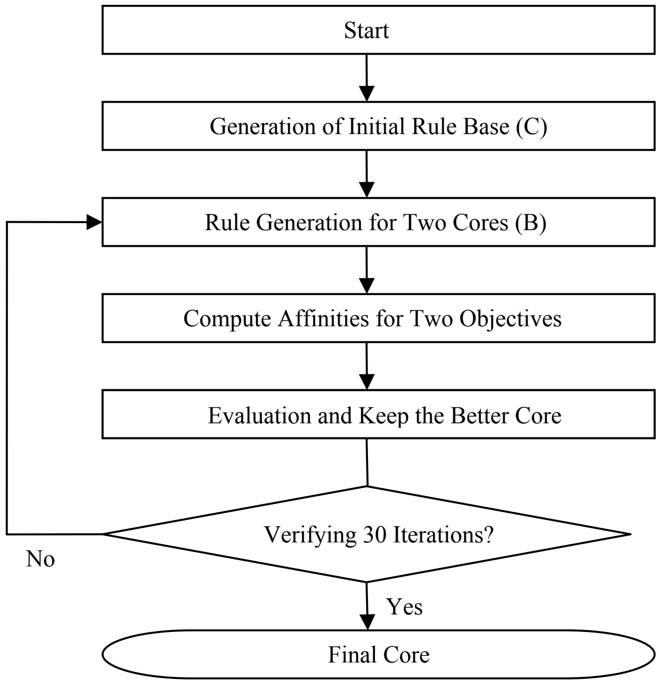

Figure 2. Process of Data Mining using the Multi-objective Affinity Model

Step 0. Subjectively set the pair (w_1, w_2). In this study, w_2 is set to 0.6 and w_1 is set to 0.4. This means we emphasize fewer rules to catch more observations. This is the *Start* stage.

Step 1. Separate the sample data into two parts; for example, 80% of data are used for training and 20% for validation. At the same time, aff_{r_i} for each rule r_i is computed in this stage and Procedure 2.1 is followed exactly to implement this step. We set a threshold to generate the initial rule base C: although thousands of rules are generated by Procedure 2.1, only the rules with the top 100 affinities are retained. This is the stage of *Generation of Initial Rule Base*.

Step 2. Randomly generate two rule sets, for instance, $B_1, B_2 \subseteq C$, to approximate the core (A). Each rule r_i in B_j, $j = 1, 2$ has its causal part (x) and consequence part (y). The size of B_j, i.e, $norm(B_j)$ is also different for each rule set, but it is included between 10 and 30. Only two cores are generated at the beginning. This is the stage of *Rule Generation for Two Cores*.

Step 3. Apply Equation (4) to compute the minimal degree of affinity d_1' for each B_j, and apply Equation (5) to compute d_2': the satisfaction felt by the decision maker with the size of B_j. After that, $d' = w_1 d_1' + w_2 d_2'$ defined in Equation (8) is used to evaluate each B_j. In this case, B_j, j = 1, 2, subsets of X, are chosen as candidates for being the core of affinity set A (where core(A) is that set B for which d' = 1). This is the stage of *Computing Affinities for Two Objectives*.

Step 4. Keep only that B_j for which *d'* is largest in Step 3 and return to ***Step 2*** to generate another B. This is the stage of *Evaluation and Keeping the Better Core*.

Step 5. Repeat the steps 1–4 until the predetermined number of iterations has been reached. Here the number of iterations is set to 30. This is the stage of *Verifying 30 Iteration*s.

Step 6. If 30 iterations are reached in ***Step 5***, then output B as the approximated core of A. This is the stage of *Final Core*.

Using these steps, 645 samples were used for training the neural network, rough set model (Rosetta), supporting vector machine (SVM), decision tree, association rule, logistic regression and the multi-objective affinity model: the performance of these models is compared in Section 5.

5 Actual Example

The objective of this research was to identify the core attributes leading to frequent revisits of emergency patients in ED within a set period of time; simply speaking, doctors expect generating useful rules for avoiding revisits. The study uses the original data from the website of Kaoping Area Medical Emergency Response Alliance (KAMERA). This site is the largest site in Taiwan for collecting trauma data of patients by more than 30 hospitals joining in an alliance. Doctors presented 645 samples of clinical data from 2008 (from Jan. to Dec.), and the samples were divided into two parts: the training set and the validation set. The training-validation ratio was established as 80%-20%, 70%-30% and 60%-40% of the data. The training set was used to derive rules from various data mining models and the validation set was used to draw the ROC curve to compare the performance of each model. On the basis of the availability of data retrieved from electronic medical records, physicians suggested nine possible influential attributes/causes $\{x\}$ leading to emergency patient revisits of (y); age (x_1), triage status (x_2), healthcare provider (x_3), time of visit (x_4), length of ED stay (x_5), breathing pattern (x_6), blood-pressure (x_7), pulse rate (x_8), physician-patient ratio, (x_9) and revisiting frequency (y), as shown in Table 3. The physician-patient ratio was defined as the number of on-duty physicians divided by the number of the patients in the ED within an 8-hour shift.

Table 3: Attributes of the Data Mining Model

Attributes	Interval	Coding
Age (x_1)	0-8	1
	9-18	2
	19-40	3
	41-65	4
	Over 66	5
Triage status (x_2)	Level 1 (Severe)	1
	Level 2 (Moderate)	2
	Level 3 (Mild)	3
Healthcare provider (x_3)	Pediatric emergency	1
	Emergency medicine	2
	Surgical emergency	3
	Others	4
Time of visiting (x_4)	00:00-08:00	1
	08:00-16:00	2
	16:00-24:00	3
Length of ED stay (x_5)	0-4 hours	1
	4-8 hours	2
	8-12 hours	3
	Over 12 hours	4
Breath pattern (x_6)	Normal	1
	Abnormal	2
Blood-pressure (x_7)	Normal	1
	Abnormal	2
Pulse rate (x_8)	Normal	1
	Abnormal	2
Physician – patient ratio (x_9)	High (1~1/20)	1
	Moderate (1/20~1/40)	2
	Low (Under 1/40)	3
Revisiting frequency (y)	One time	0
	More than one time	1

Note: the index of medical capacity is defined as the number of the available doctors divided by the number of the patients in ED.

The referential set X is defined as the vector space with the dimensionality of ten and attributes are discrete as in Table 3, $r = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, y) \in X$ by Definition 2.10. The value of each x_i (i = 1,2, … ,9) and y were randomly selected from the attribute domain in Table 3. If any x_i (i = 1,2, … ,9) had a value of zero, then this means that the corresponding attribute x_i would not be considered in the formation of rules.

Here, our new model and the popular data mining models above will be tested for their performance using the confusion matrix and ROC curve.

5.1 Confusion Matrix and ROC Curve

We employed the confusion matrix (Collett, 2003) to compare the performance of our multi-objective affinity model and of other popular data mining models. In artificial intelligence, particularly for the binary consequences of information systems, a confusion matrix is a visualization tool typically used in supervised learning. Each column of the matrix represents instances in a predicted class, while each row represents instances in an actual class. One benefit of a confusion matrix is that it is easy to observe whether the system is confusing two classes (i.e. commonly mislabeling one as another). For example, the following Table 4 shows the confusion matrix for a two-class classifier. The entries in the confusion matrix have the following meaning in the context of our study: *a* is the number of correct predictions that an instance is negative, *b* is the number of incorrect predictions that an instance is positive, *c* is the number of incorrect predictions that an instance is negative, and *d* is the number of correct predictions that an instance is positive (Collett, 2003).

Table 4: Confusion Matrix

		Predicted	
		Negative	Positive
Actual	Negative	*a*	*b*
	Positive	*c*	*d*

Several standard terms should be defined for this matrix:

- *Accuracy* (*AC*) is the proportion of the total number of predictions that were correct. It is determined using the equation:

$$AC = \frac{a+d}{a+b+c+d}$$

- The *recall* or *true positive rate* (*TP*) is the proportion of positive cases that were correctly identified, as calculated using the equation:

$$TP = \frac{d}{c+d}$$

- The *false positive rate* (*FP*) is the proportion of negatives cases that were incorrectly classified as positive, as calculated using the equation:

$$FP = \frac{b}{a+b}$$

- The *true negative rate* (*TN*) is defined as the proportion of negatives cases that were classified correctly, as calculated using the equation:

$$TN = \frac{a}{a+b}$$

- The *false negative rate* (*FN*) is the proportion of positives cases that were incorrectly classified as negative, as calculated using the equation:

$$FN = \frac{c}{c+d}$$

- Finally, *precision* (*P*) is the proportion of the predicted positive cases that were correct, as calculated using the equation:

$$P = \frac{d}{b+d}$$

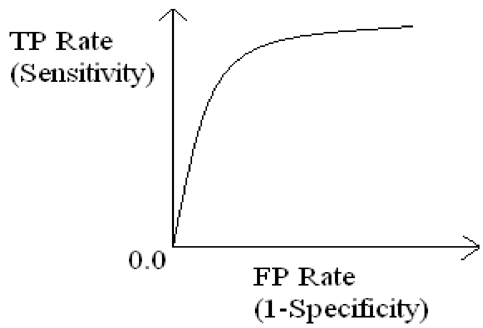

Figure 3. ROC Curve

In addition, once the confusion matrix was prepared, the ROC curve could be easily drawn. The receiver operating characteristic (ROC) curve (Zweig, Campbell, 1993) was used to compare the performance of our affinity model and of other models. In signal detection theory, a receiver operating characteristic (ROC), or simply ROC curve, is a plot of the sensitivity vs. 1 – specificity for a binary classifier system as its discrimination threshold is varied. The ROC can also be represented in the form of TP (true positive), FP (false positive), TN (true negative) and FN (false negative). For example, if a rule predicts that a patient has a high frequency of revisits (positive), and it really happens, then this is a TP case; on the contrary, if it doesn't happen then this is an FP case. The number of TPs and TNs should be reasonably large for a good prediction model. The diagnostic performance of a test or the accuracy of a test to distinguish cases of disease from normal cases was evaluated using ROC curve analysis (Zweig, Campbell, 1993). Receiver operating characteristic (ROC) curves can also be used to compare the diagnostic performance of two or more laboratory or diagnostic tests (Collett, 2003) − see Figure 3. If the plotted ROC curve of a model is more north-west skewed, or the area under the ROC curve is larger, then this model is more beneficial. The confusion matrices and the ROC curves are available in Section 4 for each data mining model.

5.2 Performance of Models

Case I, Case II and Case III show the results of training-validation rates at 80%--20%, 70%-30%, and 60%-40%, respectively. For simplicity, in the following tables, we use MA for the multi-objective affinity model, NN for the neural network model, RS for the rough set model, SVM for the model of supporting vector machine, DT for the decision tree model, AR for the model of association rules and LR for logistic regression model. For the accuracy and TP indeces the larger value the better; while for the FP index the converse is true: the smaller value the better. The ROC curve was used to compare these models in the end.

Case I: Training-validation rate of 80%-20%
The performance of each model for Case I is summarized in the following Tables 5-6.

Table 5: Confusion Matrix of Case I

Actual/Predicted	0	1
0	55(MA), 44(NN), 41 (RS), 62(SVM), 32(DT), 28(AR), 62(LR)	12(MA), 24(NN), 27(RS), 6(SVM), 36(DT), 40(AR), 6(LR)
1	13(MA), 18(NN), 24(RS), 17(SVM), 30(DT), 24(AR), 53(LR)	50(MA), 43(NN), 37(RS), 44(SVM), 31(DT), 37(AR), 8(LR)

Table 6: Performance of Case I

Model	MA	NN	RS	SVM	DT	AR	LR
Accuracy	81.6%	67.4%	60.5%	82.2%	48.8%	50.4%	54.3%
TP	78.7%	70.5%	60.7%	72.1%	50.8%	60.7%	13.1%
FP	17.6%	30.3%	39.7%	8.8%	52.9%	58.8%	8.8%

In the first case, SVM performed best (Accuracy = 82.2%), MA was a little behind SVM (Accuracy = 81.6%). In addition, the decision tree model had the poorest performance (Accuracy = 48.8%).

Case II: Training-validation rate of 70%-30%
The performance of each model for Case II is summarized in the following Tables 7-8.

Table 7: Confusion Matrix of Case II

Actual/Predicted	0	1
0	78(MA), 63(NN), 57(RS), 88(SVM), 45(DT), 43(AR), 3(LR)	16(MA), 31(NN), 37(RS), 6(SVM), 49(DT), 51(AR), 91(LR)
1	20(MA), 34(NN), 40(RS), 26(SVM), 45(DT), 51(AR), 0(LR)	81(MA), 66(NN), 60(RS), 74(SVM), 55(DT), 49(AR), 100(LR)

Table 8: Performance of Case II

Model	MA	NN	RS	SVM	DT	AR	LR
Accuracy	81.6%	66.5%	60.3%	83.5%	51.5%	47.4%	53.1%
TP	79.0%	66.0%	60.0%	74.0%	55.0%	49.0%	100%
FP	16.0%	33.1%	39.4%	6.4%	52.1%	54.3%	96.8%

In the second case, SVM performed best (Accuracy = 83.5%), and MA was still a little behind SVM (Accuracy = 81.6%). In this case, the model of association rules had the lowest accuracy of 47.4%. Furthermore, logistic regression had had uncommonly high TP and FP, which hints that the performance of this model is unstable.

Case III: Training-validation rate of 60%-40%
The performance of each model for Case III is summarized in the following Tables 9-10.

Table 9: Confusion Matrix of Case III

Actual/Predicted	0	1
0	103(MA), 80(NN), 77(RS), 119(SVM), 68(DT), 64(AR), 48(LR)	26(MA), 47(NN), 50(RS), 8(SVM), 59(DT), 63(AR), 79(LR)
1	25(MA), 43(NN), 52(RS), 33(SVM), 51(DT), 62(AR), 48(LR)	106(MA), 88(NN), 79(RS), 98(SVM), 80(DT), 69(AR), 83(LR)

Table 10: Performance of Case III

Model	MA	NN	RS	SVM	DT	AR	LR
Accuracy	82.2%	65.1%	60.5%	84.1%	57.4%	51.6%	50.7%
TP	79.9%	67.2%	60.3%	74.8%	61.1%	52.7%	63.3%
FP	21.0%	37.0%	39.4%	6.3%	53.5%	49.6%	62.2%

In the third case, SVM performed best (Accuracy = 84.1%), followed by MA (Accuracy = 82.2%). Moreover, logistic regression had the poorest accuracy of 50.7%. Finally, the ROC curves and the area under each model are presented in the following, to illustrate the computational results above.

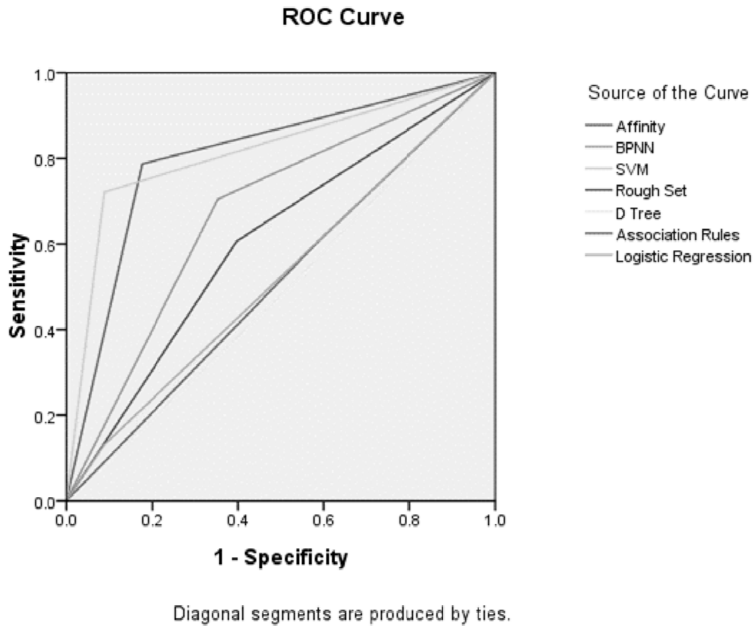

Figure 4. ROC Curve for Training-validation Ratio of 80%-20%

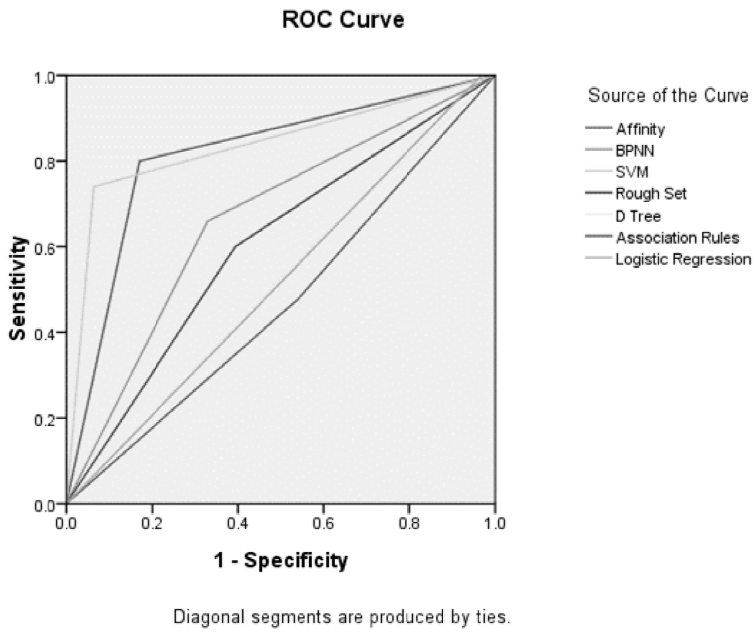

Figure 5. ROC Curve for Training-validation Ratio of 70%-30%

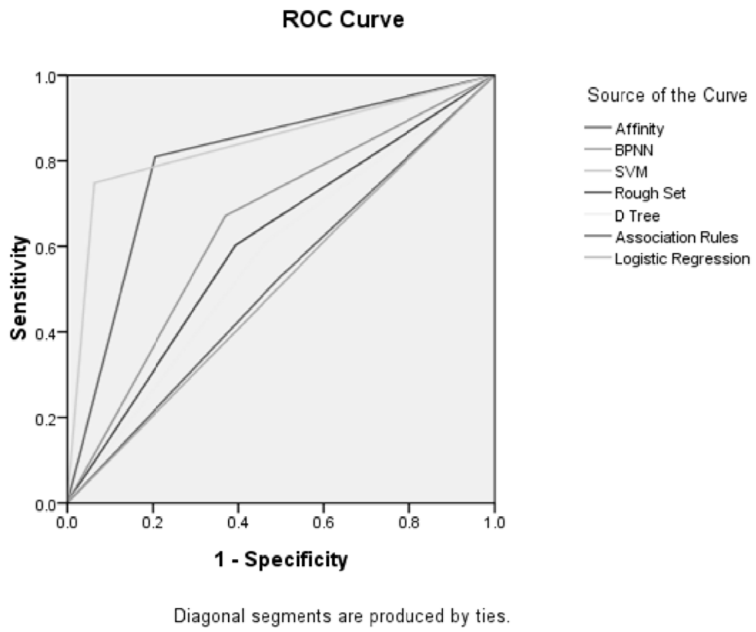

Figure 6. ROC Curve for Training-validation Ratio of 60%-40%

Table 11: Area under each Model of ROC Curves

Model	MA	NN	RS	SVM	DT	AR	LR
Case I	0.810	0.616	0.605	0.817	0.489	0.509	0.521
Case II	0.820	0.665	0.603	0.838	0.514	0.469	0.516
Case III	0.812	0.651	0.605	0.843	0.573	0.515	0.506

According to Table 11, if the area under the curve were larger, then it would have a better classification power. We simply concluded that: SVM $\succ$ MA $\succ$ NN $\succ$ RS, where "$\succ$" means "to be superior to". To summarize, these tables and ROC curves show that the multi-objective affinity model and the SVM model had significant advantages over the other data mining models. Although SVM had the best classification power, considering that the objective of this study was to find rules, we continued using the multi-objective affinity model for further explanations.

The multi-objective affinity model generated seventeen rules with a compromised d' of 0.3757. These rules are summarized in the following table, illustrating the causes leading to a high frequency of revisits.

Table 12: Generated Rules of the Multi-objective Affinity Model

Rule	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y
r_1	-	3	-	-	-	1	1	-	-	1
r_2	-	-	-	-	1	1	1	-	-	1
r_3	-	3	-	-	1	-	-	-	-	0
r_4	-	2	-	-	-	-	-	-	2	1
r_5	-	3	2	-	-	-	-	-	-	0
r_6	-	-	-	-	1	1	-	-	2	1
r_7	-	3	-	-	-	-	-	-	2	1
r_8	-	-	2	-	1	-	-	-	-	0
r_9	-	-	-	-	-	1	1	-	2	1
r_{10}	-	-	2	-	-	1	1	-	-	1
r_{11}	-	-	2	-	-	-	2	-	-	0
r_{12}	-	3	-	-	1	-	1	-	-	1
r_{13}	-	-	-	3	1	-	-	-	-	0
r_{14}	-	3	2	-	-	-	-	-	2	1
r_{15}	-	3	-	-	-	-	2	-	-	0
r_{16}	-	3	-	-	-	-	1	-	2	1
r_{17}	-	3	-	3	-	-	-	-	-	0

Note: "-" means that the corresponding attribute is ignored.

According to Table 12 and the definition of variables in Table 3, we focus on the causes $\{x_i\}$, which lead y to 1. Here x_2 ranges from 2 to 3, x_3 is at most 2, x_5 is at most 1, x_6 is at most 1, x_7 is at most 1 and x_9 is at most 2. Therefore, these rules (grey squares) could be interpreted as follows: if a patient's triage scale (x_2) is two or three, or visiting service (x_3) is in the division of emergency medicine, or stay in the ED (x_5) is less than four hours, or breath pattern (x_6) appears normal, or blood pressure (x_7) is within normal limit, or the physician-patient ratio (x_9) is in the middle level, then the revisiting frequency (y) is high. Interestingly, the mining results of $\{x_6, x_7\}$ above closely match the conclusions in Chen et al. (2009). That is, when the patient looks fine, then his/her frequency of revisiting the ED could be high.

5.3 Discussions

The following discussions are results of brain storming with the physicians using their clinical experiences. According to the results of this study, patients with abnormal blood pressure and breath patterns revisited less frequently. It is commonplace for physicians to pay more attention to patients with unstable vital signs (Aaland, Smith, 1996; Chen et al., 2009) rather than to those patients who appear normal. In such cases, more real-time, comprehensive, continuous and thorough/whole examinations tend to be performed and developed, and their problems are more likely to be addressed adequately during their initial stay in the ED, thereby avoiding possible revisits. On the other hand, the patients triaged as levels 2 or 3 are conventionally termed non-critical patients.

Our results show that a physician-patient ratio at a moderate level is associated with a higher rate of revisits. This could result from the fact that when a physician cares for too many patients, he/she will fail to provide adequate medical service for all of them. Nevertheless, a higher revisit rate was not found in the group with low physician-patient ratio.

To summarize, we propose the following issues:

1) Compared to the level 1 group in triage, groups 2 and 3 are relatively ambulatory, with less severity of illnesses. They might receive less medical treatment with fewer aggressive interventions, resulting in more unplanned revisits. The aforementioned observation tells us that the patient's situation in ED is dynamic and unpredictable, and therefore an innovative, complete and effective process for examining patients is required.
2) The low physician-patient ratio could impair the operational efficiency of the ED, thereby blocking patient's intention to revisit. Having the impression of receiving suboptimal care in the same ED, those patients may seek aid in other hospitals. However, this assumption needs more evidences to prove.
3) Humans are fallible, also in their observations of patients. If the medical personnel (doctors and nurses) is not able to pay full attention to patients in the short run, then a real-time and whole process for examining the vital signs of patients is suggested. Therefore, wearable devices for ED patients could be valuable. We could respond faster and more correctly by continuously monitoring or early alerting these patients to avoid unplanned revisits.

6 Conclusions and Recommendations

The explanatory power of the affinity model is better than that of most of the existing models. However, the data collected in this study regarding revisiting patients may have lacked some important/hidden attributes/features, detracting from the effectiveness of the mining results. The affinity model will certainly be able to provide decision makers with more satisfactory results, once the structural model is further enhanced. Other mapping/projection methods based on affinity may also generate effective rules to overcome problems associated with data mining. It is worth noting that: (a) the affinity model is quite simple, (b) it does not require explicit membership functions (Zadeh, 1965), and (c) it has significantly better performance than existing models. For further research, we propose the application of the affinity model to more complex data mining medical problems and other areas.

Acknowledgements

This research is funded by National Science Council of Taiwan (96-2416-H-212-002-MY2), and Kaohsiung Medical University (Kmuh-96-6G07, Kmuh-98-8G30). Authors would like to appreciate Prof. Chih-Hung Wu, Prof. Jerzy Michnik, Miss Chih-Min Shen and Mr. Zheng-Yen Hsieh for their serious efforts in this area.

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MULTIPLE CRITERIA DECISION MAKING
Vol. 10 2015

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A DECISION RULE FOR UNCERTAIN MULTICRITERIA MIXED DECISION MAKING BASED ON THE COEFFICIENT OF OPTIMISM

Abstract

This paper is devoted to multicriteria decision making under uncertainty with scenario planning. This topic has been explored by many researchers since almost all real-world decision problems contain multiple conflicting criteria and a deterministic criteria evaluation is often impossible.

We propose a procedure for uncertain multi-objective optimization which may be applied when a mixed strategy is sought after. A mixed strategy, as opposed to a pure strategy, allows the decision maker to select and perform a weighted combination of several accessible alternatives.

The new approach takes into account the decision maker's preference structure and attitude towards risk. This attitude is measured by the coefficient of optimism on the basis of which a set of the most probable events is suggested and an optimization problem is formulated and solved.

Keywords: multicriteria decision making, uncertainty, mixed strategy, one-shot decision, scenario planning, optimization model, coefficient of optimism, β-decision rule.

1 Introduction

This paper deals with multiple criteria decision making for cases where attribute (criterion) evaluations are uncertain. This topic has been theoretically and practically investigated by many researchers. Durbach and Stewart (2012) provide an impressive review of possible models, methods and tools used to support uncertain multicriteria decision making (e.g. models with scenarios, models using

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probabilities or probability-like quantities, models with explicit risk measures, models with fuzzy numbers). In this paper we propose a method designed for multicriteria decision making with scenario planning and one-shot decision problems. We assume that criteria payoff matrices are dependent. The goal of the new approach is to select an optimal mixed strategy. The procedure takes into consideration decision makers' objective preferences and their attitude towards risk. This attitude is measured by the coefficient of optimism on the basis of which a set of the most probable events is suggested and an optimization problem is formulated and solved.

The paper is organized as follows. Section 2 deals with the main features of multicriteria DMU (decision making under uncertainty) with scenario planning. Section 3 presents a procedure that may be used as a tool in multicriteria optimization under uncertainty for mixed strategy searching. Section 4 provides a case study. Conclusions are gathered in the last section.

2 Uncertain multicriteria decision making with scenario planning

According to the Knightian definition (Knight, 1921), we will assume that DMU is characterized by a situation where the decision maker (DM) has to choose the appropriate alternative (decision, strategy) on the basis of some scenarios (events, states of nature) whose probabilities are not known – uncertainty with unknown probabilities (Courtney et al., 1997; Dominiak, 2006; Groenewald and Pretorius, 2011; Render et al., 2006; Sikora, 2008; Trzaskalik, 2008; von Neumann and Morgenstern, 1944; Walliser, 2008; Williams et al., 1997).

There are many classical and extended decision rules designed for one-criterion DMU (Basili, 2006; Basili et al., 2008; Basili and Chateauneuf, 2011; Ellsberg, 2001; Etner et al., 2012; Gaspars, 2007; Gaspars-Wieloch, 2012, 2013, 2014a, 2014b, 2014c, 2014d, 2014, 2015a, 2015b, 2015c; Ghirardato et al., 2004; Gilboa, 2009; Gilboa and Schmeidler, 1989; Hayashi, 2008; Hurwicz, 1952; Ioan and Ioan, 2011; Marinacci, 2002; Piasecki, 1990; Savage, 1961; Schmeidler, 1986; Tversky and Kahneman, 1992; Wald, 1950) and multicriteria DMU (Aghdaie et al., 2013; Ben Amor et al., 2007; Dominiak, 2006; 2009; Durbach, 2014; Eiselt and Marianov, 2014; Gaspars-Wieloch, 2014e; Ginevičius and Zubrecovas, 2009; Goodwin and Wright, 2001; Hopfe et al., 2013; Janjic et al., 2013; Korhonen, 2001; Lee, 2012; Liu et al., 2011; Michnik, 2013; Mikhaidov and Tsvetinov, 2004; Montibeller et al., 2006; Ram et al., 2010; Ramik et al., 2008; Ravindran, 2008; Stewart, 2005; Suo et al., 2012; Triantaphyllou and Lin, 1996; Tsaur et al., 2002; Urli and Nadeau, 2004; Wang and Elhag, 2006; Wojewnik and Szapiro, 2010; Xu, 2000; Yu, 2002). Nevertheless, the majority of the extended rules refer to the probability calculus (for instance, expected utility maximization, α-maximin expected utility, cumulative prospect theory, Choquet expected utility), which is rather characteristic of DMR – decision making under risk or DMU with known probabilities. Let us recall that according to the Knight's definition uncertainty occurs when we do not know (i.e. we cannot measure) the probabilities of particular scenarios[1] (see complete uncertainty).

Some existing procedures are dedicated to searching for an optimal pure strategy, other are designed for searching for an optimal mixed strategy. In the case of pure strategies, the DM chooses and completely executes only one alternative. On the other hand, a mixed strategy implies that the DM selects and performs a weighted combination of several accessible alternatives, see e.g. bonds portfolio construction, cultivation of different plants (Guzik, 2009; Ignasiak, 1996; Officer and Anderson, 1968; Puppe and Schlag, 2009; Sikora, 2008). This paper will deal with the latter case.

We recognize both types of uncertainties: internal (related to DM's values and judgments) and external (related to imperfect knowledge of the consequences of action), but in this paper we focus on the latter (Durbach and Stewart, 2012; Stewart, 2005).

Durbach and Stewart (2012) state that uncertainties become increasingly so complex that the elicitation of measures such as probabilities, belief functions or fuzzy membership functions becomes operationally difficult for DMs to comprehend and virtually impossible to validate. Therefore, in such contexts it is useful to construct scenarios which describe possible ways in which the future might unfold. Hence, MDMU+SP (multicriteria decision making under uncertainty with scenario planning) will be considered in this paper. Scenario planning, used within the framework of DMU (Pomerol, 2001), is a technique for facilitating the identification of uncertain and uncontrollable factors which may influence the effects of decisions in the strategic management context. The construction of scenarios is described e.g. in (Dominiak, 2006; Van der Heijden, 1996). The result of the choice made under uncertainty with scenario planning depends on two factors: which decision will be selected and which scenario will occur.

The discrete version (the set of alternatives is explicitly defined and discrete in number) of MDMU+SP consists of n decisions ($D_1, \ldots, D_j, \ldots, D_n$), each evaluated on p criteria $C_1, \ldots, C_k, \ldots, C_p$ and on m mutually exclusive scenarios ($S_1, \ldots, S_i, \ldots, S_m$). The problem can be presented by means of p payoff matrices (one for each criterion) and $p \times n \times m$ evaluations. Each payoff matrix contains

[1] Of course, we are aware of the fact that many researchers apply the alternative approach according to which each non-deterministic (with known and unknown probabilities) decision problem is treated as an uncertain problem, while risk is understood as the possibility that some adverse circumstances might happen (see. e.g. Ogryczak and Sliwiński, 2009).

$n \times m$ evaluations, say a_{ij}^{k}, which denote the performance of criterion C_k resulting from the choice of decision D_j and the occurrence of scenario S_i . We assume that the distribution of payoffs related to a given decision is discrete.

Existing decision rules differ from each other with respect to the DM's attitude towards risk which can be measured, for instance, by the coefficient of pessimism (α) or the coefficient of optimism (β). Note that in this context we do not treat risk as a situation where the probability distribution of each parameter of the decision problem is known, but we admit the possibility that some adverse circumstances might happen (Dominiak, 2006, 2009; Fishburn, 1984).

It is worth emphasizing that some rules can be applied when the DM intends to perform the selected strategy only once. Others are recommended for people considering multiple realizations of the chosen variant. In the first case, the alternatives are called one-shot (one-time) decisions; in the second case, multi-shot decisions. This paper focuses on one-shot decision problems which are commonly encountered in business, economics and social systems (Guo, 2010, 2011, 2013, 2014; Liu and Zhao, 2009).

Marler and Arora (2004) divide multi-objective optimization concepts and methods into three categories: (a) methods with a priori articulation of preferences (MPAP), (b) methods with a posteriori articulation of preferences (MPSAP) and (c) methods with no articulation of preferences (MNAP). In MPAP the user indicates the relative importance of the objective functions or desired goals (by means of parameters which are coefficients, exponents, or constraint limits) before running the optimization algorithm (Chang, 2011; Churmann and Ackoff, 1954; Gaspars-Wieloch, 2011; Lotfi et al., 1997). MPSAP entail selecting a single solution from a set of mathematically equivalent solutions. This means that the DM imposes preferences directly on a set of potential final solutions. In this paper we propose an MPAP procedure with the application of weights for each attribute.

As mentioned before, the decision rule presented in this paper allows the DM to find an optimal mixed strategy, but it is worth emphasizing that the existing one-criterion and multicriteria procedures for mixed strategies are related more to game theory, i.e. game between players (Czerwiński, 1969; Gilboa, 2009; Grigorieva, 2014; Lozan and Ungureanu, 2013; Luce and Raiffa, 1957; Voorneveld et al., 1999; 2000), than to game against nature (which constitutes a neutral opponent). Therefore, the creation of an approach for uncertain multiobjective mixed decision making with scenario planning (or scenario-based MMDM) seems vital and desirable.

3 β-decision rule for uncertain multicriteria mixed decision making

When preparing a decision rule for uncertain multicriteria mixed decision making, one should answer two main questions: (1) how should DM's preferences (concerning the attitude towards risk and the importance of particular criteria) be taken into account?, and (2) how should criteria be aggregated and how should they be combined with scenarios?

Possible rules for 1-criterion mixed strategies are as follows:

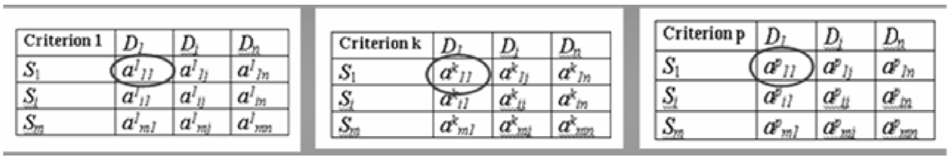
(a) Bayes' rule (the DM performs the selected plan many times) – the optimization model maximizes the average income.

(b) Wald's rule (the DM performs the chosen decision only once and behaves cautiously, the minimal guaranteed benefit is maximized) – the solution of such a problem ensures that even if the least attractive scenario takes place, the income of the DM will not be lower than y^*, i.e. the maximized minimum guaranteed revenue.

(c) Hurwicz's rule (the DM performs the selected plan only once and declares the level of pessimism or optimism) – the optimization model takes into consideration only extreme payoffs connected with a given alternative, not the frequency of the occurrence of intermediate ones (see Gaspars-Wieloch, 2012; 2014a; 2014c), which may lead to quite illogical recommendations.

The last two approaches treat nature as a conscious opponent who is altering strategies depending on the outcomes, which is strongly criticized by (Milnor, 1954; Officer and Anderson, 1968).

In connection with the fact that we analyze only one-shot decisions and that solutions recommended by the rule should vary depending on the DM's attitude towards risk, Hurwicz's rule seems the most appropriate. Nevertheless, due to some drawbacks connected with this procedure, we will refer to another method – the β-decision rule, originally designed for one-criterion mixed decision making (Gaspars-Wieloch, 2014b). In this method the number of scenarios considered in the optimization model depends on the level of the DM's optimism. If $\beta = 0$, then all states of nature are taken into account, since the DM intends to be well prepared for the uncertain future. Meanwhile, if $\beta > 0$, then the initial set of possible scenarios is appropriately reduced to a smaller set of events, because the most pessimistic states of nature may be omitted in the analysis (i.e. they are the least probable). When $\beta = 0$, the mixed strategy recommended by the β-decision rule is the optimal solution generated by the problem formulated according to Wald's rule. In Gaspars-Wieloch (2014b) it is suggested to assign the status to a given event on the basis of two measures connected with the outcomes of that state of nature. Nevertheless, the method of determining the set of the most probable scenarios may be different.



Criterion 1	D_1	D_j	D_n
S_1	a^1_{11}	a^1_{1j}	a^1_{1n}
S_i	a^1_{i1}	a^1_{ij}	a^1_{in}
S_m	a^1_{m1}	a^1_{mj}	a^1_{mn}

Criterion k	D_1	D_j	D_n
S_1	a^k_{11}	a^k_{1j}	a^k_{1n}
S_i	a^k_{i1}	a^k_{ij}	a^k_{in}
S_m	a^k_{m1}	a^k_{mj}	a^k_{mn}

Criterion p	D_1	D_j	D_n
S_1	a^p_{11}	a^p_{1j}	a^p_{1n}
S_i	a^p_{i1}	a^p_{ij}	a^p_{in}
S_m	a^p_{m1}	a^p_{mj}	a^p_{mn}

Figure 1. Payoff matrices

Now, let us check how scenarios should be combined with criteria. According to Durbach, Stewart (2012); Michnik (2013) and Stewart (2005) MDMU+SP models can be divided into two classes. The first one (A) includes two-stage models in which evaluations of particular alternatives are estimated with respect to scenarios and criteria in two separate stages. Class A contains two subclasses: A-CS and A-SC. Subclass A-CS is the set of approaches considering decisions separately in each scenario and setting an $n \times m$ table giving the aggregated (over attributes) performance of alternative D_j under scenario S_i. These evaluations are then aggregated over scenarios. In subclass A-SC the order of aggregation is reversed – performances are generated across scenarios and then measures are calculated over criteria. The second class (B) consists of one-stage procedures considering combinations of scenarios and attributes (scenario-criterion pairs) as distinct meta-criteria. In our research we will apply an A-CS model since we assume that payoff matrices are dependent, which means that if scenario S1 occurs and decision D1 is selected, then the performance of the particular criteria is as follows: a_{11}^1, a_{11}^2, …, a_{11}^p (Figure 1).

To adapt the β-decision rule for uncertain one-criterion mixed decision making to multicriteria analysis, it is necessary to combine that procedure with a multiobjective method. At first glance, there are many approaches dedicated to MDU (Trzaskalik, 2014), e.g.:

a) additive methods, such as SAW, SMART or SMARTER (Churmann and Ackoff, 1954; Edwards and Barron, 1994),
b) AHP, REMBRANDT, ANP (Saaty, 1980; 1996; Lootsma, 1993),
c) MACBETH, ZAPROS (Bana e Costa and Chagas, 2004; Larichev and Moshkovich, 1995),
d) ELECTRE (Roy and Bouyssou, 1993),
e) PROMETHEE (Brans et al., 1984),
f) TOPSIS, VIKOR, BIPOLAR (Hwang and Yoon, 1981; Opricovic, 1998; Konarzewska-Gubała, 1989).

However, it is worth noting that, due to the construction of the β-decision rule, it would be desirable if the chosen method fulfilled the following conditions:

a) it is not time-consuming since it constitutes only a stage in the whole procedure,
b) it may be applied when payoff matrices for each criterion are dependent,

c) it is applicable to problems with criteria defined in different scales and units,
d) it generates a synthetic measure for each pair: decision/scenario.

Therefore, the only methods satisfying all conditions aforementioned are SAW, SMART, SMARTER and TOPSIS. Here, the β-decision rule will be combined with SAW (Simple Additive Weighting Method).

Hence, the β-decision rule for multicriteria mixed decision making includes the following steps:

Step 1: Given a set of potential decisions and payoff matrices for each criterion, define an appropriate value of the parameter $\beta \in [0,1]$ according to your level of optimism and choose weights w^k for each attribute ($k = 1,\ldots,p$):

$$\sum_{k=1}^{p} w^k = 1 \tag{1}$$

Step 2: If necessary, normalize the evaluations (use Equation (2) for maximized criteria and Equation (3) for minimized criteria):

$$a(n)_{ij}^k = \frac{a_{ij}^k - \min\limits_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \left\{a_{ij}^k\right\}}{\max\limits_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \left\{a_{ij}^k\right\} - \min\limits_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \left\{a_{ij}^k\right\}} \qquad k = 1,\ldots,p,\ i = 1,\ldots,m,\ j = 1,\ldots,n \tag{2}$$

$$a(n)_{ij}^k = \frac{\max\limits_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \left\{a_{ij}^k\right\} - a_{ij}^k}{\max\limits_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \left\{a_{ij}^k\right\} - \min\limits_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \left\{a_{ij}^k\right\}} \qquad k = 1,\ldots,p,\ i = 1,\ldots,m,\ j = 1,\ldots,n \tag{3}$$

Step 3: Compute the aggregated measure $A(n)_{ij}$ for each pair: decision/scenario (according to the methodology of SAW):

$$A(n)_{ij} = \sum_{k=1}^{p} w^k \cdot a(n)_{ij}^k \qquad i = 1,\ldots,m,\ j = 1,\ldots,n \tag{4}$$

Step 4: Find M^* (the maximum aggregated value computed according to the max-max rule) and calculate y^* which is the maximized minimum guaranteed aggregated value computed on the basis of Wald's model (Equations 5-8):

$$y \rightarrow \max \tag{5}$$

$$\sum_{j=1}^{n} A(n)_{ij} x_j \geq y \qquad i = 1,\ldots,m \tag{6}$$

$$\sum_{j=1}^{n} x_j = 1 \tag{7}$$

$$x_j \geq 0 \qquad j = 1,\ldots,n, \tag{8}$$

where x_j is the share of alternative D_j in the mixed strategy and n stands for the number of decisions.

Step 5: Find the set of the most probable scenarios (K) with the aid of Equations (9)-(13):

$$S_i \in K \Leftrightarrow \left(\underset{A(n)_{ij}\,(j=1,\ldots,n)}{\exists} A(n)_{ij} \geq r_\beta \right) \vee \left(d_i \geq d_\beta \right) \quad (9)$$

$$r_\beta = \beta(M^* - y^*) + y^* \quad (10)$$

$$d_\beta = \beta(d_{\max} - d_{\min}) + d_{\min} \quad (11)$$

$$d_i = \sum_{j=1}^{n} d_{ij} \qquad i = 1,\ldots,m \quad (12)$$

$$d_{ij} = m - \max_i \left\{ p(A(n)_{ij}) \right\} \qquad i = 1,\ldots,m, j = 1,\ldots,n \quad (13)$$

where K is the set of the most probable events, $A(n)_{ij}$ is the synthetic value of normalized payoffs connected with decision D_j and event S_i. r_β is the expected level of the aggregated outcome dependent on β (Equation 10). d_{ij} denotes the number of aggregated values related to alternative D_j which are worse than $A(n)_{ij}$. The symbol m still denotes the number of scenarios and $p(A(n)_{ij})$ is the position of the value $A(n)_{ij}$ in the non-increasing sequence of all synthetic evaluations connected with decision D_j (if $A(n)_{ij}$ has the same value as other evaluations of a given alternative, then it is recommended to choose the farthest position of this value in the sequence – see Equation 13). d_i is the total number of "dominance cases" related to state S_i (Equation 12), d_{max} and d_{min} are the biggest and the smallest number of "dominance cases", respectively (Equation 11).

As can be seen, scenario S_i may belong to K if and only if it contains at least one aggregated payoff not lower than r_β (Equations 9 and 10) or if its number of "dominance cases" is sufficiently close to d_{max} (Equations 9 and 11). The scenario with d_{max} and with at least one aggregated payoff equal to M^* might be treated as the best state of nature (the most optimistic), but in many decision problems such an event does not exist.

Step 6: Solve the following optimization problem:

$$\sum_{i \in K} \max\{g_i, 0\} \rightarrow \min \quad (14)$$

$$\sum_{j=1}^{n} A(n)_{ij} x_j = r_\beta - g_i \quad i \in K \quad (15)$$

$$\sum_{j=1}^{n} x_j = 1 \quad (16)$$

$$x_j \geq 0 \qquad j = 1,\ldots,n \quad (17)$$

where g_i is the deviation from r_β of the aggregated income achieved by the DM if scenario S_i occurs. The optimal solution represents the multi-criteria mixed strategy reflecting the DM's level of optimism.

Both sides of condition (15) present the true aggregated revenue obtained if the shares of a particular mixed strategy equal $x_1, x_2, \ldots, x_n$ and scenario S_i takes place. The aim of the optimization model (Equation 14) is to minimize, within the set K, the sum of all deviations of the true aggregated payoffs from the expected one. Note that only positive deviations are disadvantageous since then the expected revenue exceeds the true aggregated income.

Let us call the aforementioned procedure β-MMDM, i.e. the β decision rule for multicriteria mixed decision making.

4 Case study

The method suggested in this paper will be illustrated by means of the following example. Let us assume that the DM intends to find the optimal mixed strategy on the basis of two objectives C1 and C2, which are both maximized. There are four possible decisions: D1, D2, D3 and D4. The DM is not able to define exact evaluations of both criteria, but thanks to scenario planning the list of possible states of nature (S1, S2, S3, S4, S5) has been generated. Table 1 presents payoff matrices of the analyzed decision problem.

To find the most appropriate strategy with the aid of β-MMDM, in the first step the DM is asked to declare the coefficient of optimism, let us say $\beta = 0.7$, and to set weights for each attribute, e.g. $w^1 = 0.4$ and $w^2 = 0.6$.

Table 1: Payoff matrices – initial evaluations

	C1				C2			
No	D1	D2	D3	D4	D1	D2	D3	D4
S1	2.5	4.0	4.5	3.0	20	22	15	21
S2	1.3	2.5	3.5	3.0	32	18	19	17
S3	1.6	3.0	4.3	2.0	29	19	16	18
S4	1.7	3.0	2.0	2.5	28	15	23	24
S5	1.5	3.5	4.2	4.0	30	17	16	24

Step 2 is optional, but in our case it is obligatory because the evaluations are defined in different scales. The normalized values are computed in Table 2 (Equation 2).

Table 2: Payoff matrices – normalized values

	C1				C2			
No	D1	D2	D3	D4	D1	D2	D3	D4
S1	0.38	0.84	1.00	0.53	0.29	0.41	0.00	0.35
S2	0.00	0.38	0.69	0.53	1.00	0.18	0.24	0.12
S3	0.09	0.53	0.94	0.22	0.82	0.24	0.06	0.18
S4	0.13	0.53	0.22	0.38	0.76	0.00	0.47	0.53
S5	0.06	0.69	0.91	0.84	0.88	0.12	0.06	0.53

In step 3 we refer to the A-CS approach and to SAW. The aggregated normalized values are given in Table 3 (Equation 4).

Table 3: Aggregated measures $A(n)_{ij}$

No	D1	D2	D3	D4
S1	0.326	0.585	0.400	0.424
S2	0.600	0.256	0.416	0.283
S3	0.532	0.354	0.410	0.193
S4	0.509	0.213	0.370	0.468
S5	0.554	0.346	0.398	0.655

In step 4 we find $M^* = \max\{0.600;0.585;0.416;0.655\} = 0.655$ and $y^* = 0.418$ according to the following model:

$$
\begin{gathered}
y \to \max \\
0.326x_1 + 0.585x_2 + 0.400x_3 + 0.424x_4 \geq y \\
0.600x_1 + 0.256x_2 + 0.416x_3 + 0.283x_4 \geq y \\
0.532x_1 + 0.354x_2 + 0.410x_3 + 0.193x_4 \geq y \\
0.509x_1 + 0.213x_2 + 0.370x_3 + 0.468x_4 \geq y \\
0.554x_1 + 0.346x_2 + 0.398x_3 + 0.655x_4 \geq y \\
x_1 + x_2 + x_3 + x_4 = 1 \\
x_1, x_2, x_3, x_4 \geq 0
\end{gathered}
$$

In step 5 parameters r_β (Equation 10) and d_β (Equations 11-13) are calculated in order to find the most probable scenarios:

$$r_\beta = 0.7(0.655 - 0.418) + 0.418 = 0.5838$$

$$d_\beta = 0.7(10 - 4) + 4 = 8.2$$

Table 4 contains the values of "dominance cases" and the sum of "dominance cases" for each state of nature ($d_{max} = \max\{8;10;8;4;10\} = 10$, $d_{min} = \min\{8;10;8;4;10\} = 4$).

Table 4: "Dominance cases"

No	D1	D2	D3	D4	d_i
S1	0	4	2	2	8
S2	4	1	4	1	10
S3	2	3	3	0	8
S4	1	0	0	3	4
S5	3	2	1	4	10

Hence, there are three scenarios with at least one value not lower than r_β, i.e. S1, S2 and S5. Additionally, we note that the sum of "dominance cases" for events S2 and S5 is not lower than d_β. That means that the set K contains three elements: K={S1, S2, S5}, see Equation (9).

The optimal multicriteria mixed strategy ($x_1 = 0.9489$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0.0511$) is established on the basis of the optimization model formulated below (step 6):

$$\max\{g_1,0\} + \max\{g_2,0\} + \max\{g_5,0\} \rightarrow \min$$
$$0.326x_1 + 0.585x_2 + 0.400x_3 + 0.424x_4 = 0.5838 - g_1$$
$$0.600x_1 + 0.256x_2 + 0.416x_3 + 0.283x_4 = 0.5838 - g_2$$
$$0.554x_1 + 0.346x_2 + 0.398x_3 + 0.655x_4 = 0.5838 - g_5$$
$$x_1 + x_2 + x_3 + x_4 = 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Thus, the DM should invest 94.89% of his funds in decision D1 and 5.11% in decision D4. The deviation degrees for scenarios S1, S2 and S5 are: $g_1 = 0.2528$, $g_2 = 0$, $g_5 = 0.0246$. The deviation for event S1 is the largest, but note that this state of nature does not satisfy the second condition of disjunction (9), which means that this is the least probable scenario among all scenarios belonging to K.

In this paper, the set K is formed in the paper on the basis of two criteria (the expected aggregated income r_β and the number of "dominance cases" d_β). Nevertheless, this is only a suggestion – one may choose other indices. Here, we will explain why it is recommended to consider both r_β and d_β, not only the first criterion. When the maximum aggregated payoff M^* is much higher than the remaining payoffs in the matrix, then, even for low values of β, index r_β becomes so high that only the scenario offering M^* meets the criterion r_β . This means that in such cases, regardless of the level of optimism, only one state of nature is treated as the most probable, which is not reasonable. The cardinality of the set K depends on the coefficient of optimism. The higher β is, the fewer elements the set K contains. However, it is worth emphasizing that when $\beta = 1$, the set of the most probable scenarios does not need to contain exactly one element.

5 Conclusions

In this paper, we propose a procedure for uncertain multiobjective optimization which may be applied when a mixed strategy is sought. The new approach (β-MMDM, i.e. β-decision rule for multicriteria mixed decision making) takes into account the decision maker's preference structure and attitude towards risk. This attitude is measured by the coefficient of optimism on the basis of which a set of the most probable events is suggested and an optimization problem is formulated and solved. The β-decision rule (a procedure originally designed for scenario-based one-criterion mixed decision making) is combined with the Simple Additive Weighting Method. Hence, according to the classification described in (Michnik, 2012), the β-MMDM is not a typical MCDA (multicriteria decision analysis) hybrid, since only one of its components involves multiobjective optimization (i.e. SAW), while the other one is related to one-criterion decision problems. The new decision rule has at least four significant advantages. First, it recommends different mixed strategies depending on the DM's level of optimism (in contradiction to Wald's rule or max-max rule). Second, it involves game against nature, while the existing multicriteria mixed decision making procedures are designed for games with another player). Third, is does not treat nature as a conscious opponent who is altering strategies depending on the outcomes. Fourth, it is suitable for problems with criteria defined in different scales and units. Future research should deal with the coefficient of optimism, i.e. the method of estimation of that parameter and its impact on the final decision.

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MULTIPLE CRITERIA DECISION MAKING
Vol. 10 2015

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EVALUATING THE NEGOTIATION TEMPLATE WITH SIPRES – A FUSION OF THE REVISED SIMOS' PROCEDURE AND THE ZAPROS METHOD

Abstract

In a negotiation process, building a negotiation offer scoring system consistent with the preferences of the decision-maker is a very intricate task. A variety of methods can be used to develop such a negotiation support tool, e.g. SAW and TOPSIS, but they have several disadvantages.

In this paper the issue of evaluating the negotiation template using a novel tool called SIPRES is discussed. The algorithm proposed employs the key notions of the revised Simos' procedure and ZAPROS method to elicit the negotiator's preferences over some reference solutions. On the one hand, it allows decision-makers to define their preferences in a simple and effortless way and provides a straightforward yet effective method for analyzing the trade-offs between the alternatives using selected reference alternatives only (the ZAPROS-like approach). On the other hand, the revised Simos' procedure applied in the method allows determining the cardinal scores for the alternatives. The scoring system obtained this way makes it possible to conduct a sophisticated symmetric and asymmetric negotiation analysis.

An illustrative example presented in the paper concerns the European Union's multiannual financial framework negotiations.

Keywords: European Union, multiannual financial framework (MFF), negotiations, MCDA, revised Simos' procedure, VDA, ZAPROS, SIPRES.

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1 Introduction

The theory of negotiation recommends a comprehensive preparation before negotiations commence (Stein, 1989; Zartman, 1989; Simons, Tripp, 2003) as preparation is one of the most important factors for a successful outcome. It includes recognizing the negotiation problem, knowing your needs and limits and understanding what the other party wants and anticipating their limits. It also includes the evaluation of the negotiation template.

The negotiation template describes the structure of the negotiation problem and is defined by a list of negotiation issues and their feasible options. On the basis of this list a set of potential negotiation offers may be identified by finding various combinations of options for all the issues considered. Since comparing the offers that are described by many different criteria is, in general, not easy, a negotiation offer scoring system is usually built to support negotiators in their role. This system assigns scores to the offers within the template and in doing so makes the comparisons less difficult.

Although various MCDM/A methods can be used to build a negotiation offer scoring system (see, e.g., Figuera et al., eds., 2005 and Yoon, Hwang, 1995), such a system is usually determined using SAW – simple additive weighting method[1] (Keeney, Raiffa, 1976) (for applications see, e.g., Kersten, Noronha, 1999; Schoop et al., 2003; Thiessen, Soberg, 2003). Nevertheless, recent experimental research on electronic negotiations (Wachowicz, Kersten, 2009; Wachowicz, Wu, 2010) showed that only few negotiators are able to interpret correctly the utility values and compare effectively the quality of the offers described by SAW-based scores. According to another experiment (Roszkowska, Wachowicz, 2014b), negotiators turned out to be inconsistent in evaluating and choosing the SAW-based rankings of offers that match their preferences since most of them evaluated as more useful (better) a predefined ranking that was less similar to their own subjective ranking. Another experimental study on MCDM

[1] Apart from SAW, in order to develop a negotiation support tool in the form of a negotiation offer scoring system, other methods can also be used, e.g. AHP (Saaty, 2006; Saaty, Vargas, 1991), TOPSIS (Hwang, Yoon, 1981) or MARS (Górecka et al., 2014). However, they all have some drawbacks. For instance, the application of the technique based on AHP, which is used in Web-HIPRE system (Mustajoki, Hämäläinen, 2000), where negotiators use a nine-point verbal scale and pair-wise comparisons of the elements of the negotiation template, is limited to support discrete negotiation problems only. Moreover, pair-wise comparisons may be very tedious, which is also the case in the MARS approach. Finally, the application of TOPSIS to the evaluation of the negotiation template (Roszkowska, Wachowicz, 2015; Wachowicz, Błaszczyk, 2013) limits the possibilities of defining individual preferences by the negotiators, since the concept of distance measuring to appraise the attractiveness of offers is applied there (Górecka et al., unpublished).

by Roszkowska and Wachowicz (2014a) showed that the decision-makers (DMs) often describe their preferences qualitatively, in a verbal or visual way and that they define the reference points vaguely using imprecise and qualitative categories. On the other hand, quantitative methods and models are widely used in negotiation support to elicit the negotiators' preferences and build a negotiation offer scoring system (Kersten, Noronha, 1999; Raiffa et al., 2002). It must be kept in mind that the quantitative approach is crucial in the negotiation analysis as it allows performing different analyses of the negotiation process, for instance: measuring the scale of concessions, visualizing the negotiation progress, searching for the improvements in the contract negotiated by the parties, finding the arbitration (fair) solution of the negotiation problem, and producing general conclusions of descriptive nature (Filzmoser, Vetschera, 2008; Kersten et al., 2014).

Taking all that into account it would be worth developing a tool for evaluating the negotiation template that would allow negotiators to define their preferences qualitatively but would result in a cardinal scoring system – a helpful, understandable and user-friendly tool. The aim of this paper is to propose and present just such a tool, called SIPRES. It is a novel technique that employs the key notions of the revised Simos' procedure (Figueira, Roy, 2002) and the ZAPROS method (Larichev, Moshkovich, 1995). On the one hand, it allows decision-makers to define their preferences in a simple and effortless way and provides a straightforward but effective method for analyzing the trade-offs between the alternatives using selected reference alternatives only (the ZAPROS-like approach). On the other hand, the revised Simos' procedure applied in the method allows determining the cardinal scores for the alternatives. The scoring system obtained this way makes it possible to conduct a sophisticated symmetric and asymmetric negotiation analysis.

This paper consists of an introduction, four sections and conclusions. In the second and in the third section the revised Simos' procedure and the ZAPROS method are presented as preliminaries to a new approach for scoring the negotiation template, namely the SIPRES algorithm, which is described in the fourth section. Finally, the fifth section provides an illustrative example concerning the European Union's multiannual financial framework negotiations.

2 The revised Simos' procedure

The revised Simos' procedure, introduced by Figueira and Roy (2002), is intended for the determination of the criteria weights in the ELECTRE type methods, but it can also be used to adapt or convert a scale of a given criterion into an interval or a ratio scale as well as to construct an interval or a ratio scale on any ordered set (cf. Roy, 1999).

The technique is based on a 'card-playing' procedure and consists of the following steps (Figueira, Roy, 2002):

1. We give the decision-maker a set of cards with the names of the elements (e.g. criteria) written on them; thus, we have *n* cards, where *n* is the number of elements (criteria). We also provide a set of blank cards of the same size (as many as the DM needs).
2. We ask the decision-maker to put the named cards in the ascending order, i.e. to sort the elements (criteria) from the least important (the worst) to the most important (the best) one. If, in the DM's opinion, some elements (criteria) have the same importance (and hence the same weight), the cards with their names should be placed together and held with a clip or a rubber band. As a result, we obtain a complete pre-order of the *n* elements (criteria) in which the least important (the worst) element (criterion) obtains rank 1 and the number of ranks is less than or equal to *n*.
3. We ask the decision-maker to consider whether the distances between the positions in the ranking are the same or not. In order to distinguish the importance of two successive elements (criteria) or subsets of equally important elements (criteria), we ask the DM to introduce blank cards between the subsequent cards according to the following rules:
   a) the greater the difference between the weights of the elements (criteria) or subsets of equally important elements (criteria), the greater the number of blank cards;
   b) no blank card means that the elements (criteria) do not have the same weight and the difference between the weights constitutes the unit (denoted by *u*) adopted for measuring the intervals between weights; *h* blank cards mean a difference of *h+1* units.
4. We ask the decision-maker to determine how many times the last-ranked element (criterion) is more important (better) than the first one; let *z* be the value of this ratio.
5. Let $e_r^{'}$ be the number of blank cards between the positions *r* and *r*+1. We calculate:

$$\begin{cases} e_r = 1 + e_r^{'} \qquad \forall r = 1,...,n^* - 1 \\ e = \sum_{r=1}^{n^*-1} e_r \\ u = \dfrac{z-1}{e} \end{cases}$$

retaining six decimal places for u. Subsequently, we determine the non-normalized weight $p(r)$ for each position in the ranking: $p(r) = 1 + u \cdot (e_0 + \ldots + e_{r-1})$, where $e_0 = 0$. We round these weights to two decimal places. If there are several elements (criteria) in the same position r, all of them obtain the same weight $p(r)$.

6. Let g_k be an element (criterion) in the position r, and p'_k – the non-normalized weight of this element (criterion), $p'_k = p(r)$. We calculate:

$$\begin{cases} P' = \sum_{k=1}^{n} p'_k \\ p^*_k = \dfrac{100 \cdot p'_k}{P'} \end{cases}$$

Subsequently, we determine p''_k by deleting some of the decimal digits from p^*_k. Let s be the number of decimal places taken into account. We compute:

$$\begin{cases} P'' = \sum_{k=1}^{n} p''_k \leq 100 \\ \varepsilon = 100 - P'' \leq 10^{-s} \cdot n \\ v = 10^{s} \cdot \varepsilon \end{cases}$$

Finally, we set $p_k = p''_k + 10^{-s}$ for v suitably selected elements (criteria) and $p_k = p''_k$ for the other $n - v$ elements (criteria). We obtain $\sum_{k=1}^{n} p_k = 100$, where p_k is the normalized weight of the element (criterion) g_k, with the required number of decimal places.

The choice of the v elements (criteria), whose weights will be rounded, is performed using the following algorithm:

1. For each element (criterion) g_k we determine the ratios:

$$d_k = \frac{10^{-s} - (p^*_k - p''_k)}{p^*_k}$$

$$d^*_k = \frac{(p^*_k - p''_k)}{p^*_k}$$

2. We create two lists, R and R^*:
   a) the R list, consisting of the pairs (k, d_k) sorted in the ascending order of d_k,
   b) the R^* list, consisting of the pairs (k, d^*_k) sorted in the descending order of d^*_k.

3. We set $M = \{k : d_k > d_k^*\}$, $|M| = m$.
4. We partition the set of n elements (criteria) into two subsets: F^+ and F^-, where $|F^+| = v$ and $|F^-| = n - v$, as follows:
   - if $m + v \leq n$, then F^- consists of the m elements (criteria) of M and the last $n - v - m$ elements (criteria) of R^* which are not in M; while F^+ consists of the first v elements (criteria) of R^* which are not in M;
   - if $m + v > n$, then F^+ consists of the $n - m$ elements (criteria) not belonging to M and the first $v + m - n$ elements (criteria) of R which are in M; while F^- consists of the last $n - v$ elements (criteria) of R which are in M.

3 The ZAPROS method

The ZAPROS method (Larichev, Moshkovich, 1995) is intended for decision-making problems in which it is required to order a fairly large number of alternatives. The set of the alternatives may change while the decision rules remain constant.

The technique is based on Verbal Decision Analysis (VDA). The term ‘Verbal Decision Analysis’ had not been introduced by Larichev and Moshkovich until 1997 (Larichev, Moshkovich, 1997), even though research within this approach had already started in the 1980s (see, e.g., Larichev, Moshkovich, 1988).

VDA is a framework for designing MCDA methods by using preferential information obtained from the decision makers in the ordinal form (for instance ‘more preferable’, ‘less preferable’ or ‘equally preferable’). This type of judgments seems stable and reliable according to the results of psychological experiments. Moreover, the judgments are verified by testing their consistency (Ashikhmin, Furems, 2005; Moshkovich, Mechitov, 2013).

VDA is based on cognitive psychology, applied mathematics and computer science, and it was proposed for unstructured decision-making problems[2] which are problems with mostly qualitative parameters and no objective model for their aggregation. Examples of such tasks can be found in policy making and strategic

[2] The general features of unstructured problems are as follows (Larichev, 2001; Moshkovich et al., 2005):
- they are unique in the sense that each problem is new to the decision-maker and has characteristics not previously experienced;
- the criteria in these problems are mostly qualitative in nature, most often formulated in a natural language;
- in many cases, the evaluations of alternatives according to the criteria may be obtained only from human beings (experts or decision-makers);
- the degrees of the criterion scales are defined verbally and represent subjective assessments by the decision-maker.

planning in different fields, as well as in personal decisions. For instance, the ZAPROS method (and its variations) has been used in R&D planning (see Larichev, Moshkovich, 1995 and 1997), applicant selection (see Moshkovich et al., 1998), job selection and pipeline selection (Moshkovich et al., 2005).

VDA takes into account peculiarities and constraints of the human information processing system. The key idea of the VDA approach is that there is a need for a decision aiding tools, which enable the decision maker to express his/her evaluations and preferences verbally, and this linguistic, non-numerical form should not be transformed into a quantitative one in any arbitrary way (Moshkovich, Mechitov, 2013). Techniques based on VDA do not use quantitative information on the importance of criteria, only verbal estimates, and no quantitative operations are performed on them. Hence, all operations are clear and understandable to decision-makers (Ashikhmin, Furems, 2005).

Table 1: VDA approach – summary

Verbal decision analysis
Application
Designed to elicit a sound preference relationship that can be applied to future cases; especially useful when a decision is made under new circumstances or in conditions of high ambiguity
Decision-making problem
More oriented to tasks with a fairly large number of alternatives, while the number of criteria is usually relatively small so as to reduce the number of comparisons required
Methodology
Bases its outranking on axiomatic relationships, to include direct assessment, dominance, transitivity and preferential independence Based on the same principles as MAUT but oriented toward using the verbal form of preference elicitation and toward evaluation of alternatives without resorting to numbers; as in MAUT, the idea is to construct universal decision rules in the criteria space and then use them on any set of actual alternatives
Decision-makers
Does not require any special knowledge of decision analysis on the part of the decision-makers

Source: Moshkovich et al. (2005).

In 1997 three methods were introduced as a VDA toolkit – one for each major type of decision-making problems, namely (Moshkovich, Mechitov, 2013):

- PARK (Berkeley et al., 1991) – for selecting the best alternative,
- ORCLASS (Larichev, Moshkovich, 1994) – for classifying alternatives,
- ZAPROS (Larichev, Moshkovich, 1995) – for ordering alternatives.

As regards ZAPROS, preference elicitation consists in comparisons of pairs of hypothetical alternatives (each with the best evaluations for all the criteria but one) differing in performance with respect to two criteria only. The results of these comparisons are transformed into the so-called Joint Ordinal Scale (JOS), which is subsequently used to compare actual decision-making alternatives (Ashikhmin, Furems, 2005).

The ZAPROS procedure consists of the following steps (Moshkovich et al., 2005):

1. We determine the evaluation scale for each criterion considered in the decision-making problem.
2. We compare pair-wise the hypothetical alternatives, each with the best possible values for all the criteria but one, using the ordinal scale (more preferable, less preferable, and equally preferable).
3. We construct the JOS, which is a complete rank order of the hypothetical alternatives with the best evaluations for all the criteria but one.
4. We compare pair-wise the actual decision-making alternatives using the JOS and construct a partial order on their set.

4 The SIPRES method

From the point of view of the negotiation analysis and evaluation of the negotiation template ZAPROS has a few advantages:

- it allows comparing complete packages (offers), which is a natural way of evaluating the concessions between the offers by the negotiators;
- it does not require evaluating the weights of negotiation issues separately, but derives them from package-to-package comparisons;
- it compares quasi-ideal packages, which are close to aspiration levels defined usually by the negotiators.

Unfortunately, it has also one serious disadvantage, namely a relatively low comparison power, which makes the occurrence of incomparability of alternatives (offers) almost unavoidable. Moreover, the outcome is represented on a graph showing the preference relations and ranking only which might be insufficient for the negotiators expecting numerical information on differences between the global attractiveness of the alternatives (offers).

Taking these drawbacks into account, a new approach called SIPRES is proposed. The acronym **SIPRES** stands for: **Si**mos' **p**rocedure for **Re**ference **S**ituations. It is based on two methods: revised Simos' procedure and ZAPROS, and aims at obtaining a complete ranking of the alternatives with scores measured on a cardinal scale.

Let $F = \{f_1, f_2, \ldots, f_n\}$ be a finite set of n evaluation criteria (issues); X_k – a finite set of possible verbal values on the scale of criterion $k = 1,2,\ldots,n$, where $|X_k| = n_k$; $X = \prod_{k=1}^{n} X_k$ is the set of all possible vectors in the decision (negotiation) space of n criteria; and $A = \{a_1, a_2,\ldots, a_m\} \subseteq X$ is a subset of X describing the alternatives (offers) considered.

The SIPRES procedure consists of the following steps:

1. We determine the evaluation scale for each criterion considered in the negotiation problem.
2. We prepare a set of blank cards and a set of cards with hypothetical alternatives (each with the best resolution level for all the criteria but one) as well as the ideal and anti-ideal reference vectors (with the best and the worst evaluations for all the criteria, respectively) and rank them from the worst to the best one.
3. We introduce blank cards between two successive cards if necessary. The greater the difference between the evaluations of the alternatives, the greater the number of blank cards:
   a) no blank card means that the alternatives do not have the same evaluation and that the difference between the evaluations is equal to one unit u used for measuring the intervals between evaluations,
   b) one blank card means a difference of two units, two blank cards mean a difference of three units, etc.
4. We determine how many times the best alternative is better than the worst one in the ranking.
5. We process the information obtained as in the revised Simos' procedure in order to obtain the normalized scores for the elements compared, i.e. to form the Joint Cardinal Scale (JCS).
6. We substitute the resolution levels in each vector describing the alternative from the negotiation template by the corresponding scores from the JCS. For each alternative we define the distance from the ideal alternative using the formula:

$$L_i = \sum_{k=1}^{n} (p_k^{\max} - p_{ik})$$

   where p_{ik} is the score from the JCS substituting the assessment of alternative a_i according to criterion f_k and $p_k^{\max}$ is the score for the best possible assessment for a given criterion.
7. We construct the complete final ranking of the alternatives according to the distance values L_i in ascending order.

5 Illustrative example

The usefulness of the SIPRES method for the facilitation of the negotiation process, namely for building a negotiation offer scoring system, will be illustrated by an example which concerns the European Union's multiannual financial framework negotiations.

The multiannual financial framework (MFF) is a spending plan that translates the EU priorities into financial terms. It sets the limits for the general annual budgets of the EU ('ceilings') as it determines how much in total and how much for different broad policy areas ('headings') the EU may spend each year over a period of at least 5 years. The previous MFF period started in 2007 and ended in 2013; the current one covers the years from 2014 to 2020 (www 1; www 4). The MFF ensures that EU spending is predictable. Besides, it allows the EU to conduct common policies over a long enough period to make them work. This long-term vision is important for potential beneficiaries of EU financial support, co-financing authorities, as well as national treasuries (www 3). The MFF regulation is proposed by the European Commission. It is adopted by the Council in a unanimous vote and after having obtained the consent of the European Parliament (www 1).

The negotiations on the MFF are one of the key issues for the Member States since they determine the possibility of obtaining funds from the EU for at least 5 years. The history of the MFF negotiations demonstrates that this process is long and complicated. It consists of three stages carried out at different levels. The first stage, lasting 1-2 years on average, consists of the negotiations in the Council, during which the final outline of the MFF is determined. The second stage consists of the negotiations with the European Parliament. Stage three, which consists of the negotiations of dozens of acts that constitute the legal basis for the implementation of the policies and mobilization of the previously negotiated funds, is carried out in parallel to the first two stages and lasts 1-1.5 years (www 5). Hence, the MFF 2007-2013 negotiations, conducted after the Eastern enlargement, were launched in 2004 and concluded in 2007, and the MFF 2014-2020 negotiations, taking place in a difficult situation for the EU, both economically (recession, increasing unemployment, sovereign debt crisis) and politically (the rise of Euroscepticism, dominance of the national interests, Member States' unwillingness to contribute to the EU budget), began in 2011 and proceeded two and a half years (www 2; www 5).

Let us assume that in the European Union's multiannual financial framework negotiations, a Member State decides to formalize and evaluate the negotiation template to obtain the negotiation offer scoring system.

The following negotiation issues are discussed:

- f_1 – the size of the European budget,
- f_2 – the allocation of the resources under the EU budget,
- f_3 – the way of financing the expenditures.

The negotiation template is defined linguistically for all the issues considered by means of the following sets of the reference salient options:

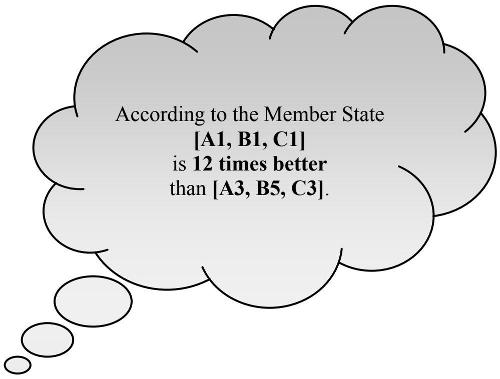
Table 2: Negotiation template

Issues		Options
f_1	**Budget size**	**A1**. Increased
		A2. Unchanged
		A3. Decreased
f_2	**Allocation of the resources**	**B1**. Very favorable (fully consistent with the position of the Member State)
		B2. Favorable (highly consistent with the position of the Member State)
		B3. Neutral (partially consistent, partially inconsistent with the Member State's position)
		B4. Adverse (highly inconsistent with the position of the Member State)
		B5. Very adverse (fully inconsistent with the position of the Member State)
f_3	**Financing of expenditures**	**C1**. Favorable (consistent with the expectations)
		C2. Neutral
		C3. Adverse (inconsistent with the expectations)

Table 3 presents the ranking of cards with hypothetical alternatives (offers), determined by the Member State in accordance with steps 2 and 3 of the SIPRES algorithm. The ranking includes the offers with the best resolution level for all the criteria but one along with the ideal and anti-ideal alternatives. Additionally, in the cloud, the information required by step 4 of the algorithm is provided on how many times, in the Member State's opinion, the best alternative is better than the worst one.

Table 3: Member State's preferences based on the card play procedure

A3	**B5**	**C3**
Blank card		
Blank card		
Blank card		
Blank card		
Blank card		
A1	**B5**	**C1**
A3	**B1**	**C1**
A1	**B4**	**C1**
Blank card		
A1	**B3**	**C1**
Blank card		
A1	**B1**	**C3**
A2	**B1**	**C1**
Blank card		
A1	**B2**	**C1**
A1	**B1**	**C2**
A1	**B1**	**C1**



Following step 5 of our algorithm, the information on Member State's preferences is processed as described in the revised Simos' procedure to obtain the normalized evaluations for the elements compared, i.e. to form the Joint Cardinal Scale (JCS). The calculations are shown in the tables below.

Table 4: Determining the non-normalized evaluations of the hypothetical alternatives (z = 12)

Position r	Alternatives in the position r			Number of blank cards between the positions r and r + 1	e_r	Non-normalized evaluations p(r)
	f_1	f_2	f_3			
1	A3	B5	C3	5	6	1.00
2	A1	B5	C1	0	1	4.88
3	A3	B1	C1	0	1	5.53
4	A1	B4	C1	1	2	6.18
5	A1	B3	C1	1	2	7.47
6	A1	B1	C3	0	1	8.76
7	A2	B1	C1	1	2	9.41
8	A1	B2	C1	0	1	10.71
9	A1	B1	C2	0	1	11.35
10	A1	B1	C1	…	…	12.00
Sum				**8**	**17**	**77.29**

Table 5: Determining the normalized evaluations of the hypothetical alternatives (s = 2, z = 12)

Position r	Alternatives in the position r			p_k^*	p_k''	d_k	d_k^*	Set M	p_k
	f_1	f_2	f_3						
1	A3	B5	C3	1.293828	1.29	0.004770	0.002959	(M)	1.29
2	A1	B5	C1	6.313883	6.31	0.000969	0.000615	(M)	6.31
3	A3	B1	C1	7.154871	7.15	0.000717	0.000681	(M)	7.15
4	A1	B4	C1	7.995860	7.99	0.000518	0.000733		8.00
5	A1	B3	C1	9.664898	9.66	0.000528	0.000507	(M)	9.66
6	A1	B1	C3	11.333937	11.33	0.000535	0.000347	(M)	11.33
7	A2	B1	C1	12.174926	12.17	0.000417	0.000405	(M)	12.18
8	A1	B2	C1	13.856903	13.85	0.000224	0.000498		13.86
9	A1	B1	C2	14.684953	14.68	0.000344	0.000337	(M)	14.69
10	A1	B1	C1	15.525941	15.52	0.000261	0.000383		15.53
Sum				**100**	**99.95**				**100**

Table 6: R and R^* lists (s = 2, v = 5, m = 7, n = 10)

List R					List R^*				
Position r	Alternatives			d_k	Position r	Alternatives			d_k^*
	f_1	f_2	f_3			f_1	f_2	f_3	
8	A1	B2	C1	0.000224	**1**	A3	B5	C3	0.002959
10	A1	B1	C1	0.000261	**4**	A1	B4	C1	0.000733
9	A1	B1	C2	0.000344	**3**	A3	B1	C1	0.000681
7	A2	B1	C1	0.000417	**2**	A1	B5	C1	0.000615
4	A1	B4	C1	0.000518	**5**	A1	B3	C1	0.000507
5	A1	B3	C1	0.000528	**8**	A1	B2	C1	0.000498
6	A1	B1	C3	0.000535	**7**	A2	B1	C1	0.000405
3	A3	B1	C1	0.000717	**10**	A1	B1	C1	0.000383
2	A1	B5	C1	0.000969	**6**	A1	B1	C3	0.000347
1	A3	B5	C3	0.004770	**9**	A1	B1	C2	0.000337
F^+ = {4, 8, 10, 9, 7}; F^- = {1, 2, 3, 6, 5}									

Tables 7 and 8 present the normalized scores for the hypothetical reference alternatives and the Joint Cardinal Scale respectively. The normalized scores reflect the scale of concessions required, when the ideal option is replaced by the option under consideration.

Table 7: Normalized scores of the hypothetical alternatives

Alternatives			p_k
f_1	f_2	f_3	
A3	B5	C3	1.29
A1	**B5**	C1	6.31
A3	B1	C1	7.15
A1	**B4**	C1	8.00
A1	**B3**	C1	9.66
A1	B1	**C3**	11.33
A2	B1	C1	12.18
A1	**B2**	C1	13.86
A1	B1	**C2**	14.69
A1	**B1**	**C1**	15.53

Table 8: Joint Cardinal Scale

JCS	
Resolution level	**Score**
B5	6.31
A3	7.15
B4	8.00
B3	9.66
C3	11.33
A2	12.18
B2	13.86
C2	14.69
A1	15.53
B1	15.53
C1	15.53

Following step 6 of the SIPRES algorithm we substitute the resolution levels in each vector describing the alternative from the negotiation template by the corresponding scores from the JCS. For each alternative we define the distance from the ideal alternative and on this basis we build the ranking of the alternatives. The distances to the ideal alternative for each of the 45 packages that can be built within the negotiation template as well as their ranks are given in Table 9.

Table 9: Packages, their distances to the ideal alternative and ranks

Criterion value			Score			Distance L_i	Rank
f_1	f_2	f_3	p_{i1}	p_{i2}	p_{i3}		
A1	**B1**	**C1**	15.53	15.53	15.53	**0.00**	**1**
A1	**B1**	**C2**	15.53	15.53	14.69	**0.84**	**2**
A1	**B2**	**C1**	15.53	13.86	15.53	**1.67**	**3**
A1	**B2**	**C2**	15.53	13.86	14.69	**2.51**	**4**
A2	**B1**	**C1**	12.18	15.53	15.53	**3.35**	**5**
A2	**B1**	**C2**	12.18	15.53	14.69	**4.19**	**6**
A1	**B1**	**C3**	15.53	15.53	11.33	**4.20**	**7**
A2	**B2**	**C1**	12.18	13.86	15.53	**5.02**	**8**
A2	**B2**	**C2**	12.18	13.86	14.69	**5.86**	**9**
A1	**B2**	**C3**	15.53	13.86	11.33	**5.87**	**10.5**
A1	**B3**	**C1**	15.53	9.66	15.53	**5.87**	
A1	**B3**	**C2**	15.53	9.66	14.69	**6.71**	**12**
A1	**B4**	**C1**	15.53	8.00	15.53	**7.53**	**13**
A2	**B1**	**C3**	12.18	15.53	11.33	**7.55**	**14**
A1	**B4**	**C2**	15.53	8.00	14.69	**8.37**	**15**
A3	**B1**	**C1**	7.15	15.53	15.53	**8.38**	**16**
A1	**B5**	**C1**	15.53	6.31	15.53	**9.22**	**18.5**
A2	**B2**	**C3**	12.18	13.86	11.33	**9.22**	
A2	**B3**	**C1**	12.18	9.66	15.53	**9.22**	
A3	**B1**	**C2**	7.15	15.53	14.69	**9.22**	
A3	**B2**	**C1**	7.15	13.86	15.53	**10.05**	**21**
A1	**B5**	**C2**	15.53	6.31	14.69	**10.06**	**22.5**
A2	**B3**	**C2**	12.18	9.66	14.69	**10.06**	
A1	**B3**	**C3**	15.53	9.66	11.33	**10.07**	**24**
A2	**B4**	**C1**	12.18	8.00	15.53	**10.88**	**25**
A3	**B2**	**C2**	7.15	13.86	14.69	**10.89**	**26**
A2	**B4**	**C2**	12.18	8.00	14.69	**11.72**	**27**
A1	**B4**	**C3**	15.53	8.00	11.33	**11.73**	**28**
A2	**B5**	**C1**	12.18	6.31	15.53	**12.57**	**29**
A3	**B1**	**C3**	7.15	15.53	11.33	**12.58**	**30**
A2	**B5**	**C2**	12.18	6.31	14.69	**13.41**	**31**
A1	**B5**	**C3**	15.53	6.31	11.33	**13.42**	**32.5**
A2	**B3**	**C3**	12.18	9.66	11.33	**13.42**	
A3	**B3**	**C1**	7.15	9.66	15.53	**14.25**	**34.5**
A3	**B2**	**C3**	7.15	13.86	11.33	**14.25**	
A2	**B4**	**C3**	12.18	8.00	11.33	**15.08**	**36**
A3	**B3**	**C2**	7.15	9.66	14.69	**15.09**	**37**
A3	**B4**	**C1**	7.15	8.00	15.53	**15.91**	**38**
A3	**B4**	**C2**	7.15	8.00	14.69	**16.75**	**39**
A2	**B5**	**C3**	12.18	6.31	11.33	**16.77**	**40**
A3	**B5**	**C1**	7.15	6.31	15.53	**17.60**	**41**
A3	**B5**	**C2**	7.15	6.31	14.69	**18.44**	**42**
A3	**B3**	**C3**	7.15	9.66	11.33	**18.45**	**43**
A3	**B4**	**C3**	7.15	8.00	11.33	**20.11**	**44**
A3	**B5**	**C3**	7.15	6.31	11.33	**21.80**	**45**
f_1	f_2	f_3	p_{i1}	p_{i2}	p_{i3}	Distance L_i	Rank
Criterion value			Score				

6 Conclusions

The SIPRES method proposed in this paper is an uncomplicated and functional technique which should improve the decision-making process. It requires the negotiators to supply the basic preferential information only – they need to evaluate trade-offs only, which seems natural for them since this is similar to the actual decision making analysis conducted in a real-life negotiations. Moreover, when defining preferences, the negotiators use an intuitively interpreted card tool. As a result, a cardinal negotiation offer scoring system is built, in which no two alternatives are incomparable.

Additionally, it should be noted that the SIPRES method can be applied not only in negotiation support to build a negotiation offer scoring system but also in other multi-criteria decision aiding contexts, such as policy-making, strategic planning, transportation or environmental problems to order the alternatives or to select the best one.

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MULTIPLE CRITERIA DECISION MAKING
Vol. 10 2015

Michał Jakubczyk*

USING A FUZZY APPROACH IN MULTI-CRITERIA DECISION MAKING WITH MULTIPLE ALTERNATIVES IN HEALTH CARE

Abstract

One of the responsibilities of the health care sector regulator is to decide which health technologies (drugs, procedures, diagnostic tests, etc.) should be financed using public resources. That requires taking into account multiple criteria, of which two important ones are: cost and effectiveness of a technology (others being, e.g., prevalence, safety, ethical and social implications). Hence, health and wealth need to be traded off against each other, and hence the willingness-to-pay (WTP) has to be determined. Various approaches to setting WTP have been taken, yet the results differ substantially. In the present paper I claim that the proper approach is to treat WTP as a fuzzy concept (the decision maker may not be able to decidedly state that a given health-wealth trade-off coefficient is acceptable/unacceptable – an idea backed up by the survey presented in the paper). Previous research shows how this fuzzy approach can be embedded in defining the preference relation and pairwise comparisons. In the present paper I account for the fact that there are often more than two alternatives available. To avoid difficulties that might arise (e.g., incompleteness or intransitivity of preferences) I show how the fuzzy approach can be used to define a fuzzy choice function based on the axiomatic approach. Some properties are discussed (e.g., how the approach handles the dominance and extended dominance), and the directions of further research are hinted at.

Keywords: health technology assessment, cost-effectiveness analysis, fuzziness, choice functions, willingness to pay, net benefit.

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1 Introduction

The health care market is often regulated due to its specificity as pointed out by Arrow (1963). The regulation encompasses, e.g., the decisions on which technologies should be financed using public resources. To make such decisions, the public regulator should analyse the clinical and financial consequences of using the technology. Health technology assessment (HTA) is an interdisciplinary approach (linking medicine, economics, statistics) developing methods allowing to define and measure these consequences. HTA is used more and more often, e.g., in Poland (Ustawa z dnia 12 maja 2011 r…).

Using financial and clinical criteria requires, explicitly or implicitly, a trade-off between money and health: the willingness to pay (WTP) of the decision maker needs to be determined. There have been various approaches to setting WTP (cf. Section 3), yet the results differ substantially. I argue here that determining the WTP is difficult due to the peculiarity of health as economic good and results from an inherent reluctance to report a precise price for health. The problem with determining WTP is not of statistical nature and requires a particular approach. Fuzzy-set modelling is suggested below.

The goal of the present paper is to show, from the theoretical point of view, how the fuzzy approach can be embedded in the decision making process. Jakubczyk and Kamiński (2015) showed how fuzzy preference relations can be used to model comparisons between two alternatives under uncertainty. Here I extend these ideas in one direction, modifying them to support choice from among multiple alternatives (I neglect the uncertainty, however). To avoid technical difficulties (e.g., lack of completeness or transitivity) I approach this problem by defining a fuzzy choice function. After all, ultimately the decision maker needs to make a choice, rather than simply express her preferences.

In Section 2 I present the typical approach to decision making in HTA and the concepts defined therein. In Section 3, I briefly discuss the attempts to determine the value of WTP presented in the literature and the results of the survey conducted by Jakubczyk and Kamiński (2015). In Section 4 I present the proposed model for decision making – axioms, properties, and the decision making approach. The last section is a summary. The proofs are in the appendix.

2 Decision making in health technology assessment

2.1 Nature of decision problems in HTA

Arrow (1963) pointed out that the health care services sector has many peculiarities: e.g., the demand for health care services is stochastic; there is a strong asymmetry of information between the recipients and the providers; the product quality is uncertain and difficult to verify; there are externalities, related, for instance, to ethical issues. Partly for these reasons the health care markets are often regulated with the goal to improve the efficiency in their functioning. Numerous decisions have to be made centrally, and as public money is spent, there must be a clear rationale behind the decisions. One of them is the choice of health technologies to be financed using public resources (e.g., which drugs should be reimbursed). The public regulator needs to weigh benefits and costs in a process called health technology assessment, and there are multiple criteria to be used. Reimbursing drugs uses the limited public resources, and hence the total cost needs to be assessed. Obviously, the public regulator wants to maximize the positive impact on health, and hence the clinical effectiveness of technologies is measured. As reimbursement decisions are performed across various illnesses, and the drugs compete for a single budget, the varying clinical effects have to be measured along one scale to allow comparisons. Usually a so called quality-adjusted life years (QALYs) are used, the concept combining the duration of life with its quality (cf. Pliskin et al., 1980; Bleichrodt et al., 1997). Other criteria may also be used, e.g., ethical aspects (e.g., providing extra care for patients with rare or ultra-rare diseases). Still, in the present paper I restrict my attention to two criteria only: cost and effect per single treated patient.

Let us assume we are interested in the average values of these two, i.e, the decision maker is risk neutral. Risk neutrality for cost results from averaging out the actual cost for many patients treated. Risk neutrality with respect to the effect stems from QALY being defined *à la* von Neumann-Morgenstern utility, for which the expectation is maximized. Thus, we can neglect the cost & effect variability among individual patients (first-order uncertainty). In the current study, due to space limitation, I neglect also the second-order uncertainty (average values of cost and effect being given only as estimates).

2.2 Decision analysis in HTA

To make the paper more self-contained, I present here the standard approach used in HTA and introduce the most important definitions and notation. These concepts are then redefined when fuzziness has been introduced. The interested reader might consult Gold et al. (1996), Karlsson and Johannesson (1996), or Garber (2000) for more details.

Under certainty, the decision maker knows the expected costs and effects of decision alternatives: c_i, e_i, respectively, where $i = 1, \ldots, n$ enumerates the alternatives. When referring to technologies being compared, we will use $T_1, T_2, \ldots$ (or capital letters A, B, …). I assume that the cost and effect are measured relative to some null option, denoting, depending on the context, no treatment, basic supportive care only, or a standard treatment. Importantly, I assume that all $e_i \geq 0$, and hence the considered technologies can only increase the effectiveness as compared to the null option. I do not impose, however, $c_i \geq 0$, as it may be the case that active treatment allows to avoid, e.g., the cost of treating complications.

If $n = 2$ then we can simply calculate the *incremental cost-effectiveness ratio*:

$$ICER = \frac{c_2 - c_1}{e_2 - e_1},$$

assuming that T_2 is more effective and more costly (otherwise the choice is trivial, or we reverse the notation). ICER measures the additional cost of obtaining an additional unit of effect. It is natural then to treat ICER as a price of health that we can pay when switching from T_1 to T_2. Hence, we should compare it to the decision maker's WTP and switch if $ICER < WTP$, the interpretation being that the market price is smaller than our reservation price. This is, in turn, algebraically equivalent to calculating *net benefit* (NB):

$$NB_i = WTP \times e_i - c_i,$$

and selecting T_i maximizing this expression. The net benefit approach is mathematically more convenient as we don't have to worry about possible dominance (when ICER is meaningless).

In the case of $n > 2$ alternatives we need to decide which pairwise comparisons to make to calculate ICERs. It has been shown that this should be done in the form of a league table, i.e., first removing some technologies, then sorting the remaining ones according to effectiveness, and finally calculating ICERs between consecutive technologies in the table (e.g., Table 1). We remove dominated technologies; we should, e.g., remove T_1 from Table 1 (dominated by T_2, i.e., is more costly and less effective). We also disregard technologies subject to extended dominance, i.e., dominated by convex combinations of two other alternatives. We should remove T_3 from Table 1 (dominated by a simple average of T_2 and T_4). Another rationale is that the ICER between T_3 and T_2 amounts to 2, and the ICER between T_4 and T_3 amounts to 1, and hence if it makes sense to upgrade from T_2 to T_3, it makes even more sense to upgrade further to T_4. We then sort the technologies by effectiveness (sorting by cost yields the same results after removing the dominated alternatives), and calculate the ICERs between consecutive technologies (the ICER for the first technology is calculated with respect to the null option).

The decision making rule for a known WTP is to proceed in this table as long as $ICER < WTP$. E.g., if $WTP = 1.8$ in our example, then we should adopt technology T_4.

Table 1: Health technologies comparison in the form of a league table

Alternative	Effect	Cost	Comment	ICER
T_1	1	3	dominated	n.a.
T_2	2	2	compared with null	1
T_3	3	4	ext. dominated	n.a.
T_4	4	5	compared with T_2	1.5
T_5	7	11	compared with T_4	2

In the actual decision making (e_i, c_i) are almost never known precisely. They are based on estimates from randomized controlled trials (RCTs), observational trials, patients' registries, etc., and hence are based on parameters given with statistical error. The values of (e_i, c_i) are often calculated using modelling, combining different parameters, extending the time horizon of the RCTs, etc. (Buxton et al., 1997). Often a Bayesian interpretation is used, in which the *a posteriori* distribution of (e_i, c_i) is available to the decision maker (Hoch and Blume, 2008). Various tools for sensitivity analysis have been proposed in the HTA literature, e.g., confidence intervals for ICER, cost-effectiveness acceptability curves (CEACs), expected value of perfect information (EVPI), cost-disutility plane, and others (cf. e.g., Eckermann and Willan, 2011). It was also pointed out that the situation becomes more complicated when more than two alternatives are considered (Barton et al., 2008; Sadatsafavi et al., 2008; Jakubczyk and Kamiński, 2010). Introducing fuzziness may complicate this further, and hence in the present paper I develop the model not accounting for uncertainty, leaving it for further research.

As can be seen in the above presentation, it is crucial to know the value of WTP to proceed with the decision making. Should the WTP be subject to (statistical type) estimation, the resulting uncertainty would be no different than parameters uncertainty and could be merged therewith and accounted for using standard techniques. In the next section, however, I argue that WTP should rather be defined using fuzzy sets concepts and hence requires a new toolbox.

3 Willingness to pay for health

3.1 Elicitation methods and results – a review

When estimating WTP we should differentiate between the willingness-to-pay to avoid certain death, the willingness-to-pay to reduce the risk of own death, and the willingness-(of the society we are part of)-to-pay to reduce the risk of somebody's death. In the first case, almost by definition, we should be willing to sacrifice all our resources (as not having sacrificed them we are certain not to profit from them). We may be willing to take a loan to pay more, or not to pay and let our children come into our wealth. One way or another, the answer to this question is both very subjective (depends on the wealth, family situation) and very emotions-driven (facing immediate death).

In the second case we are considering only marginal impact on the risk of death, and that is referred to as measuring the value of statistical life (VSL). We may try to estimate this value using revealed preferences approach, i.e., assuming that people's choices affecting their wealth and risk of death are rational and based on optimisation, hence they reveal the trade-off between life and money. An example might be the analysis of the tendency to accept risky employments (or an employment in a city that generates additional risk, e.g., due to the pollution, etc.) accounting for the wage differences. Another approach would be to see the revealed preference of the public for safety precautions, e.g., smoke detectors, burglar alarms, or airbags. Viscusi and Aldy (2003) present the results of a systematic review of the values reported in the literature. They report, for the US labour market data, VSL in the range as wide as 0.5-20.8 million USD (year 2000 value). For the US housing and product markets, they report the values in the range of 0.77-9.9 million USD. Obviously, using non-USA data further widens the range. In a newer meta-analysis Bellavance et al. (2009) present average values of VSL (along with standard deviations) calculated based on studies identified for several countries – e.g. (in million USD), for USA: 6.27 (5.04); for Canada: 9.16 (10.39); for the UK: 17 (12.59); for Australia: 11.17 (9.62). Notably the standard deviations are in the same range as the averages, proving it is difficult to come up with a reliable estimate.

Yet another question is: *'how much do you think the society you are part of should be willing to pay to save somebody's life'*. In the early 2000s in Poland the answer used to be approximated by the revealed preferences of the public payer, taking the kidney dialysis as the procedure that, as is widely accepted, ought to be provided and financed from public resources, clearly prolongs life, and has a determined cost for the public payer. Lee et al. (2009) present a quantitative analysis of this approach, showing that this translates to the implicit willingness to pay ca. $130,000 for a QALY or $61,000 for a year of life in the USA.

In the UK, where HTA is a well-established method of making a choice regarding the availability of health technologies, no official threshold is given. There were attempts to deduce this threshold via econometric analysis based on the past choices, that located WTP to be around 35,000 GBP (Devlin and Parkin, 2004; Dakin et al., 2006). A similar analysis in Poland, conducted for HTA decisions made until the end of 2011, yielded no clear conclusions on WTP (Niewada et al., 2013).

Currently in Poland the value of one QALY was set to the triple annual gross domestic product per capita, as of now ca. 120,000 PLN/QALY (based on the idea presented by Tan-Torres Edejer et al., 2003; WHO, 2001). Even though the limit is officially stated, proving a technology to offer one additional QALY at a lower cost does not guarantee reimbursement, which, in practice, makes the official threshold more of an upper acceptable bound.

Claxton et al. (2015) present another approach to estimate the WTP and combine data on health care spending and changes in mortality in the UK. They end up with lower values, of around 13,000 GBP.

As can be seen from this brief review, various methodologies can be applied, and even a single methodology can lead to varying results. The interpretation motivating the present paper is that this is exactly what should be expected based on the nature of the question. First, health cannot actually be purchased in the market so that the society can learn its monetary value. It is the health services that are bought, but the actual impact of these services on health is uncertain. The question about WTP, therefore, does not refer to any direct past experience.

Second, there is most likely a great ethically-based reluctance to define a precise threshold, if that would mean that health would not be purchased for someone, if the price exceeded the threshold by some negligible amount. That is why giving a precise answer (or behaving consistently in life-decisions, so that a revealed preferences method yields consistent results) is not possible. At the same time, as members of the society, we may feel that some values are definitely too high (we shouldn't be spending that much, and should rather direct the financial resources somewhere else) and some other values are definitely acceptable. That is what motivates the use of the fuzzy set theory to model the attitudes towards WTP.

3.2 Fuzzy description of preferences – a survey

To better justify the use of fuzzy set theory, I present the results of a survey on the perception of WTP among Polish HTA experts. Jakubczyk and Kamiński (2015) conducted a survey to verify how difficult it is for the public to decide about the WTP that should be used to ration health care services. The aim was not to come up with the ultimate estimate of the WTP, but rather to see how crisp the opinions of individuals regarding the concrete value of WTP are. In order to make it easier to understand the question the HTA experts were surveyed (27 experts participated; three answered '*no*' to the Q1 and were removed according to the survey protocol; two showed pre-defined logical inconsistencies – increasing enthusiasm in Q4 – and were removed), working in pharmaceutical companies, HTA consulting companies, and public agencies. To reduce the impact of unmentioned factors, the respondents were asked to think in terms of diabetes-related treatment. Table 2 presents the questions asked and a summary of answers (the actual questions were asked in Polish).

Table 2: The results of a survey on the willingness-to-pay in Poland

ID	Question	Answer type	Results
Q1	Cost should also be used as a criterion		88% agree/strongly agree
Q2	Exact WTP should be used in decision making		90% agree/strongly agree
Q3	This threshold should be publicly known	5-point Likert	100% agree/strongly agree
Q4	If $e_2 - e_1 = 1$, is i=2 better for various $c_2 - c_1$		(see Figure 1)
Q5	(similar to Q4, willingness to accept)		(irrelevant to this paper)
Q6	What range contains your WTP (PLN/QALY)	a range	ca. 89,000-125,000
Q7	What value equals to your WTP (PLN/QALY)	a number	ca. 105,000
Q8	How convinced are you by the answer to Q7's	4-point scale	45% level 1&2

A 5-point Likert scale used in Q1-Q5 contains the categories: *completely disagree*, *rather disagree*, *no opinion*, *rather agree*, *completely agree.* As can be seen, the respondents strongly supported the use of some kind of WTP parameter, that should be defined and publicly known in the decision making process (Q1-Q3). Hence, our respondents may be regarded as motivated to try to pinpoint the exact value of WTP.

In Q4 the respondents were asked to decide whether or not the technology that yields an additional unit of effect should be adopted if it also involves additional cost, depending on the exact value of this cost. The results are depicted in Figure 1: the fraction of respondents selecting a given answer is proportional to the area of the circle; the median answers are marked in black. We can see that there are differences between the respondents, as shown by the vertical span of responses for various suggested levels of WTP. This is especially the case for values between 100,000 and 150,000 PLN/QALY, but to a lesser degree for as wide a range as 75,000-300,000 PLN/QALY. Second, the individual respondents quite often have absolutely no opinion whether a given value should be regarded as a WTP (e.g., for WTPs = 125,000 27.3% neither agreed nor disagreed). For all the values in the range 125,000-175,000 less than a third had a definite opinion (either completely disagreed, or completely agreed).

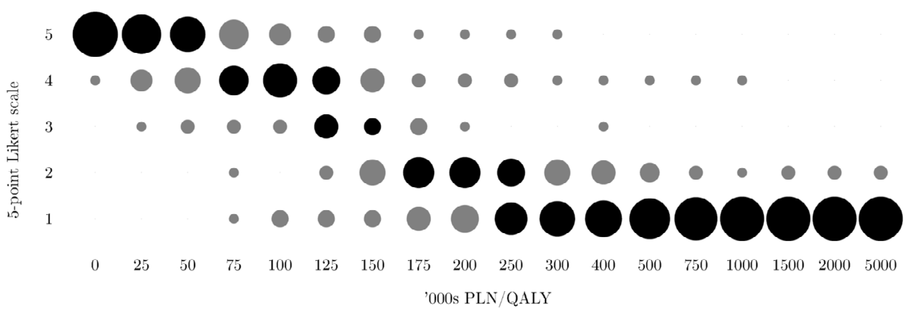

Figure 1. Respondents opinions about various levels of WTP (responses on a 5-point Likert scale: 1 = completely disagree, 5 = completely agree). The area of the circle is proportional to the percentage of responses. Median responses are in black

Third, when we combine the two above phenomena, and try to base the WTP assessment on a median voter approach, for some values of WTP we have no opinion as a society, i.e., for 125,000 and 150,000 PLN/QALY the median answer lies in the middle of the Likert scale. That means that we, as a society, are undecided whether or not the currently valid threshold (ca. 120,000 PLN/QALY) is correct.

In the survey an analogous question (Q5) was asked for the willingness-to-accept (WTA), when effectiveness was reduced, but that is of no relevance to the present study. In Q6 & Q7 the respondents were asked to give a value and a range that present their WTP. In Q8 they were asked to evaluate their satisfaction with their own answer, and almost a half was less than half-satisfied.

The results of the survey confirm that it is rather difficult, even for people with a large expertise in the area, to present a single estimate of WTP, and hence a fuzzy approach is appropriate.

4 Fuzzy decision making with multiple criteria and many alternatives – a formal model

4.1 Axioms for preferences

The axiomatic approach presented below follows the one of Jakubczyk and Kamiński (2015), but here I consider the case of more than two alternatives. To avoid difficulties with directly modelling preferences between any two alternatives (e.g., lack of transitivity), I assume that each alternative is compared to the null option only. The results of these individual comparisons are then used to select the best alternative, using a choice function approach. As all the alternatives are assumed more effective than the null option, we do not consider the relation between the WTP and willingness-to-accept (cf. Jakubczyk and Kamiński, 2015).

Let us assume that the decision maker can express her preference for each alternative $(e, c) \in \Re_+ \times \Re$, as compared to the null option. We assume that this preference is fuzzy, i.e., it is defined as $\mu(e, c) \to [0,1]$, where $\mu(e, c)$ measures the conviction that (e, c) is (weakly) preferred, i.e., is at least as good as the null option. Putting it differently, $\mu(e, c)$ is a fuzzy assessment that the sentence: *I'd like to use this technology* is true. I assume the following axioms.

Axiom 1 (reflexivity). We assume $\mu(0,0) = 1$, i.e., (something equivalent to) no treatment is as good as no treatment.

Axiom 1 serves only to clearly identify $\mu(\cdot,\cdot)$ as a fuzzy weak preference relation.

Axiom 2 (crisp preference for individual criteria). $\forall x > 0 \colon \mu(x, 0) = 1$, $\mu(0, x) = 0$, i.e., even small gains in effect (cost) are liked (disliked) in a crisp fashion.

Axiom 3 (monotonicity). $\mu(\cdot,\cdot)$ is non-decreasing (non-increasing) in the first (second) argument.

Axioms 1-3 together imply that $\mu(e,c) = 1$ for $c \leq 0$.

Axiom 4 (limit behaviour). $\forall e \in R_{+} \exists c \in R: \mu(e,c) = 0$; $\forall c \in R \exists e \in R_{+}: \mu(e,c) = 1$

Axiom 4, being quite natural, is at the same time not vital, and is introduced mainly to make the proofs easier in borderline cases.

Axiom 5 (radiality). $\forall \alpha > 0\ \mu(\alpha e, \alpha c)$ is constant.

Axiom 5 states that the decision maker is insensitive to scale, i.e., if she finds some technology (e,c) somewhat attractive, then a proportional scaling of effects and costs does not change her opinion. It might be interpreted that knowing the number of patients in which the technology might be used does not impact the evaluation. This is probably the least intuitive axiom and the first one to be dropped in further research.

4.2 Fuzzy willingness-to-pay and fuzzy net benefit

Based on the axioms presented in the previous subsection we can define the fuzzy WTP and the fuzzy net benefit. The former can be used to elicit the complete preference structure more easily (e.g., via surveys as presented in Section 3.2); the latter allows to compare alternatives with each other (even though originally the preferences are defined only between each alternative and the null option) and to define a choice function.

Note that $\mu(e,c)$ is defined trivially for $e = 0$ and for $c \leq 0$. Then, for all $(e,c), e > 0, c > 0$, $\mu(e,c) = \mu(1, \frac{c}{e})$. The value of $\mu(1,x)$ can be interpreted as the conviction that it is worth to pay x to get an additional unit of effect. Let us interpret the values of $\mu(1,x)$ as the membership function of a fuzzy set whose elements are values that are considered to be an acceptable cost to incur so as to gain one unit of effect. Hence, $\mu(1,x)$ defines the fuzzy willingness-to-pay.

Definition 1 (fuzzy willingness-to-pay, fWTP). Consider a preference structure as defined by axioms 1-5. Define the fuzzy set fWTP over the whole real axis by defining its membership function $\mu(1,x): \mathfrak{R} \to [0,1]$.

It is immediate to show that fWTP is a normal and convex fuzzy set, and that $\mu(1,x) = 1$ for $x \leq 0$. For brevity take $\mu(1,x) = fWTP(x)$. Note that under our axioms the whole preference structure can be rebuilt using fWTP as a starting point. That implies that questions like Q4 (section 0) could help to elicit fWTP, and hence the complete preference structure.

It is important that $\mu(\cdot,\cdot)$ allows to compare alternatives with the null option, but not with each other, and hence it cannot directly help to make a choice. I suggest an approach in which we measure the attractiveness of each alternative resulting from the comparison with the null option, and then make a choice using these measures of attractiveness for the individual alternatives. I suggest using the fuzzy net benefit measure, defined as in Jakubczyk and Kamiński (2015).

Definition 2 (fuzzy net benefit, fNB, of an alternative (e,c)). Consider a preference structure as defined by axioms 1-5 and a given alternative (e,c). Define a fuzzy set fNB over the whole real axis by defining its membership function $fNB_{(e,c)}(x): \Re \to [0,1]$ as $fNB_{(e,c)}(x) = \mu(e, c+x)$ (the subscript will be omitted or replaced by another symbol denoting a technology when convenient)

The fNB measures the conviction that by adopting (e,c), instead of the null option, the decision maker effectively gains x (in monetary terms), i.e., would be indifferent to adopt (e,c) for an additional cost of x. We could alternatively define $fNB(x) = fWTP(\frac{c+b}{e})$. I will denote by $fNB^{\alpha}_{(e,c)}$ the α-cuts of fNB.

4.3 Choosing with fNB

In the previous subsection I defined the fNB that can be calculated for each alternative. Comparing two technologies could then be reduced to comparing two fuzzy sets, fNBs. Choosing a technology from a larger set can, in turns, be defined as maximizing fNB, treated as a fuzzy number. It is important that the choice method should not violate intuition, and the following proposition says that fNB meets the basic properties.

Proposition 1 (fNB respects dominance). Assume axioms 1-5. Consider any two alternatives: $(e_1,c_1),(e_2,c_2)$, such that $e_2 \le e_1 \wedge c_2 \ge c_1$ and at least one inequality is strict. Then fNB_{e_2,c_2} is strictly smaller than fNB_{e_1,c_1} in the sense that: $\forall\alpha > 0\ fNB^{\alpha}_{e_2,c_2} \subset fNB^{\alpha}_{e_1,c_1}$ and $\exists\alpha > 0\ fNB^{\alpha}_{e_2,c_2} \neq fNB^{\alpha}_{e_1,c_1}$.

The above proposition guarantees that fuzzy approach to net benefit allows to maintain the information that a dominance holds, and hence the dominated alternative is not worth considering. The next proposition extends it to the extended dominance case.

Proposition 2 (fNB respects extended dominance). Assume axioms 1-5. Consider any three alternatives: $(e_1,c_1),(e_2,c_2),(e_3,c_3)$, such that $\exists\lambda \in (0,1)$ that $e_3 < \lambda e_1 + (1-\lambda)e_2 \wedge c_3 > \lambda c_1 + (1-\lambda)c_2$. Then for all $\forall\alpha > 0\ fNB^{\alpha}_{(e_3,c_3)} \subset (fNB^{\alpha}_{(e_1,c_1)} \cup fNB^{\alpha}_{(e_2,c_2)})$ and for some α it is a proper subset.

Propositions 1-2 justify the omission of the dominated or extended dominated alternatives in the comparisons. And vice versa: they suggests that comparing alternatives can be attempted by comparing the α-cuts of fNB sets, and, in particular, the suprema of the α-cuts. I propose the following choice function.

Definition 3 (fuzzy choice function, fC). Consider a finite set of alternatives $T_1, \dots, T_n$ described by (e_i, c_i), and the preference structure as defined by axioms 1-5. For each alternative T_i calculate the set A_i containing such an α that $fNB^{\alpha}_{e_i,c_i}$ is the largest of (or equal to) all α-cuts:

$$A_i = \left\{\alpha \in [0,1]: \forall_{j\in\{1,\dots,n\}} fNB^{\alpha}_{(e_j,c_j)} \subset fNB^{\alpha}_{(e_i,c_i)}\right\}.$$

A fuzzy choice function is then defined as:

$$fC(T_1, \dots, T_n) = (|A_1|, |A_2|, \dots, |A_n|),$$

where $|A_i|$ denotes the Lebesgue measure of A_i.

The fuzzy choice function returns then an ordered n-tuple of numbers between 0 and 1 that we will interpret as the conviction that a given alternative is the best choice. The next proposition claims that Definition 3 can be actually used, i.e., the resulting $|A_i|$ are intervals and hence have a well-defined measure.

Proposition 3 (definition of fC is formally correct). The sets A_i defined in Definition 3 are (perhaps empty) intervals, and hence the Lebesgue measure is well defined (and is, trivially, their length).

We can justify the use of fC appealing to intuition in several ways. First, it is in agreement with dominance and extended dominance as stated in Propositions 1-2. Second, consider crisp preferences, i.e., such that $\mu(\cdot,\cdot) \in \{0,1\}$ and take $WTP^* = \sup\{x \in \Re: fWTP(x) = 1\}$. Consider two technologies only: (e_1, c_1) and (e_2, c_2), and $ICER = \frac{e_2-e_1}{c_2-c_1}$. Then, if $ICER < WTP^*$ we get $fC(T_1, T_2) = (0,1)$, and hence T_2 is recommended. If $ICER > WTP^*$, $fC(T_1, T_2) = (1,0)$. In the limit case of $ICER = WTP^*$ we have $fC(T_1, T_2) = (1,1)$, and hence the decision maker can safely choose any alternative.

Third, let us return to fuzzy preferences, and compare two technologies: (e_1, c_1) and $(e_1 + e_2, c_1 + c_2)$. Using the additivity of fNB (cf. the proof of Proposition 3) it is interesting to measure the conviction that (e_2, c_2) offers a positive NB, and hence let $\alpha^* = \mu(e_2, c_2)$. It is easy to verify that fC yields exactly α^* as the conviction that $(e_1 + e_2, c_1 + c_2)$ should be chosen.

Thus, using fNB allows to define a fuzzy choice function that returns a (possibly non-normal) fuzzy set over the universe of all *a priori* alternatives. The membership function of fC combines the complete available information on the decision maker's (fuzzy) preferences and relative attractiveness of alternatives accounting for both criteria: effect and cost.

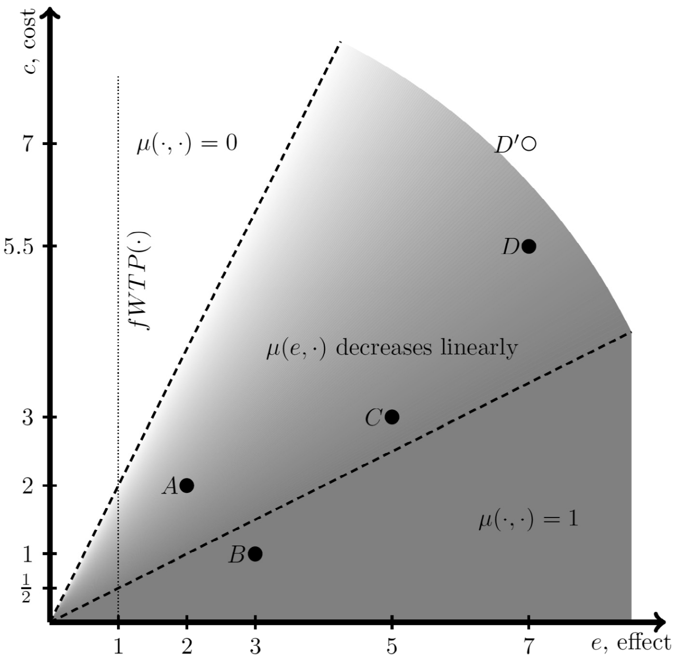

Figure 2. An example: four technologies shown in the cost-effect plane: A = (2,2), B = (3,1), C = (5,3), D = (7,5.5). We additionally consider D' = (7,7). Shades of grey represent the values of $\mu(e,c)$

Figure 2 and Figure 3 present an example. Figure 2 shows sample technologies A-D (and, additionally, D'). Note that A is dominated. I assume that $\mu(e,c) = 1$ below the line $2c = e$, and $\mu(e,c) = 0$ above the line $c = 2e$. Between these lines $\mu(e,c)$ decreases linearly with c, as shown by changing shades of grey. Specific values can also be projected radially from the membership function of $fWTP(\cdot)$, drawn as a horizontal line through (1,0). Figure 3 presents the membership functions of fNB for technologies A-D. fNB_A is moved to the left as compared to fNB_B due to the dominance. All other technologies offer the greatest net benefit with some conviction, while D maximizes the net benefit for the largest range of α's, which is reflected by the values of fC: $fC(A,B,C,D) = (0,\frac{1}{3},\frac{1}{6},\frac{1}{2})$. Note that if we considered D' instead of D, we would have to move fNB_D left by 1.5. Then $fC(A,B,C,D') = \left(0,\frac{1}{3},\frac{2}{3},0\right)$, and hence the technology D' is not recommended (not being dominated) as $ICER_{DvsC} = 2$, and $\mu(1,2) = 0$.

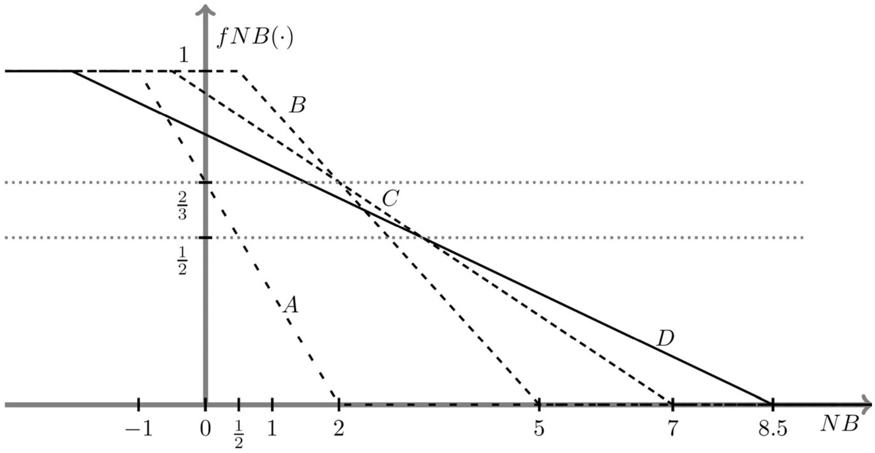

Figure 3. fNB for technologies presented in Figure 2. Horizontal dotted lines show crossings, and hence $fC(A,B,C,D) = (0,\frac{1}{3},\frac{1}{6},\frac{1}{2})$

5 Final remarks

The motivation for the present paper was the conviction that fuzzy approach is natural to WTP. Luckily, the fuzzy approach can be operationalized, i.e., axiomatically based, elicited using surveys, and used for decision making. The main outcome of the present paper is a conceptual framework allowing to use this fuzzy approach to compare several alternatives – health technologies. The paper is focused on the technical aspects of this framework, i.e., it is consistent with intuitive properties (e.g., respecting the dominance). Once the framework is developed (e.g., to encompass uncertainty) it can be used in the HTA process, i.e., in comparing health technologies, to inform the decision maker about the attractiveness of decision alternatives at hand.

One might be disappointed that the outcome is only a fuzzy choice function, i.e., a statement that, e.g., we are 0.4 convicted that T_1 should be selected, and 0.6 convicted that T_2 should be selected. It is important to stress that the goal was to show how far the fuzzy preferences, being the departure point, can be taken without forcibly changing fuzzy opinions into crisp ones. Obviously, the ultimate decision requires crispification, e.g., taking the argmax of the $fC(\cdot)$ choice function (and selecting D in the example in Figure 3).

Note that the current approach, i.e., comparing all the alternatives with the null option, allows to disregard the potential technical difficulties with the preference relation not being a total pre-order and also allows to focus on positive effects, and to disregard the potential difficulties with $WTP \neq WTA$, lack of transitivity, etc.

Further research should, in my opinion, focus on the following issues: i), discussing other possible approaches to making a crisp choice based on the fuzzy choice function outcomes (and to verifying their properties); ii), introducing uncertainty into the model; iii) trying to discuss and perhaps relax some axioms, e.g., radiality. Also, the present paper is a theoretical one, and further research should also present some sample applications of this methodology to actual decision problems.

Appendix

Proof of proposition 1

Let us start with a quick proof of the non-strict version. Take any $x \in \mathfrak{R}$. Then $fNB_{e_2,c_2}(x) = \mu(e_2, c_2 + x)$. Using the monotonicity axiom we immediately get that $\mu(e_1, c_1 + x)$ is not smaller. Now, let us proceed with the strict version, which we will prove for $\alpha = 1$. First, note that for $e_1 = 0$ we have also $e_2 = 0$, and hence $c_2 > c_1$ (for dominance to hold), which immediately gives the desired result, as α-cuts will be translated horizontally by the difference in cost. Assume henceforth that $e_1 > 0$. Denote $y = \sup\{x \in \mathfrak{R}: \mu(e_1, c_1 + x) \geq 1\}$, and hence y is the supremum of the considered α-cut (here $\alpha = 1$). Limit behaviour and radiality imply that $c_1 + y > 0$. Radiality further implies that $\sup\{x \in \mathfrak{R}: \mu(e_2, c_2 + x) \geq 1\} = (c_1 + y)\frac{e_2}{e_1} - c_2 = y\frac{e_2}{e_1} + \left(c_1\frac{e_2}{e_1} - c_2\right)$, where either $\frac{e_2}{e_1} < 1$ or the second term is negative, which finishes the proof for $\alpha = 1$. The proof for other $\alpha > 0$ follows analogously.

Proof of proposition 2

Let us consider, non-trivially, $e_2 > e_1 \wedge c_2 > c_1$ and $c_3 < c_2 \wedge e_3 > e_1$, as otherwise (e_3, c_3) is simply dominated by one of the other two alternatives. Note that $ICER_{3vs1} > ICER_{2vs3}$. Take any $\alpha \in (0,1]$. Denote $y = \sup\{x \in \mathfrak{R}: \mu(e_3, c_3 + x) \geq \alpha\}$. Limit behaviour, monotonicity, and radiality imply that $c_3 + y > 0$. Consider the slope of the line passing through the origin and the point $(e_3, c_3 + y)$, i.e., $\frac{c_3+y}{e_3}$. Assume that $\text{ICER}_{3vs1} > \frac{c_3+y}{e_3}$. Simple algebraic transformations yield that: $(c_3 + y)\frac{e_1}{e_3} - c_1 > y$, and hence the respective α-cut for technology 1 is larger than that for technology 3. If $\text{ICER}_{3vs1} \leq \frac{c_3+y}{e_3}$, then $\text{ICER}_{2vs3} < \frac{c_3+y}{e_3}$, and we get the required result for the α-cut for technology 2.

Proof of proposition 3

Consider any $(e_1, c_1), (e_2, c_2)$, $e_1 > 0, e_2 > 0$. Using radiality we can easily notice that fNB is additive, i.e., for any $\alpha > 0$, we have $\sup fNB^{\alpha}_{e_1,c_1} + + \sup fNB^{\alpha}_{e_2,c_2} = \sup fNB^{\alpha}_{(e_1+e_2),(c_1+c_2)}$. The monotonicity axiom implies that $\sup fNB^{\alpha}_{e_2,c_2}$ is non-increasing in . These two further imply that if for any α^* we have $\sup fNB^{\alpha^*}_{(e_1+e_2),(c_1+c_2)} \geq \sup fNB^{\alpha^*}_{e_1,c_1}$ then also for any $\alpha < \alpha^*$ we have $\sup fNB^{\alpha}_{(e_1+e_2),(c_1+c_2)} \geq \sup fNB^{\alpha}_{e_1,c_1}$. This yields the result.

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MULTIPLE CRITERIA DECISION MAKING
Vol. 10 2015

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EVOLUTIONARY MULTIOBJECTIVE OPTIMIZATION FOR INTENSITY MODULATED RADIATION THERAPY

Abstract

As cancer diseases take nowadays a heavy toll on societies worldwide, extensive research is being conducted to provide more accurate diagnoses and more effective treatments. In particular, Multiobjective Optimization has turned out to be an appropriate and efficient framework for timely and accurate radiotherapy planning.

In the paper, we sketch briefly the background of Multiobjective Optimization research to Intensity Modulated Radiation Therapy, and next we present a rudimentary formulation of the problem. We also present a generic methodology we developed for Multiple Criteria Decision Making, and we present preliminary results with it when applied to radiation treatment planning.

Keywords: Evolutionary multiobjective optimization, multiple criteria decision making, Intensity Modulated Radiation Therapy planning.

1 Introduction

Around the mid 1990s, precise techniques to deliver radiation to malicious tissues became available and then optimization techniques were harnessed to produce patient treatment plans timely and accurately. This resulted in a flow of research papers on the subject, estimated in several hundreds. About a decade later, the multiobjective optimization quite naturally turned out to be an adequate framework to represent trade-offs between the goal to irradiate the tumorous re-

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gions of the body with sufficiently high levels of radiation, and the requirement to protect healthy organs as much as possible.

The principle of radiation therapy is as follows. A number of high energy *beamlets* (rays), of order of tens of thousands (depending of the equipment), are radiated from a linear accelerator towards a patient positioned on a couch. The beamlets deposit radiation doses in the patient's tissue causing its ionization. When the radiation dose is over a certain level, the tissue is killed.

In the early stage of oncological radiation therapy (in the sequel, for short, *radiotherapy*), conformal radiotherapy was used. In this technique, all points of the radiated field receive the same dose and the shape of the field is formed with physical reflectors and dumpers (Bortfeld et al., 1994).

The first delivery using the *intensity modulated radiation therapy* (IMRT), was reported in 1994. From that time clinical evidences have been collected and reported in the literature that IMRT is remarkably well suited to multiobjective optimization (Küfer et al., 2005; Craft et al., 2012; Breedveld et al., 2012).

2 Intensity Modulated Radiation Therapy

The energy which can be deposited in a tissue by a beamlet is proportional to the time the beamlet is radiated. This time is controlled by a *collimator* – a set of iron blades which slide across a rectangular aperture in the *radiation head* (with a linear accelerator inside) with varied speed. When the aperture is fully open, all the beamlets carry the same energy. On the other end, when the aperture is fully closed, no radiation is emitted. In between, the collimator allows for a whole range of radiation energy patterns, called *fluency maps*. An example of a fluency map for 4 × 5 beamlets is given in Figure 1.

A collection of beamlets radiated from one position is called a *beam*. The radiation head, mounted on the *rotating gantry*, can be a source of many beams (say 36 beams with a 10° angle step).

The problem is to produce fluency maps whose superposition kills the malicious (tumor) tissue with the least harm to the organs which have to be especially protected (Organs At Risk) and limited doses to the normal tissue (not tumor or any OAR) of the patient. This is schematically presented in Figure 2.

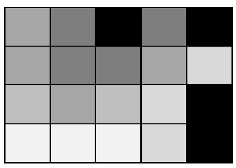

Figure 1. An example of a fluency map for 4 × 5 beamlets. The darker the colour is, the higher dose is deposited in a voxel

To control the radiation dose deposition in the irradiated region of the patient's body, this region is divided into small cubes (say, depending on the accuracy required, 2.5 mm × 2.5 mm × 10 mm), called *voxels*. The radiation dose deposited in a voxel by a beamlet radiated for one unit of time is specific to that voxel (this is calculated from a physical model) and denoted by d_{ij}, where i is the index of the voxel and j is the index of the beamlet. Thus, the dose deposited in voxel i is $d_i = \sum_j d_{ij}\, x_j$, where x_j is radiation time for beamlet j. This can be represented in the matrix form:

$$Dx = d\,,$$

where $D = \{d_{ij}\}$ is the dose-influence matrix for all beams. Additivity of radiation doses deposited by individual beamlets is the standard assumption in the oncology radiotherapy.

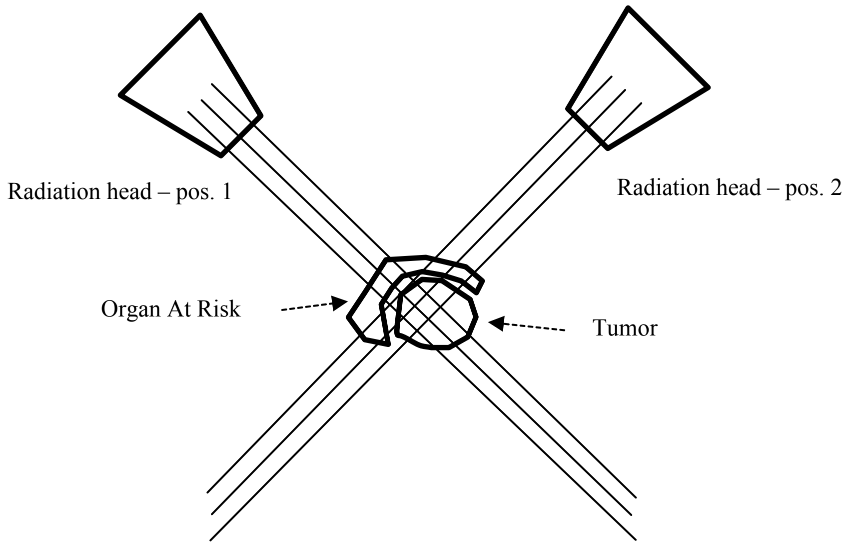

Figure 2. A schematic representation of radiation delivery to tumor and OAR by two beams

3 Multiobjective Optimization in IMRT

The distribution of energy doses to tumor, OARs and normal tissue, is the subject of optimization.

The rudimentary multiobjective optimization (specified up to objective functions) model for radiotherapy treatment planning is as follows:

$$\begin{gathered} f_l(d) \ \rightarrow \max\ (or\ min) \quad l = 1, \dots, k\,, \\ Dx = d, \\ l_{tumor} \ \leq \ d_i \leq \ u_{tumor}\,, i \in I_{tumor}\,, \\ d_i \ \leq \ u_{OAR_t}\,, i \in I_{OAR_t}\,, t = 1, \dots, s\,, \\ d_i \leq \ u_{normal\ tissue}\,, i \in I_{normal\ tissue}\,, \end{gathered} \tag{1}$$

where x is the vector of beamlet radiation times, I_{tumor}, I_{OAR_t} and $I_{normal\ tissue}$ are the sets of indices of voxels belonging to the respective areas, s is the number of OARs. Radiation doses deposited in tumor voxels are bounded from below by l_{tumor} and from above by u_{tumor} . For voxels of OARs and of the normal tissue only upper bounds u_{OAR_t} and $u_{normal\ tissue}$, respectively, are imposed.

The interplay between objective function values defines doses delivered to the tumor, to OARs and to the normal tissue. Dosed delivered to tumor should be maximized and doses delivered to OAR and the normal tissue should be minimized. To fulfil these general goals, various objective functions are used.

As an alternative, two-sided constraints on doses deposited in tumor voxels can be replaced by a weaker requirement, namely that the deviation of the average dose deposited in a voxel of the tumor from the dose prescribed be within a band around zero.

It should be stressed here that multiobjective optimization models solved for optimization of radiotherapy planning are large-scale, with the number of voxels reaching hundreds of thousands and the number of beamlets reaching tens of thousands.

As we see from the rudimentary multiobjective optimization model, the only element which can differentiate between models are objective functions. In the radiology literature there are many objective functions proposed. They can be of the statistical type, i.e. describing the dose distributions in organs considered, and of the biophysical type, describing the effect of radiation on the radiated cells. The latter are as a rule nonlinear. Taking this in mind, and wanting to be independent of solvers devoted to a particular class of problems and to have freedom to switch from one type of criteria function to another without having to pay attention to their analytical properties, one can opt for multiobjective evolutionary optimization (Deb, 2001; Deb et al., 2003; Coello Coello et al., 2002; Bokrantz, 2013). Bellow we follow this option.

However, switching to evolutionary computations, which are in principle heuristics with no performance guarantee, one loses a grip on the concept of optimality. In the next section, we show how one can cope with this issue.

4 The Proposed Multiple Criteria Decision Making Methodology

For the sake of consistency, we present here the proposed methodology in terms specific to radiotherapy planning. However, the method, originally described in Kaliszewski et al. (2012), is general and can be applied to any multiple criteria decision making problem.

Let x denote a vector of beamlet intensities of length n . By the physical interpretation, set X_0 of feasible x is a subset of R^n_+, the nonnegative orthant of R^n .

The underlying Multiobjective Optimization model for *Multiple Criteria Decision Making* is[1]:

$$"max"f(x), \tag{2}$$

where $f: R^n_+ \to R^k$, $f = (f_1, \dots, f_k)$, $f_l : R^n_+ \to R, l = 1, \dots, k, k \geq 2$, f_l are *objective functions, "max"* denotes the operator of deriving all Pareto optimal solutions (in the sense of Pareto) in X_0.

We assume that Pareto optimal solutions are derived by solving the following optimization problem:

$$min_{x \epsilon X_0}\; max_l\; \lambda_l(y^* - f_l(x)), \tag{3}$$

where y^* is such that $f(x) < y^*$ for any $x \in X_0$. The set $f(x)$, where x are all Pareto optimal solutions, is called the *Pareto front.*

We have selected optimization problem (3) as a Pareto optimal solution generator because it has the ability to provide all Pareto optimal solutions[2] to a given problem, the only condition being the existence of element y^* (for details, see e.g. Kaliszewski et al., 2012; Kaliszewski, 2006; Ehrgott, 2005; Miettinen, 1999).

Under the assumption that all objectives are of the "max" type, for a given element y^*, the optimization problem realizes a line search along the so-called *compromise half line* (Kaliszewski et al., 2012), provided that the compromise half line $y = y^* - t\,\tau, \tau_l > 0, t \geq 0$, intersects set $f(X_0)$, but it yields a Pareto optimal solution in any case. This argument is graphically represented in Figure 3.

The relation between search directions τ (called in Kaliszewski et al., 2012 *directions of concessions*) and parameters λ in the objective function of optimization problem (2), is given by formula:

$$\lambda_l = (\tau_l)^{-1}, l = 1, \dots, k. \tag{4}$$

All components of search directions τ_l are positive, hence $\lambda_l > 0, l = 1, \dots, k$ (Kaliszewski et al., 2012).

Formula (4) establishes a clear relationship between technical parameters λ in the optimization problem (3), and the realm of decision making where vectors of concessions τ are easily interpretable. Indeed, vector τ represents a simple form of *preference carrier* which can be used to encapsulate the radiotherapy planner's (in general: the decision maker's) preferences[3].

---

1 For the sake of brevity of presentation we assume here that all objectives are of, or are converted to, the "max" type.

2 In fact, this optimization problem provides a characterization of weakly Pareto optimal solutions, but for the problems considered in this work such a distinction is immaterial.

3 For the complete treatment of this problem see: Kaliszewski (2006); Ehrgot (2005); Miettinen (1999).

With vectors of concessions we can do more than that. With two sets of elements: feasible S_L – *lower shell* and infeasible A_U – *upper approximation*, with images $f(\cdot)$ located, respectively, below and above set $f(x)$, where x are Pareto optimal solutions, for a given vector of concessions τ we can calculate $L_l(\tau, S_L)$ and $U_l(\tau, A_U)$, such that:

$$L_l(\tau, S_L) \le f_l(x) \le U_l(\tau, A_U),\ l = 1,\dots,k,$$

where x is the solution which would be derived if the optimization problem was solved with $\lambda_l = (\tau_l)^{-1}$, $l = 1,\dots, k$. We can now estimate unknown $f_l(x)$ indirectly, by lower and upper bounds $L_l(\tau, S_L)$ and $U_l(\tau, A_U)$, $l = 1,\dots, k$ (Kaliszewski et al., 2012).

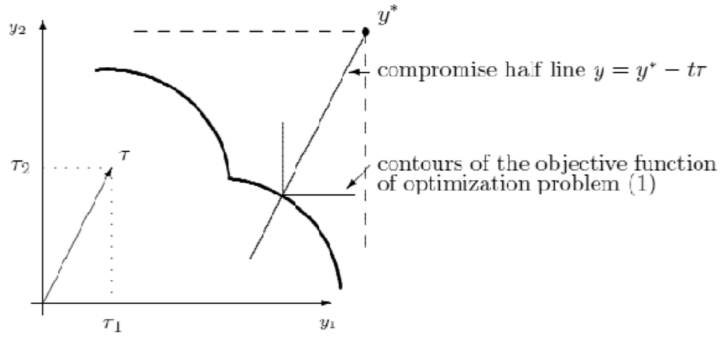

Figure 3. A graphical interpretation of vector of concessions τ and optimization problem (2)

Sets S_L and A_U are to be derived by specific evolutionary multiobjective optimization algorithms (Kaliszewski et al., 2012).

5 Preliminary Results

We have solved a number of test problems extracted from anonymized clinical data. The largest problem solved (Head & Neck tumor case) has 3 beams, 8064 beamlets, 181292 voxels. An approximation of the Pareto front was obtained with the NSGA-II evolutionary algorithm with 300 iterations and population size equal to 100. The number of elements in the approximation was 76. Computations on an AMD Dual Core E2-1800, 1.7 GHz desktop computer running under the Linux operating system took 10 min. In this particular case, the formulation of the rudimentary problem (1) presented in Section 3 was as follows:

$$\frac{1}{|I_{tumor}|}\sum_{i \in I_{tumor}} d_i \to max,$$

$$\max\{max_{i \in I_{SPINE}} d_i,\ max_{i \in I_{JAW}} d_i\} \to min,$$

$$Dx = d,$$

$$0.18 \times 66\ Gy \le d_i \le 1.15 \times 66\ Gy,\ i \in I_{tumor},$$

$$d_i \le 45\ Gy,\ i \in I_{SPINE},$$

$$d_i \le 70\ Gy,\ i \in I_{JAW},$$

where Gy (gray) is a unit of radiation dose, SPINE and JAW are OARs, $|\cdot|$ denotes the cardinality of a set. The first objective function represents the average dose per voxel deposited in the tumor, and this value has to be maximized. The second objective function represents the maximum of doses deposited in voxels of two organs to be spared, namely spine and jaw, and this value has to be minimized.

It is worth observing that even for the size of the largest problem it was possible to derive an approximation of the whole Pareto front (in Figure 4, to be consistent with multiobjective optimization model (2), the second objective function is multiplied by -1 and maximized). To our best knowledge, solving multiobjective optimization problems of such sizes have not been reported in the literature. The only paper which discusses the issue of solving large-scale multiobjective optimization problems by evolutionary computations is Antonio, Coello Coello (2013). However, that paper presents results for artificial test problems scalable to any size. The paper reports solving problems with up to 5000 variables and only box constraints.

With elements of Pareto front approximations derived, we are able to apply our methodology, as outlined in Section 4. Let us illustrate how it can work with the largest problem solved for the Head & Neck cancer case. We can proceed according to two scenarios. Both scenarios are hypothetical because the preliminary results we have obtained are of no real clinical value. Radiotherapy plans with the IMRT technique applied to patients involve at least five beams. Therefore the results we have obtained so far are to be regarded only as a proof of the concept.

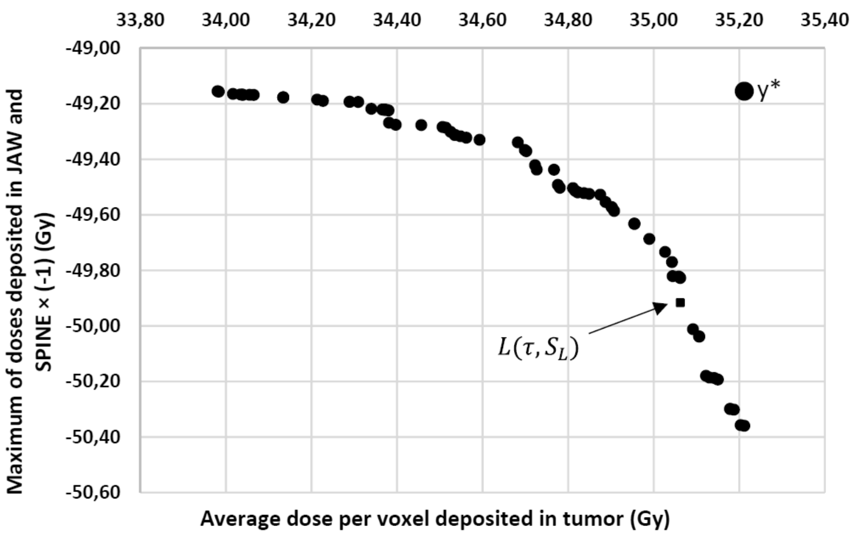

Figure 4. A Pareto front approximation of Head & Neck tumor case with three beams

Decision making scenario 1

Let us regard the approximation of the Pareto front derived as an accurate representation of the Pareto front, sufficient for radiotherapy planning.

Presenting 76 elements of the Pareto front to the radiotherapy planner (medical physicist) or the oncology physician leaves him unsupported.

Here comes the proposed multiple criteria decision making methodology presented in Section 4. We calculate element $y_l^* = max_{x \in \{Pareto\ front\}} f_l(x) + \varepsilon, \varepsilon > 0, l = 1{,}2$, and for $\varepsilon = 0.5$ (selected arbitrarily) we get $y^* = (35.71, -49.15)$.

For selected vectors of concessions τ, using formula (4) we can find, in the 76-element representation of the Pareto front, the element with the minimal value of the objective function in problem (3). With τ representing the decision maker's preferences, the selected elements correspond best to those preferences (in the sense of the objective function in problem (3)).

Table 1 presents selected elements for five vectors τ (in the table vectors τ are normalized). To be consistent with the assumption made in Section 4, the second objective function was converted to the "max" type by multiplication of its values by -1.

Table 1: Selected elements for a pair of vectors τ

τ_1	τ_2	$f_1(x)$	$f_2(x)$	$L_1(\tau, S_L)$	$L_2(\tau, S_L)$
0.1	0.9	35.21	-50.36	35.21	-53.15
0.25	0.75	35.18	-50.30	35.16	-50.30
0.5	0.5	34.85	-49.52	34.84	-49.52
0.75	0.25	34.13	-49.18	34.13	-49.18
0.90	0.10	33.98	-49.15	31.21	-49.15

Table 1 also provides bounds on solutions of problem (3) which would be derived if problem (3) was solved with a given τ. It is of interest to note, that, in full accordance with the methodology, in some cases lower bounds on components can be higher than components of minimizers of the objective in problem function (2). For example, for $\tau = (0.1, 0.9)$, the lower bound on the second component is -53.15, whereas the second component of the element derived for that τ is -53.36.

Decision making scenario 2

Let us regard the approximation of the Pareto front derived as an inaccurate representation of the Pareto front, insufficient for radiotherapy planning. In that case we can use it as a shell S_L (see Section 4) to calculate lower bounds on components of unknown $f(x)$, selected implicitly by DM's preferences represented by vectors τ. For example, in Figure 4 there is a region not well covered by elements of S_L in the segment $[-50.00, 49.80]$ of the horizontal axis. We can probe that region with the compromise half line with $\tau = (0.34, 0.66)$. Without solving problem (3) we get lower bounds for the solution of this problem for $\lambda_l = (\tau_l)^{-1}, l = 1,2$, as shown in Table 2. In this way, we can probe any fragment of the Pareto front.

We could also get an upper bound for this solution, but for this aim we would need an upper approximation A_U. As the problem considered here has no clinical value and is used here as an illustration, we did not calculate upper bounds. But for more realistic, hence larger problems, we will calculate two-sided bounds which is a reasonable way to avoid solving problem (3) explicitly.

Table 2: Two-sided bounds

τ_1	τ_2	$f_1(x)$	$f_2(x)$	$L_1(\tau, S_L)$	$L_2(\tau, S_L)$
0.34	0.66	*unknown*	*unknown*	35.06	-49.92

6 Concluding Remarks and Direction of Further Research

This paper reports on our efforts to establish practical connections between multiobjective optimization and radiotherapy planning. To this aim we are strongly supported by cooperating radiotherapy planners who:

1) have shown deep interest in the issue,
2) provided us with clinical data,
3) verify results of our computations,
4) declare to use our results in clinical practice if we provide comparative or better results than those produced by treatment planning systems currently in use.

The preliminary results we have obtained indicate that problems with a limited number of beams, but nevertheless large-scale problems, can be solved with general purpose Evolutionary Multiobjective Optimization methods, where the solution takes the form of a (hopefully fair) representation of the Pareto front. That is a novelty in the literature on the multiobjective optimization.

As radiotherapy plans quality increases with the increasing number of beams, we expect that the derivation of representations of the whole Pareto front, given a reasonable time budget, will not be possible. In fact, we have never intended to propose this. Instead, with the relation (4) the radiotherapy planner is in the position to direct the derivation of radiotherapy treatment plans to the regions of the Pareto front of his/her direct interest. To arrive at a feasible and Pareto optimal treatment plan, the planner has to compromise on unattainable values of components of y^* and he/she can easily do so in terms of vectors of concessions.

Encapsulation of preferences in terms of vectors of concessions is a simple but sufficient tool to interface the decision making realm with optimization engines, the latter of no or little interest for a general decision maker. The approach and tool we propose and advocate sets a very low cognitive barrier for entering into Multiple Criteria Decision Making. In radiotherapy planning, where planners (medical physicists by profession) work in the regime of daily routines, under stress and time pressure to deliver patient radiation plans timely, this is a key factor for the successful adoption of the multiple criteria perspective.

However, the ultimate goal, as it is suggested by radiotherapy practitioners we cooperate with, should be to actively include physicians-oncologists, who are the last and decisive link in the decision making chain, in the multiple criteria decision making processes. For this aim, a low cognitive barrier to enter will be of paramount importance.

Providing clean radiotherapy data, in formats suitable for optimization, requires a great amount of work. It has taken us two years to produce preliminary results. In addition, some physical models have been built to provide data, which cannot be otherwise obtained from commercial systems currently in operation in oncology centres.

We would like to stress again that the approach to multiple criteria decision making outlined in this paper, being general, is applicable to any problem with multiobjective optimization as the underlying model. It has been already successfully applied to problems in engineering design (Kaliszewski et al., 2015) and to the airport gate assignment problem (Kaliszewski et al., 2013).

Acknowledgements

The Authors wish to express their gratitude to Paweł Kukołowicz and Anna Zawadzka from the Department of Medicine Physics, Cancer Center and Institute of Oncology, Warsaw, and Bartosz Chaber and Jacek Starzyński from the Faculty of Electrical Engineering, Warsaw University of Technology, Warsaw, for valuable discussions on the topic of this paper and their assistance in providing data for numerical tests.

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Vol. 10 2015

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FUZZY PARETO DOMINANCE IN MULTIPLE CRITERIA PROJECT SCHEDULING PROBLEM

Abstract

Planning is one of the most important aspects of project management. A project plan defines objectives, activities and timeframe for project realization. To be able to define the required timeframe for project realization it is important to prepare its schedule.

The purpose of this paper is to present the project scheduling problem as a multiple criteria decision making problem and to solve it using two evolutionary algorithms: SPEA2 and an evolutionary algorithm driven by the fuzzification of Pareto dominance. A comparison of these two approaches is conducted to investigate if it is reasonable to use the fuzzification of the Pareto dominance relation in evolutionary algorithms for the multiple criteria project scheduling problem.

Keywords: fuzzy Pareto dominance, project scheduling problem, multiple criteria optimization, evolutionary algorithms.

1 Introduction

A company's success depends on how it adapts to the changes in its current dynamic environment. Changes are conducted under pressure of time and cost and with limited access to the resources. Those changes should be managed as projects. In the current environment, when companies have to adapt to changes quickly, the number of projects conducted in companies is increasing. We can say that currently more than 25% of companies' activities should be managed as projects (Brilman, 2002). This is the case in such areas as engineering or IT. Projects managed properly lead to situations when companies' goals are met on time within the assumed budget and with limited resources.

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One of the most important phases of project management is project planning. Scheduling is one of the most important elements of a project plan. The most popular techniques used by companies for project scheduling are CPM and PERT which provide schedules optimal in terms of time. In real-life applications a project schedule should be optimized also in terms of other elements such as resources or cash flows generated in the project.

The multiple criteria project scheduling problem is not frequently discussed in the literature.

An example of describing and solving the multiple criteria project scheduling problem is presented in Viana, de Sousa (2000). The authors have considered a resource constrained problem whose objectives are: minimization of the project completion time, minimization of project delay, and minimization of the violation of resource constraints. They have presented two multiple objective techniques to solve this problem: Pareto Simulated Annealing and Multiple Objective Taboo Search.

Also Hapke et al. (1998) considered the multiple criteria project scheduling problem. They have described a problem using four components: the set of resources **R**, the set of activities **Z**, the set of precedence relationships on **Z,** and the set of objectives **C**. The project scheduling problem is a problem of allocation of resources from the set **R** to activities from the set **Z**, so that all activities can be completed, constraints can be met and the best compromise between the objectives from **C** is reached. The authors have considered a problem with three criteria: project cost minimization, project delay minimization, and resource usage optimization. To solve the problem, Pareto Simulated Annealing was used in the first stage and the interactive local search method was used to identify the final solution from the set of solutions obtained in the first stage.

The multiple criteria project scheduling problem is also considered in Krzeszowska (2013). The author has proposed a mathematical model with three objectives: minimization of penalty for project delay, minimization of the cost of additional resource usage, and maximization of NPV. The problem was solved in two stages. In the first stage the SPEA2 algorithm was used to find the set of non-dominated solutions. In the second stage an interactive method was used to identify the final solution from the set of solutions obtained in the first stage.

Another example is described in Leu et al. (1999). The authors have considered a resource constrained problem with three objectives: time, cost, and resource usage optimization. The problem is solved in two stages. In the first stage a compromise between time and cost is considered and resources are allocated to the project. In the second stage resource leveling is applied.

In the present paper the multiple criteria project scheduling problem is considered. Three objectives are taken into account: minimization of the penalty for project delay, minimization of the penalty for resource over-usage, and maximization of NP. The problem is solved with the fuzzy dominance-driven evolutionary algorithm. The results obtained are compared with the result obtained by the SPEA2 algorithm.

2 Multiple criteria project scheduling problem

We consider a project for which a schedule should be prepared. By scheduling we understand setting the start and finish times for each activity of the project. We are looking for a schedule which meets constraints and is the best compromise between the objectives.

For the problem described above the following assumption have been made:

– the project consists of J activities $j = 1,\dots,J$,
– the project has been described on an AON network (Activity On Node – using this type of network allows to use all precedence relationship types),
– each activity is described by three elements: duration, type and amount of required resources, cash flows generated by each activity,
– deterministic time is considered,
– if the project is finished with delay, a penalty is foreseen for each unit of delay,
– cash flows are generated at activity completion,
– only renewable resources are constrained (with the assumption that the amount of nonrenewable resources is sufficient to complete the project),
– we consider internal resources available for the project and external resources whose usage leads to penalty.

The following notation is used:

J – number of all activities of the project ($j = 1,\dots,J$),
T – number of all time units ($t = 1,..,T$),
K – number of renewable resources ($k = 1,\dots,K$),
F^J – project completion time,
LF^J – project completion time defined by the decision maker,
Z^J – penalty for unit of project delay,
cf_j – net cash flows generated by activity j,
α – discount rate,
x_{jt} – decision variable,
d_j – duration of activity j,
F_j – completion time of activity j,
F_i – completion time of predecessor i,
S_j – start time of activity j,
S_i – start time of predecessor i,
r_{jk} – amount of kth resource required by activity j,

R_{kt} – amount of kth resource available at time unit t (both internal and external),
R_{kt}^{w} – amount of kth internal resource available at time unit t,
V_k – penalty for using external renewable resources,
A^{I}_{j}, A^{II}_{j}, A^{III}_{j}, A^{IV}_{j} – predecessors of activity j (precedence relationships are as follows: finish to start, start to start, start to finish, finish to finish).

A multiple objective model for the project scheduling problem can be formulated as follows:

Objectives

$$\max\{F^{J} - LF^{J}, 0\} \cdot Z^{J} \to \min \tag{2.1}$$

$$\sum_{t=1}^{T}[\sum_{k=1}^{K}[\max\{\sum_{j=1}^{J}(r_{jk} \cdot x_{jt}) - R_{kt}^{w}, 0\} \cdot V_k]] \to \min \tag{2.2}$$

$$\sum_{j=1}^{J} cf_j \cdot e^{-\alpha F_j} \to \max \tag{2.3}$$

Constraints

$$x_{jt} = \{0,1\} \tag{2.4}$$

$$\bigwedge_{j=1,\ldots,J} \sum_{t=1}^{T} x_{jt} = d_j \tag{2.5}$$

$$F_j = \max_{t=1,\ldots,T}(t \cdot x_{jt}) \tag{2.6}$$

$$\bigwedge_{x_{jt} \neq 0} S_j = \min_{t=1,\ldots,T}(t \cdot x_{jt}) - 1 \tag{2.7}$$

$$F_j = S_j + d_j \tag{2.8}$$

$$S_j \geq F_i \qquad (i \in A^{I}_{j}) \tag{2.9}$$

$$S_j \geq S_i \qquad (i \in A^{II}_{j}) \tag{2.10}$$

$$F_j \geq S_i \qquad (i \in A^{III}_{j}) \tag{2.11}$$

$$F_j \geq F_i \qquad (i \in A^{IV}_{j}) \tag{2.12}$$

$$\bigwedge_{k=1,\ldots,K} \bigwedge_{t=1,\ldots,T} \sum_{j=1}^{J}(r_{jk} \cdot x_{jt}) \leq R_{kt} \tag{2.13}$$

The purpose of the criterion function (2.1) is to minimize the penalty for project delay. Delay is defined as a situation in which the project finishes later than it was assumed by the decision maker. If the decision maker did not provide a due date for project completion, then delay is calculated with respect to the latest finish time using the critical path method (CPM). The purpose of criterion (2.2) is to leverage resource usage. The project has its own resources available, but if needed, it can use other resources in the company; using those resources, however, leads to penalty. The criterion function (2.3) describes NPV maximization.

Constraint (2.4) defines a binary decision variable. This decision variable is equal to 1 when activity j lasts in time t, otherwise it is equal to 0. Each activity can be performed only once, and its duration is defined by (2.5). Equations (2.6) and (2.7) are used to calculate the activity completion and start time, respectively. An activity which has started cannot be stopped until it is completed (2.8). The lines (2.9)-(2.12) define precedence relationships of various types and (2.13) is the resource availability constraint.

3 Fuzzy Pareto dominance and fuzzy ranking

The subset of all vectors of a set *A* which are not dominated by any other vector of *A* is the Pareto set. The Pareto set for univariate data (single objective) contains solely the maximum of the data (Köppen et al., 2005).

Given two vectors *a* and *b* we say that *a* (Pareto-) dominates *b* when each component of *a* is less than or equal to the corresponding component of *b*, and at least one component is smaller:

$$a >_D b \leftrightarrow \forall i \ (a_i \leq b_i) \ \wedge \ \exists k (a_k < b_k) \,. \tag{3.1}$$

The fuzzification of the Pareto dominance relation is given by the following definition:

We say that vector *a* dominates vector *b* with degree μ_a given by the formula:

$$\mu_a(a,b) = \frac{\Pi_i \min(a_i, b_i)}{\Pi_i a_i} \tag{3.2}$$

and that vector *a* is dominated by vector *b* with degree μ_p given by the formula:

$$\mu_p(a,b) = \frac{\Pi_i \min(a_i, b_i)}{\Pi_i b_i} \,. \tag{3.3}$$

The definition of Fuzzy Pareto Dominance is illustrated in Figure 1.

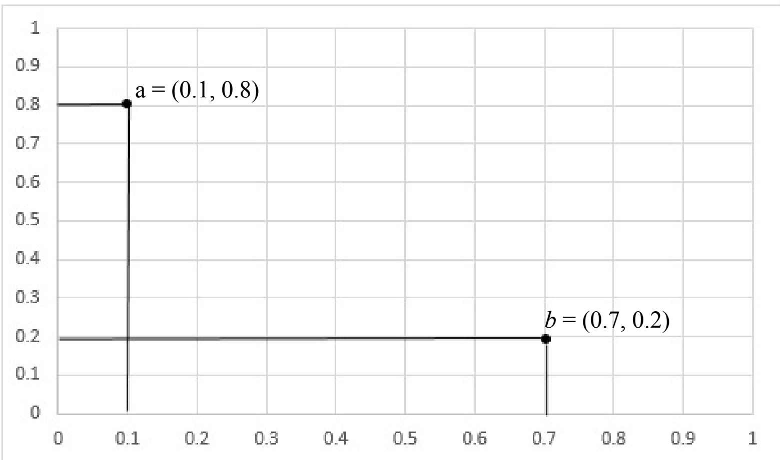

Figure 1. Definition of Fuzzy Pareto Dominance

Source: Based on: Köppen et al. (2005).

Of the two vectors $a = (0.1, 0.8)$ and $b = (0.7, 0.2)$, vector a dominates vector b with degree:

$$\mu_a(a,b) = \frac{0.1 \cdot 0.2}{0.1 \cdot 0.8} = 0.25,$$

and vector a is dominated by vector b with degree:

$$\mu_p(a,b) = \frac{0.1 \cdot 0.2}{0.7 \cdot 0.2} \approx 0.143.$$

We may use these dominance degrees to rank the elements of a set A of multivariate data (vectors) such as the fitness values of a multiple objective optimization problem.

Each element of A is assigned the maximum degree of being dominated by any other element of A:

$$r_A(a) = \max_{b \in A \setminus \{a\}} \mu_p(a,b). \tag{3.4}$$

Next, the elements of A are sorted in increasing order according to the ranking values.

4 Comparison of the Fuzzy Pareto Dominance-Driven Evolutionary Algorithm with the SPEA2 algorithm

The fuzzy Pareto dominance-driven algorithm has been developed on the basis of the SPEA2 (*Strength Pareto Evolutionary Approach 2*) algorithm, which is an elitist algorithm. As research shows (Zitzler, 1999), elitism in evolutionary algorithms can improve the results obtained.

4.1 Evolutionary algorithm scheme

The SPEA2 algorithm consists of the following steps (Zitzler et al., 2001):

Input:
N – population size,
$\overline{N}$ – size of external set,
G − maximum number of generations.

Output:
A − set of non-dominated solutions.

Step 1: Initialization

The initial population P_0 is generated and an empty external set $\overline{P_0}$ is created.

Step 2: Performance

Fitness assignment is performed for individuals from the sets P_0 and $\overline{P_0}$.

Step 3: Selection and external set updating

All non-dominated solutions are copied from the sets $\overline{P_g}$ and P_g to the set $\overline{P_{g+1}}$.

Step 4: Termination

If the stopping criterion is satisfied then set A is a set of decision vectors represented by the non-dominated individuals in $\overline{P_{g+1}}$.

Step 5: Mating selection

A tournament selection with replacement on $\overline{P_{g+1}}$ to fill the mating pool is conducted.

Step 6: Variation

Genetic operators are applied to individuals from the mating pool. The population P_{g+1} is the result of the variation.

4.2 Characteristics of the algorithm

The fuzzy Pareto dominance-driven evolutionary algorithm differs from the SPEA2 algorithm in two respects: performance and environmental selection.

In the SPEA2 algorithm the performance *F(i)* is calculated using the following equation:

$$F(i) = R(i) + D(i)\,. \tag{4.1}$$

At first a strength value *S(i)* is assigned to each individual. It represents the number of individuals that the individual *i* dominates:

$$S(i) = |\{j \mid j \in P_g + \overline{P_g} \wedge i \succ j\}|\,. \tag{4.2}$$

Then a raw fitness of individual *i* is calculated:

$$R(i) = \sum_{j \in P_g + \overline{P_g},\, j \succ i} S(j)\,. \tag{4.3}$$

Individuals are discriminated from each other using density information. The density estimation technique is an adaptation of the *k-th* nearest neighbor method (Silverman,1986), where the density at any point is a (decreasing) function of the distance to the *k-th* nearest data point. For each individual the distances (in objective space) to all individuals in archive and population are calculated and stored in a list. Once the list is sorted in increasing order, the *k-th* element gives the distance sought, denoted by σ_i^k.

The density is defined by:

$$D(i) = \frac{1}{\sigma_i^k + 2}\,. \tag{4.4}$$

Individuals with the fitness value *F(i)* lower than 1 are non-dominated.

In the fuzzy Pareto dominance-driven evolutionary algorithm a ranking of all individuals is calculated (according to the scheme described in section 3). After assigning to each element of A the maximum degree of being dominated by any other elements of A, we sort the individuals in increasing order:

$$r_A(a) = \max_{b \in A \setminus \{a\}} \mu_p(a,b) . \tag{4.5}$$

The higher the position in the ranking, the better the individual performance.

The next aspect in which the SPEA2 algorithm and the fuzzy Pareto dominance-driven evolutionary algorithm differ is environmental selection.

In the SPEA2 algorithm the individuals are selected to the external set according to the following rule:

$$\overline{P}_{g+1} = \{i \mid i \in P_g + \overline{P_g} \wedge F(i) < 1\} . \tag{4.6}$$

If $\overline{P_{g+1}}$ is larger than the external set, it is reduced; if it is smaller, it is filled with dominated individuals from $\overline{P_g}$ and P_g.

In the fuzzy Pareto dominance-driven evolutionary algorithm the first N individuals from the ranking are copied to the external set. No additional set reduction or selection of individuals to the external set is required.

4.3 Other elements of algorithm

Individual

Binary variables are used in the scheduling problem described in section 2. In this paper an individual is a binary matrix with J rows and T columns. Activities are presented in rows and time units are presented in columns. The individual i can be presented as follows:

$$Ch_i = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,T} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,T} \\ \vdots & \vdots & \vdots & \vdots \\ x_{J,1} & x_{J,2} & \dots & x_{J,T} \end{bmatrix} .$$

For the initial population only feasible solutions are generated.

Crossover

A crossover process proposed in this paper is conducted in two phases. In the first phase the individuals for which crossover will be performed are randomly chosen from the population and the crossover point is chosen, also randomly. The crossover point is the row number. In the second stage the chosen row is exchanged between the two individuals.

Mutation

In the proposed solution, mutation is a process of delaying a randomly chosen activity. The activity is delayed by one time unit.

Constraints considering

The mathematical model presented in section 2 contains constraints which should be taken into account in the algorithm. In the approach proposed in this paper a penalty is foreseen for each not feasible solution. The penalty makes the individual performance dramatically worse, to reduce the probability of such individual reproduction.

5 Experimental results

In this section the fuzzy Pareto dominance-driven evolutionary algorithm will be used to solve an example of the multiple objective project scheduling problem.

A project consisting of 13 activities (Figure 2) is to be scheduled.

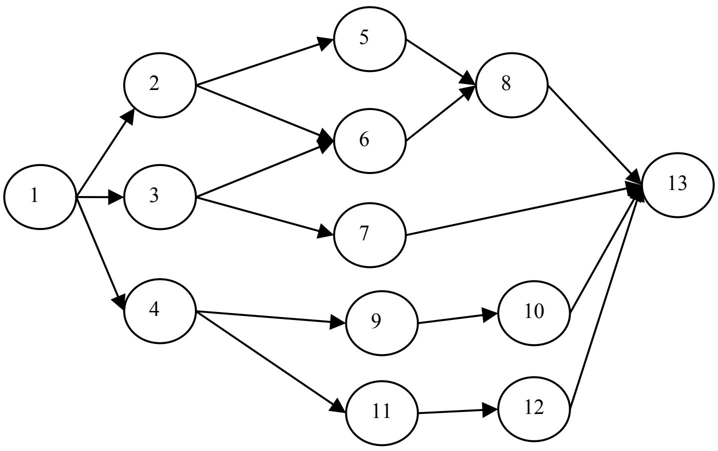

Figure 2. AON network for the example

Source: Prepared by the author.

For each activity a deterministic duration is given (Table 1). For the realization of the project two resource types are required: *k1* and *k2*. The amount of resources required by each activity is given in Table 1. The availability of resource *k1* is restricted and equal to 1 units in the project and to 2 in the company (in each time unit). The availability of resource *k2* is restricted and equal to 3 units in the project (in each time unit) and to 5 in the company. For each activity the cash flows generated by it are determined.

Table 1: Data for the example

Activity	Duration	Resource *k1* requirement	Resource *k2* requirement	Net cash flows
1	4	1	3	-2000
2	9	1	2	-1000
3	10	0	3	-2000
4	8	0	3	-1000
5	13	0	2	-2000
6	8	1	1	-2000
7	4	0	2	2000
8	5	0	3	4000
9	13	1	2	6000
10	12	1	2	8000
11	10	0	2	10000
12	12	1	0	12000
13	10	1	5	15000

The following parameters have been set for computations:

- Population size: 10 individuals,
- Crossover probability: 90%,
- Mutation probability: 10%,
- Number of generations: 100,
- Size of external set: 5.

After 100 generations the following set has been obtained (Table 2):

Table 2: Set of solutions after 100 iterations of the fuzzy Pareto dominance-driven evolutionary algorithm

Solution	Objective 1 (min)	Objective 2 (min)	Objective 3 (max)
1A	1 500	144	7 537
2A	1 300	152	19 891
3A	1 100	150	8 334
4A	1 100	150	8 633
5A	1 100	141	8 846

The solutions are ordered according to their ranking, so we can assume that the first solution is the best one. Its maximum value with which this solution is dominated by the other solutions in this set is the smallest. During the analysis of these solutions, we can conclude that:

- solution 1A is dominated by solution 5A,
- solution 3A is dominated by solutions 4A and 5A,
- solution 4A is dominated by solution 5A.

Solutions 2A and 5A are non-dominated.

In the next step we have performed 100 iterations of the SPEA2 algorithm and the following solutions have been obtained (Table 3).

Table 3: Set of solutions after 100 iterations of SPEA2

Solution	Objective 1 (min)	Objective 2 (min)	Objective 3 (max)
1B	1 300	152	19 891
2B	1 000	151	12 256
3B	700	169	14 955
4B	1 100	156	13 702
5B	1 100	141	8 846

Comparing Tables 2 and 3 we can see that solution 1B is identical with 2A and solution 5B, with 5A. Solutions 1A, 3A and 4A are dominated by 2A and 5A, but not by 2B, 3B and 4B.

Now we will mutually compare all solutions using fuzzy Pareto dominance (Table 4).

Table 4: Comparison of all solutions

Solution	max
1A	0.0017629
2A, 1B	0.0015279
3A	0.0012928
4A	0.0012928
5A, 5B	0.0012928
4B	0.0012928
2B	0.0011753
3B	0.0008227

From Table 4 we can see that solutions 2B, 3B and 4B identified by the SPEA2 algorithm but not by the fuzzy Pareto dominance-driven algorithm are on the last 3 positions in the fuzzy ranking. What is interesting, solutions 1A, 3A and 4A are dominated by solutions 2A (1B) and 5A (5B), but solutions 2B, 3B and 4B are not dominated by solutions 2A (1B) and 5A (5B). Comparing solutions 1A, 3A and 4A with solutions 2B, 3B and 4B we are unable to find any dominance relationship between them.

6 Summary

In this paper a project scheduling problem has been described as a multiple objective decision making problem. It has been solved using the fuzzy Pareto dominance-driven evolutionary algorithm.

Applying fuzzy Pareto dominance in an evolutionary algorithm seems to make the performance of the individuals and environment selection (also selection to the external set) better. Additionally, thanks to the fuzzy ranking scheme it is clear which solution should be chosen as the final one – we should always choose the highest-ranking solution. In other evolutionary algorithms for multiple objective problems we obtain a set of solutions, and then we should choose one of them. In the case of the SPEA2 algorithm we can choose any solution from the set of solutions with the objective function $F(i)$ lower than 1, as those are non-dominated solutions.

Both algorithms ended with similar solutions, and even though some papers report that the evolutionary algorithm using fuzzy Pareto dominance is more effective (Köppen et al., 2005), it is difficult to conclude the same from the present paper. This may be caused by the small size of the example presented in this paper and that is why additional experiments should be conducted. Therefore, in a future study a larger example will be considered.

Acknowledgments

This research was supported by National Science Centre on basis of the decision number DEC-2012/07/N/HS4/02736.

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MULTIPLE CRITERIA DECISION MAKING
Vol. 10 2015

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STRUCTURAL ANALYSIS OF PROBLEMS IN PUBLIC RELATIONS

Abstract

The literature on the activities of public relations (PR) is getting richer. Also, numerous empirical studies on the PR process, methods and techniques are conducted, as well as analyses on the effectiveness of PR and ethics in this field. There is a relatively small number of studies that examine decision-making processes by PR practitioners. Despite numerous discussions on the issue of decision-making, methods of decision making in public relations are not a subject of research and debate. Most decisions in this area are probably made unsystematically and in a very individual way. However, the introduction of effective methods, proven in other areas, which support decision making practice related to communication processes, can help to improve efficiency and effectiveness of the organization in the field of building relationships with the stakeholders. The authors show how the use of cognitive maps and the WINGS method can help PR consultants to choose a PR strategy in situations which can seriously jeopardize the organization's reputation.

Keywords: cognitive maps, communication models, multiple criteria decision aiding, public relations, structural methods, WINGS.

1 Introduction

This paper is an attempt to identify opportunities for using cognitive maps for making decisions in public relations (PR) activities as a method which supports

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decisions of practitioners (communication managers). The decision-making process is understood here as a situation in which a decision maker as an independent individual wants and has the authority to decide and solve a given problem (Michnik, 2013b, p. 15).

The literature on public relations definitions is very wide. Rühl highlights three perspectives of PR and three types of definitions. They are: lay (non-expert PR), professional and scholarly perspectives (Rühl, 2008, p. 22-25). One of the theories, which tries to define PR, involves the concept of a system (Pieczka, 2006, p. 333; Greenwood, 2010, p. 459). Another important approach tries to explain PR in terms of rhetoric and persuasion theory (Heath, 2000, p. 31; L'Etang, 2006, p. 359). Wojcik classifies definitions taking into account language and their cultural origin (Wojcik, 2015, p. 21-29). There is a wide range of PR definitions. The one that comes from James Grunig is the most frequently used. It is simple and clearly explains the core of PR activities: "public relations is the management of communication between an organization and its publics" (Grunig, 1984, p. 6).

In the analysis the authors refer to the definition of PR introduced by Krystyna Wojcik, which puts strong emphasis on the decision making process[1]: "Public relations are systematic and procedural activities – a system of actions in the field of social communication, a social process of a constructive dialogue, oriented towards a consensus" (Wojcik, 2013, p. 26).

The systematic and enumerating definition of PR quoted above, is very strongly rooted in management sciences and specifically underlines the importance of the decision-making process in PR, pointing at some of its essential features such as being methodical, planned, regular. It also refers to all disciplines "that create opportunities for effectiveness". The definition quoted indirectly indicates the need to formalize the decision making process in public relations, so as to achieve better results (greater effectiveness) of the selected action.

Current changes in communication technology as well as the increasing role of communication in society challenge organizational decision-making. What is more, decisions need communication for better understanding among organization's publics (Mykkanen, Tampere, 2014, p. 132) and communication needs decisions to be made. Organizations define how much communication is required for every decision and they state how a particular decision should be communicated, but at the same time they do not outline how decisions about means of communication should be made. Luhmann points out that a decision is a specific form of communication: decisions are not first made and then communicated, but decisions are decision communication (Luhmann, 2005). Every single deci-

[1] The original definition by Wojcik is much longer. She stressed that PR activity should be conscious, methodical, planned, systematic and permanent (Wojcik, 2013, p. 26).

sion serves as a decision premise for later decisions (Seidl, Becker, 2006, p. 27). Decision is a medium and a form of communication (Mykkanen, Tampere, 2014, p. 135).

Although decision problems that appear in PR are complicated and connected with the firm's strategy, there are no formal methods in this field. In this paper structural approaches based on cognitive maps and on the WINGS method have been proposed to aid in PR decision-making. A real-life practical problem of organizing a PR campaign when the firm's reputation is in jeopardy serves as an illustrative example.

The authors propose to begin with structuring the problem using a generic cognitive map that represents the qualitative approach. This map models the problem as a system of concepts linked by causal relations. During the construction of the map the decision maker gains a deeper understanding of the nature of the problem. The conflicting objectives and potential options of solving the problem are recognized. Drawing the cognitive map helps to find the important relations along paths linking the options with the objectives (Michnik, 2014).

In situations when the cognitive map does not provide convincing arguments for making a decision, an extended approach is proposed. A model that is capable of making more informed decisions is introduced. It is based on quantitative assessments and can better differentiate among the potential options of action at the cost of greater effort to provide quantitative data about causal relations between elements. This model is grounded in the WINGS method which provides greater flexibility in a decision process. WINGS includes, in a natural way, the strength (importance) of system elements so it can better represent the decision maker's preferences.

To the best knowledge of the authors, the solution presented in this paper is the first attempt to apply a structural approach to assist in solving a PR problem.

The remainder of the paper is organized as follows. The next section presents general models of PR. In Section 3 key decisions in public relations are characterized. Section 4 describes a decision model based on a cognitive map. It is followed by a discussion of a cognitive map with quantitative assessments (Section 5). The application of WINGS is presented in Section 6. Summary (Section 7) and conclusions (Section 8) complete the paper.

2 Models of public relations practices

In their classic publication *Managing Public Relations*, James E. Grunig and Todd T. Hunt proposed four models of PR (Grunig, Hunt, 1984, p. 21ff.): press agentry and public information, which are based on one-way transmission from the sender to the recipient, and two-way communication models: asymmetrical and symmetrical. These four models result from the analysis of the practitioners' experience, but they are also useful tools for the practice of PR, or directly for practical use when selecting a strategy (Grunig, 2001, p. 11ff.). These models of communication in public relations can be characterized as follows: (1) press agentry model, in which communication is used to disseminate information in order to convince public opinion; the purpose of this model is propaganda, persuasion, and communication as a one-way flow of information from the sender to the recipient; (2) public information model which, like the previous model, is a one-way communication technique but insists on truth, precision and clarity; (3) two-way asymmetrical model which assumes the use of persuasion (what is called by the authors "scientific persuasion") and of psychographic and demographic information in the practice of communication; in this model important values, attitudes and opinions are studied before a specific message is prepared. In other words, the model focuses on the use of persuasion through the understanding of stakeholders with whom the organization is planning to build relationships to create the most convincing message; (4) two-way symmetrical model which uses interactive communication by seeking ways to adapt a message to both the organization and its stakeholders; interactions rely on an honest exchange of information and efforts toward a better understanding of the various stakeholders of the organization. The purpose of this model is to use research to pursue a dialogue that is mutually beneficial for the organization and its environment, and that might change ideas, attitudes or behavior (Grunig, 2001).

These four models of PR are used in practice, though quite often without conscious reflection on their pros and cons. PR consultants use certain principles of communication intuitively; they are guided by well-known and publicized cases, rather than by reliable academic research. The use of certain models requires a prior analysis of the specific situation, problem, the current image of the organization, specific audience (stakeholders), as well as financial and organizational capabilities. Each model may find its practical application depending on the results of this analysis. As James Grunig stresses, the quality of relationships between an organization and its publics depends on the model of public relations used (Grunig, 1993).

3 Key decisions in public relations

Some decisions related to PR are strategic and require a large amount of information to identify and evaluate potential options for decision making, in the context of the desired goals. Because a PR consultant deals with multiple (at least two), usually conflicting objectives, the selection of the preferred option is not obvious. Usually it is also the case that these options are not mutually exclusive. There are situations when it is possible to implement mixed options. They occur when an organization can distribute its available resources in specific proportions for different variants. The problem considered in this paper is exactly such a situation.

One of the situations requiring a strategic decision is a crisis, when communicating dramatic events to stakeholders can cause panic, but lack of such information will be regarded as deceitful and unethical. That is why it is so important to perform a systemic analysis of the given situation, and in particular to determine the desired aims and their mutual relations. It is also important to identify possible options for the implementation of the action. Both the amount of the data involved, and sometimes its inaccessibility, raise doubts that can be an obstacle to make an appropriate and efficient decision in a critical situation. It will be much harder to deal with a high degree of uncertainty, which often happens in PR work. As it is stated in the literature and PR practice, crisis communication is perceived as a part of the public relations field (Fitzpatrick, 1995). Furthermore, it seems that the top management is influenced much more by their PR officers than by their legal counselors (Lee, Jares, Heath, 1999, p. 266). That is why it is so crucial in the process of crisis management to make excellent decisions which do not raise doubts. In the remainder of the paper we present three formal models that can serve as useful tools for supporting PR decisions in a reputation crisis.

4 Cognitive map of a strategic problem in PR

A cognitive map is a useful tool that can facilitate analysis and solution of a complicated problem (Eden, 2004). It is constructed by an individual or a group to better understand the nature of the problem and potential ways of solving it. As such, a cognitive map is a subjective picture of an actual problem, as seen by subjects involved in its solution. In spite of being a simplified model of an actual situation, a cognitive map helps its users to better understand the problem, to structure it and to find the best possible (or at least satisfactory) solution.

Formally, a cognitive map is a digraph in which nodes represent concepts pertaining to a problem and arrows represent causal relations between concepts (because of this feature some researchers prefer to call such a map a ‘causal map’). Arrows are labeled with plus or minus signs showing the character of relations. A plus sign means that when the source concept increases (becomes stronger), the result concept increases (becomes stronger), too. A minus sign has the opposite meaning: when the source concept increases (becomes stronger), the result concept decreases (becomes weaker) (cf. Montibeller and Belton, 2006).

Typically, a concept without any outgoing arrow is called ‘head’ and represents an objective, while a concept with no incoming arrow is called ‘tail’ and denotes an option (a decision alternative) (Eden, 2004). Usually, heads are located at the top of a map, and tails at its bottom. Between tails and heads there are a number of intermediate concepts that provide causal paths linking options with objectives.

In the case of a serious reputation crisis the main objective of the PR campaign is to re-build and strengthen the firm's reputation. A substantial cost will be another – non-desired – effect of PR activities. It is represented by the second top node on the map and can be regarded as a negative objective. The map developed for this case may look as the one shown in Figure 1.

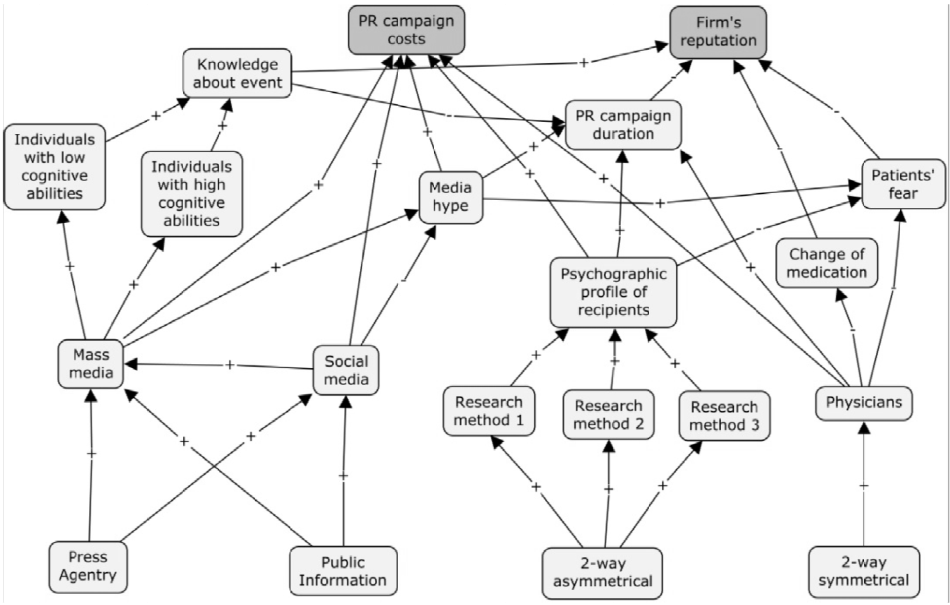

Figure 1. Cognitive map of a PR campaign

At the bottom of the map there are four options – four Grunig's PR models. In this case they can be characterized as follows:

Press Agentry. This option consists in the maximal use of mass media in order to inform the highest possible number of people, regardless of their knowledge and cognitive ability. Using mass media as a communication channel generates high cost because of traditional advertising techniques required.

Public Information. In the case of saving the firm's reputation this model applies persuasion techniques that can be used through social media. This option can lower the cost in comparison with mass media.

Two-way Asymmetrical. The main activity is a study of audiences in order to adapt communication to their profiles. This model is time-consuming and costly.

Two-way Symmetrical. This model is based on a dialog with the public. In our case the main tool is the dialog with physicians to convince them about the credibility of the firm.

A cognitive map is helpful not only in better understanding and structuring of a problem, but it can also be used to perform some qualitative analysis. The most often analyzed feature is the topological characteristics of the map. As we are interested in an evaluation of the options, we would like to determine the causal effect of each tail on each head. Two indices are used for this purpose. The first one is called partial *effect* and is the product of all signs along the path from tail to head. It is positive if the number of minus signs along the path is even, otherwise it is negative. The second is *total effect* of a tail on a head. It is positive if all partial effects of a tail on a head are positive, negative if all partial effects are negative, otherwise it is undetermined.

A map with a small number of nodes can be analyzed manually. For a larger map this may be difficult, so it is better to use a correspondence between digraphs and square matrices (Kaveh, 2013). The *adjacency matrix* for a digraph with *n* nodes is defined as an $n \times n$ square matrix $\mathbf{E} = [e_{ij}]$, where:

$$e_{ij} = \begin{cases} 0, \text{ if there is no arrow from } i \text{ to } j, \\ 1, \text{ if the arrow from } i \text{ to } j \text{ has + sign}, \\ -1, \text{ if the arrow from } i \text{ to } j \text{ has } - \text{ sign}. \end{cases}$$

With the adjacency matrix, the partial effect of node i on node j can be defined as the product of the elements of the adjacency matrix along the path from node i to node j. A path that consists of k arrows has length k. The element of the k-th power of matrix **E**, $[\mathbf{E}^k]_{ij}$ is equal to the algebraic sum of partial effects calculated along all paths of length k from node i to node j. Additionally, we can use the matrix of absolute values $|e_{ij}|$ to calculate the number of different paths of any length going from i to j.

For the map presented in Figure 1, the partial and total effects of the four options on the two objectives are shown in Table 1.

Table 1: Partial and total effects of options on objectives for the cognitive map of a PR campaign

Option	Firm's reputation			PR campaign's costs		
	No of (+) paths	No of (−) paths	Total effect	No of (+) paths	No of (−) paths	Total effect
Press Agentry	10	2	Undefined	3	1	Undefined
Public Info.	10	2	Undefined	3	1	Undefined
2-way asym.	3	0	Positive	3	0	Positive
2-way symm.	2	1	Undefined	1	0	Positive

Three of the options have undefined total effect on the firm's reputation and only one – the 2-way asymmetrical – a positive effect. But this option has also a positive total effect on the PR campaign's costs. Both Press Agentry and Public Information have also undefined total effect on the PR campaign's costs. Thus, the comparison of total effects does not make clear the differences between the options.

The other topological characteristics such as *potency* or *shortest path* do not help much in our case, either. The *potency* of an option is defined as the number of objectives it influences (Eden, 2004). In our case all options influence both the positive objective (reputation) and the negative one (costs). The option with the shortest path to the objectives can be considered as the most influential (Hall, 2002). In our case the two-way symmetrical model has the shortest path to the firm's reputation (three paths of length 3), but it also has the shortest path (of length 2) to costs (Press Agentry and Public Information have paths of the same length to costs).

Since the qualitative assessment does not give enough information to differentiate among the options[2], we can try to extend our analysis by incorporating some quantitative characteristics into our model. The use of quantitative assessment of causal influences is described in the next section.

5 Aiding PR Decisions with Quantitative Cognitive Map

In the previous section we discussed the use of a cognitive map for deeper understanding and structuring of the problem of a PR campaign. We also analyzed some additional characteristics of the options developed from the topological structure of the cognitive map. However, it turned out that a cognitive map in its original form does not provide enough information to make a well-founded decision. This is not unusual, and researchers tried to develop more extended models to evaluate decision options (Roberts, 1976; Kosko, 1986; Montibeller et al., 2008).

We propose to introduce to the map developed in the previous section, a quantitative assessment of the influence of a source concept on a result concept. For this purpose we use a numerical 9-point scale in which 1 means the weakest influence, and 9, the strongest, with the appropriate sign. Figure 2 presents the cognitive map with the numerical assessments based on the experience of one of the authors (AAM).

[2] In the paper Montibeller and Belton (2006) the authors call this effect 'indistinction'.

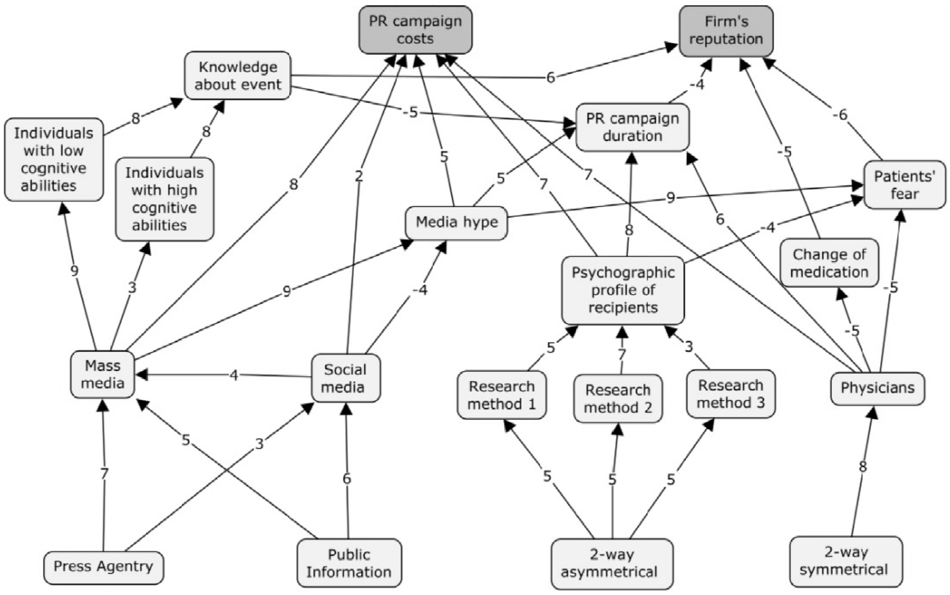

Figure 2. Quantitative Cognitive Map of a PR campaign

We introduce a matrix similar to the adjacency matrix used in Section 4. This matrix differs from the adjacency matrix in that it has numbers from 1 to 9 (with a sign) instead of +1 and −1 only (Roberts, 1976). In this case partial effects will change. In the adjacency matrix they can have three values only: 0, −1, +1, while now they can have many different values, being products of numbers from 1 to 9 (with signs). Consequently, the element (i, j) of the k-th power of this matrix is an algebraic sum of partial effects along all paths of length k. Now we can sum the partial effects along all paths of different lengths to evaluate the influence of each option on each objective. This model has one important disadvantage. In the type of problems considered here, one can expect that the influence along a longer path will be weaker than along a shorter one. But with numbers larger than 1 the effect is opposite[3]. This is why we propose to normalize the evaluations by dividing each of them by 10. After this transformation, all elements of the matrix are lower than 1 and we achieve the desired effect: the influence along a longer path is weaker.

The values of total effects calculated using the normalized matrix are presented in Table 2. They are re-normalized so that the sum of evaluations for each objective is equal to 1. The option ‘Public Information’ received the highest ef-

[3] If a map contained loops (cycles) the partial effect could be even infinite. However, as it is advised to avoid loops in cognitive maps, this effect does not occur. In our case there are no loops in the map and the longest paths contain five segments.

fect on the firm's reputation. The order of the remaining options is: 2) 2-way symmetrical, 3) Press Agentry, 4) 2-way asymmetrical. The ranking changes when we take into account costs (a negative objective): 1) 2-way symmetrical, 2) 2-way asymmetrical, 3) Public Information, 4) Press Agentry.

Table 2: Total effects of the options on the objectives

Option	Firm's reputation	PR campaign's costs
Press Agentry	0,209	0,361
Public Info.	0,352	0,258
2-way asym.	0,127	0,216
2-way symm.	0,312	0,165

To better see the relationships between the options we can draw a 2--dimensional graph with Costs on the horizontal axis and Reputation on the vertical one (see Figure 3). Now it is clearly visible that the 2-way symmetrical option dominates both the 2-way asymmetrical and Public Agentry ones. Also, Public Information dominates Public Agentry.

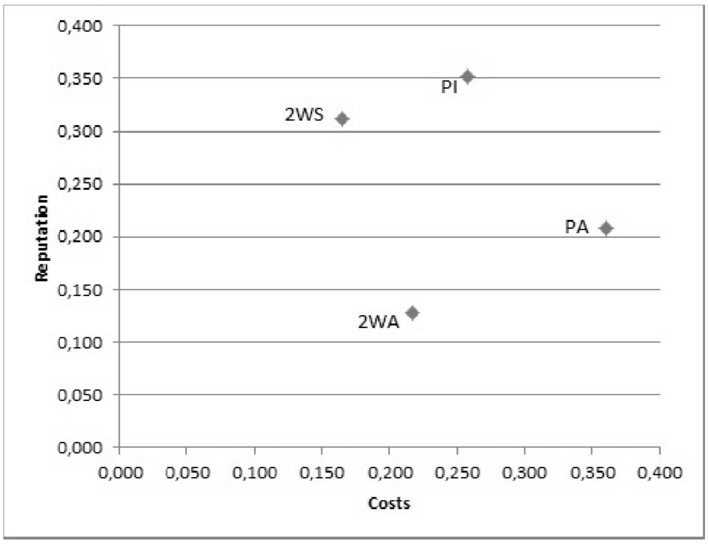

Figure 3. The graph of the options: Reputation vs. Costs

With this model the decision maker can make a more informed decision. For instance, she/he can decide to use the most resources for more effective options: 2-way symmetrical and Public Information, and only a very small part for the other two (in a PR campaign it is practically impossible to completely neglect any of the options).

6 Aiding the PR decision with WINGS

6.1 The WINGS procedure

Here we present the WINGS procedure which is based on the original paper (Michnik, 2013a).

Stage 1. Construction of the model of a problem

At the beginning the user selects n components that constitute the system and analyzes the important interdependencies among them. The result of this step is presented as a digraph in which nodes represent components and arrows represent their mutual influences. The WINGS digraph is a network similar to a cognitive map with quantitative evaluations (see Section 5), but different in two important features: 1) loops (cycles) are allowed; 2) there are only positive influences in the network[4].

Stage 2. Input of data (feeding the model with data)

In the initial phase, the user chooses also verbal scales for both strength of components and their influences. The number of points on the scale depends on the user's intuition. The minimal number suggested is three or four, e.g., low, medium, high, very high (importance/strength or influence). The scale can be expanded by adding, e.g., "very low" and/or other verbal descriptions, depending on the user's needs. Since the scale represents subjective assessments of the user it is not recommended to use a scale with too many points.

Next, the user assigns numerical values to verbal evaluations. This assignment depends on the user's assessment, but for simplicity and to preserve a balance between strength and influence, it is best to use integer values and the same mapping for both measures. The lowest non-zero point on the verbal scale is mapped to 1, which is a natural unit. Since we apply a ratio scale here, the higher points are mapped to the ratios of the corresponding numerical values to the first-level (unit) value. The mapping can be linear or non-linear, depending on the user's evaluation of the relations between concepts in the system.

Stage 3. Calculations

All numbers assigned are inserted into the direct strength-influence matrix $\mathbf{D} = [d_{ij}]$, $i, j = 1, \ldots, n$.

– Strengths of components constitute the main diagonal: d_{ii} = strength of component i.
– Influences are the remaining elements: for $i \neq j$, d_{ij} = influence of component i on component j; $i, j = 1, \ldots, n$.

[4] WINGS shares this feature with other similar methods, such as DEMATEL and ANP.

1. Matrix **D** is scaled according to the following formula:
$$\mathbf{S} = \frac{1}{s}\mathbf{D}, \tag{1}$$
where **S** is the *scaled strength-influence matrix* and the scaling factor is the sum of all elements of matrix **D**:
$$s = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}. \tag{2}$$
2. The *total strength-influence matrix* **T** is calculated from the following formula:
$$\mathbf{T} = \mathbf{S} + \mathbf{S}^2 + \mathbf{S}^3 + \cdots = \frac{\mathbf{S}}{\mathbf{I} - \mathbf{S}}. \tag{3}$$

Thanks to the scaling defined in Eq. (2) the series in the following formula converges, and thus matrix **T** is well defined (mathematical details can be found in Michnik (2013)).

As already mentioned in Section 4, the correspondence between matrices and digraphs allows an obvious interpretation of the above formulas. The *ij*-th element of $\mathbf{S}^k$ (the *k*-th power of matrix **S)** is the product of influences of component *i* on component *j* taken along the path of length *k* (if there is no such path, that element is equal to zero). Matrix T, as the sum of all powers of matrix S, comprises influences along all paths of any length. An important feature of WINGS is that a non-zero strength of the component also contributes to its total impact. The inclusion of the strength of a component introduces a self-loop into the model. As a result, paths of any length occur in the system and the sum in Eq. (3) contains infinitely many of terms.

Stage 4. Output of the model

Total impact

It represents the influence of component *i* on all other components in the system and is equal to the sum of the elements of matrix **T** from row *i*.
$$I_i = \sum_{i=1}^{n} t_{ij}. \tag{4}$$

Total receptivity

It represents the influence of all other components in the system on component *i* and is equal to the sum of the elements of matrix **T** from column *i*.
$$R_i = \sum_{j=1}^{n} t_{ij}. \tag{5}$$

Total involvement
The sum of all influences exerted on and received by component i, that is, $I_i + R_i$, determines the total involvement of component i in the system.

Role (position) of the component in the system
The difference between all influences exerted on and received by component i indicates its role (position) in the system: if it is positive, component i belongs to the *influencing (cause) group*; if it is negative, component i belongs to the *influenced (result) group*.

The analysis performed with WINGS gives the user synthetic profiles of the system components. They result from a combination of two values assigned to each component: its intrinsic (initial) strength and its influence on other components. The values of total impact, total receptivity, total involvement and role allow ranking of the system components.

6.2 Solving the PR problem with the WINGS procedure

The cognitive map developed in Section 4 is a point of departure for the WINGS model of a PR campaign. Since the problem contains opposite objectives, we separate them into two networks. The first network contains beneficial objectives, in our case: strengthening the firm's reputation. The second network contains detriments, in our case: campaign costs and weaker effects of a lengthy campaign. This procedure has been developed by T. Saaty for applications of his ANP method (Saaty, 2005). Both networks are presented in Figure 4 and Figure 5, respectively.

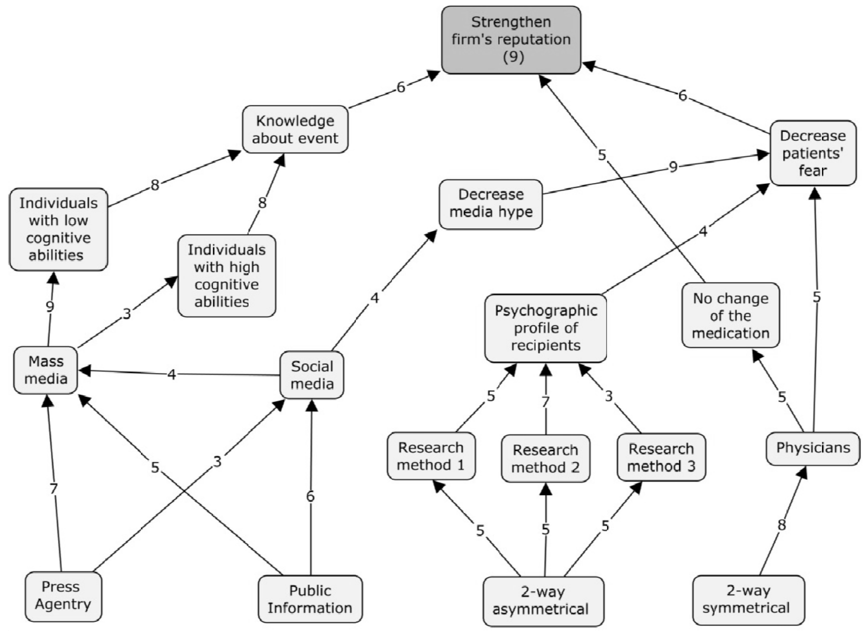

Figure 4. WINGS network of PR campaign – benefits

The interrelations and values are copied from the quantitative cognitive map – this allows, despite obvious differences, to make reliable comparisons between these methods. The difference is in the possibility to include the importance (strength) of some selected concepts. Obviously, this applies to the objectives. 'Strengthen the firm's reputation' obtained the highest value 9 (although it should be noted that this does not change the final result because this is the only objective in the benefits network). The detriments network contains two objectives and here different importance values lead to different results, as they play the role of relative weights. In our case the user assigned the lowest non-zero value to costs (1) and a very high value (8) to the weaker effects of a lengthy campaign. Calculations made according to Stage 3 of the WINGS procedure give the output presented in Table 3.

Table 3: Total Impacts of the options in Benefits and Detriments networks

Option	Benefits	Detriments
Press Agentry	0,231	0,233
Public Info.	0,253	0,252
2-way asym.	0,333	0,329
2-way symm.	0,184	0,185

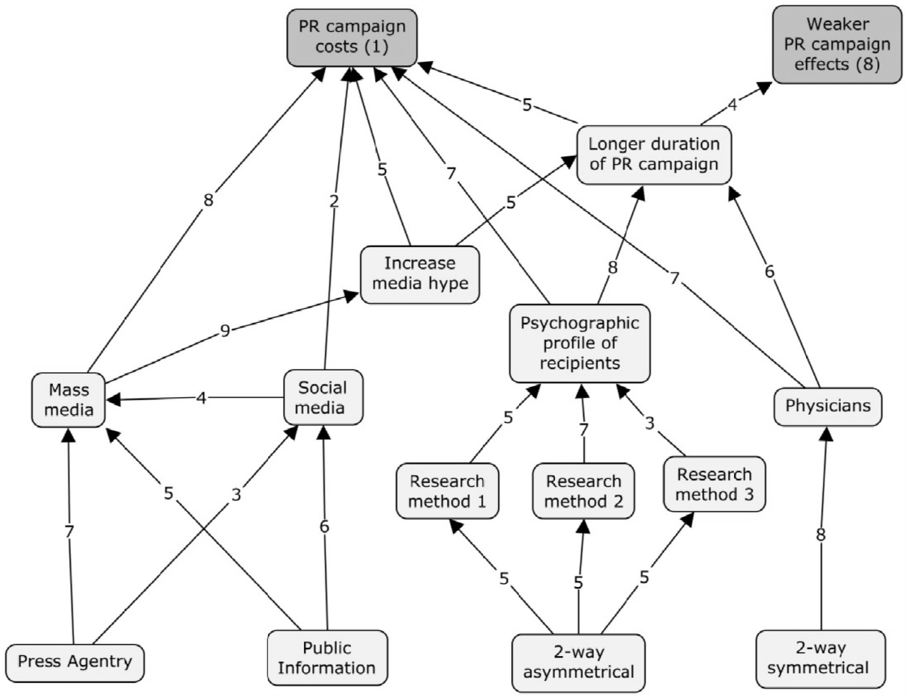

Figure 5. WINGS network of PR campaign – detriments

Similarly as in Section 5, the results can be illustrated with a 2-dimensional graph (Figure 6). A comparison with the quantitative cognitive map reveals the essential difference between the two solutions. First of all, in WINGS, no option dominates the other ones (this may be regarded as a more realistic result). Now the 2-way asymmetrical model is the most effective, but it also has the highest detriment value (in terms of costs and negative effects of the campaign). The 2-way symmetrical model has the smallest value in both dimensions.

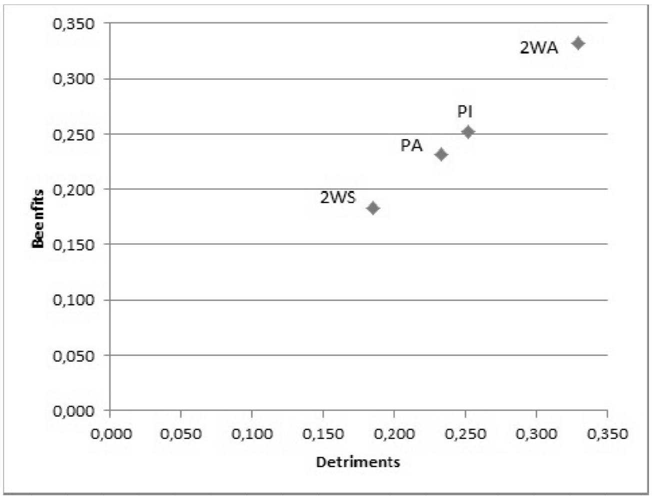

Figure 6. The graph of options: Benefits vs. Detriments

This difference can have several causes, the main one being the method of calculating the final outcome of an option. In the quantitative cognitive map the total effect of an option is the sum of all partial effects on a given objective. In WINGS it is the total impact which comprise the influence on all components of the system. The calculation of a similar measure for the cognitive map is not applicable because the map contains opposite objectives.

With WINGS we are able to aggregate benefits and detriments into a single measure that provides a ranking of options. There are several alternative ways of doing this (Saaty, 2005; Wijnmalen, 2007). In both networks, benefits and detriments, the scales are normalized (the sum of evaluations is equal to 1), so the weights assigned by the user to benefits and detriments directly reflect their relative importance (they also sum up to 1). We propose the following formula:

$$AS(i) = w_b b_i - w_d d_i, \tag{6}$$

where:

$AS(i)$ – aggregated score of option i,

w_b (w_d) – weight of benefits (detriments); $w_b + w_d = 1$,

b_i (d_i) – benefits (detriments) score of option i.

The aggregated scores for the full range of weights are presented in Figure 7. For the decision maker who is focused on benefits, the 2-way asymmetric model ranks first, followed by Public Information. The decision maker who is more sensitive to costs and to the weaker effects of a lengthy campaign will prefer to concentrate on the 2-way symmetrical model (ranking first) and Public Agentry which ranks second.

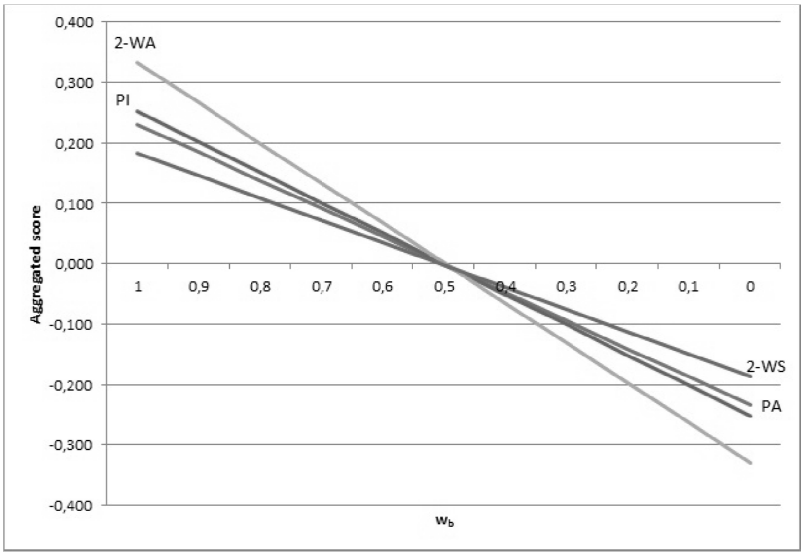

Figure 7. Dependence of aggregated scores of options on the relative weights of benefits and detriments

7 Summary

In this paper we have proposed a structural approach to the decision problem in public relations. This approach draws from formal analysis based on structuring a problem with the aid of causal reasoning and graphical tools. First, the original version of the cognitive map was used to help to determine the important concepts involved in the problem and to analyze the causal relations between them. The cognitive map is a relatively simple tool based on the qualitative assessment of the relations between its components. However, it may be not sufficient to help in decision making because in many cases its results are ambiguous. To obtain a more definite advice a quantitative evaluation may be needed.

An extension of the generic cognitive map with quantitative evaluation is proposed as a more sophisticated approach to obtain evaluations of potential decision options. This is done at the cost of providing more input data – the quantitative assessments of influence. With some technical manipulations (such as normalization), such an approach is possible (the cognitive map has no loops), but even then a ranking of the options is not easy to obtain.

Finally we presented the WINGS method, a general systemic approach that can be applied to solve a variety of complicated problems. Its main distinguishing feature is the ability to evaluate both the strength of the acting factor and the intensity of its influence. When WINGS is used as a tool of multiple criteria decision aiding, the strength (or importance) of the factor plays the role of a criterion weight. WINGS allows the evaluation of alternatives when interrelations between the criteria cannot be neglected. To perform a comprehensive analysis of the PR problem we proposed to use separate networks for benefits and detriments. This approach facilitates the structuring of the problem allowing the user to analyze the positive and negative consequences of the chosen options separately. The outputs of the WINGS network have been aggregated to assign a single score to each option and to rank them.

8 Conclusions

The authors are aware of some simplifications applied in the presented case. However, the aim of this article was to show a practical application of methods for supporting decision-making process in specific PR activities. The chosen example of a reputation crisis is widely known, not only among PR specialists. This case not only shows a method for selecting a communication model appropriate in such a situation, but it also reveals the complexity of the decision-making process, even though it involves one of the most common and best-known processes, which is communication. The task of identifying not only models of communication, but also its means and techniques, is tackled only in a limited way by practitioners and researchers in public relations. Most often it is assumed that the choice depends on the purpose and audience of communication. Proposing the cognitive map as a possible tool is only one example of the use of structural analysis in the practice of PR. Decision making in PR, in times of significant dynamics of the environment and of the development of new communication techniques, becomes increasingly complex and at the same time demands higher responsibility. Therefore, methods and techniques of decision making developed by operational research experts who are supported by information techniques can become increasingly important in the practice of PR.

Acknowledgments

The research of the second author (JM) was supported by National Science Centre, Poland under the decision No. DEC-2013/09/B/HS4/01102.

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Vol. 10 2015

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INCOMPLETE PREFERENCE MATRIX ON ALO-GROUP AND ITS APPLICATION TO RANKING OF ALTERNATIVES

Abstract

Pairwise comparison is a powerful method in multi-criteria optimization. When comparing two elements, the decision maker assigns a value from the given scale which is an Abelian linearly ordered group (Alo-group) of the real line to any pair of alternatives representing an element of the preference matrix (P-matrix). Both non-fuzzy and fuzzy multiplicative and additive preference matrices are generalized. Then we focus on situations where some elements of the P-matrix are missing. We propose a general method for completing fuzzy matrix with missing elements, called the extension of the P-matrix, and investigate some important particular cases of fuzzy preference matrix with missing elements. Eight illustrative numerical examples are included.

Keywords: multi-criteria optimization, pairwise comparison, preference matrix, incomplete matrix, Alo-group.

1 Introduction

In various selection and prioritization processes the decision maker(s) (DM) try to find the best alternative(s) from a finite set of alternatives. DM problems and procedures have been established to combine opinions about alternatives related to different DM criteria. These procedures are often based on pairwise comparisons, in the sense that the processes are linked to some preference values from a given scale of one alternative over another. According to the nature

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of the information expressed by the DM, for every pair of alternatives different representation formats can be used to express preferences, e.g. multiplicative preference relations, Herrera-Viedma et al. (2001), fuzzy preference relations, see Chiclana et al. (2009), Herrera-Viedma et al. (2004), Ma et al. (2008), interval-valued preference relations, Xu (2008), and also linguistic preference relations, see Alonso et al. (2008).

In this paper we consider pairwise comparison matrices over an *Abelian linearly ordered group* (*Alo-group*) and, in this way, we provide a general framework for all the above mentioned cases. By introducing this more general setting, we provide a consistency measure that has a natural meaning: it corresponds to the consistency indices presented in the literature, see e.g. Ramik (2014); it is easy to calculate it in the additive, multiplicative and fuzzy cases. This setting is based on the papers of Cavallo et al. (2009), Cavallo et al. (2012), and Ramik (2014).

Usually, experts are characterized by their personal background and experience of the problem to be solved. Expert opinions may differ substantially: some of them would not be able to efficiently express a preference degree between two or more of the available options. This may be due to an expert's not possessing a precise or sufficient level of knowledge of part of the problem, or because these experts are unable to estimate the degree to which some options are better than others. In these situations an expert will provide an incomplete preference matrix, see Alonso et al. (2008), Kim et al. (1999), Xu (2008).

Usual procedures for DM problems correct this lack of knowledge of a particular expert using the information provided by the other experts together with aggregation procedures, see Saaty (2008). In the literature, see Xu et al. (2008), the problem is solved by using the least deviation method to obtain a priority vector of the corresponding preference relation. In this paper, we put forward a general procedure that attempts to estimate the missing information in any of the above formats of incomplete preference relations. Our proposal is different to the above mentioned procedures in Alonso et al. (2008), Kim et al. (1999), Xu (2008) because the estimation of missing values in an expert incomplete preference matrix is done using only the preference values provided by these particular experts. By doing this, we assume that the reconstruction of the incomplete preference matrix is compatible with the rest of the information provided by the experts.

The paper is organized as follows. Some basic information on Alo-groups is summarized in Section 2. In Section 3, preference matrices with elements from an Alo-grup are investigated, and reciprocity and consistency conditions are defined as well as the inconsistency index of the P-matrix. The priority vector for ranking the alternatives is also defined. In Section 4, a special notation for the matrix with missing elements is introduced and the concept of the extension of a P-matrix with missing elements is defined. This concept is based on a particular representation of the consistent matrix; the missing elements of the extended matrix are calculated by applying the generalized least squares method. In Section 5, two special cases of the P-matrix with missing elements are investigated. Here, for an $n \times n$ P-matrix the expert evaluates only $n-1$ pairs of alternatives. In this section, two numerical examples illustrating the necessary and sufficient conditions for elements to be evaluated in the P-matrix are presented. In Section 6, some concluding considerations and remarks are presented.

2 Abelian linearly ordered groups

In this section we summarize basic information on Abelian linearly ordered groups (Alo-groups). The content of this section is based mainly on Cavallo et al. (2012), and Bourbaki (1990).

An *Abelian group* is a set $\mathbf{G}$, together with an operation $\odot$ (read: operation *odot*) that combines any two elements $a, b \in \mathbf{G}$ to form another element denoted by $a \odot b$. The symbol $\odot$ is a placeholder for a concrete operation. The set and the operation $(\mathbf{G}, \odot)$, satisfy the following requirements known as the *Abelian group axioms*:

- If $a, b \in \mathbf{G}$, then $a \odot b \in \mathbf{G}$ (*closure*).
- If $a, b, c \in \mathbf{G}$, then $(a \odot b) \odot c = a \odot (b \odot c)$ (*associativity*).
- There exists an element $e \in \mathbf{G}$ called the *identity element*, such that for all $a \in \mathbf{G}$, $e \odot a = a \odot e = a$ (*identity element*).
- If $a \in \mathbf{G}$, then there exists an element $a^{(-1)} \in \mathbf{G}$ called the *inverse element to* a such that $a \odot a^{(-1)} = a^{(-1)} \odot a = e$ (*inverse element*).
- If $a, b \in \mathbf{G}$, then $a \odot b = b \odot a$ (*commutativity*).

The *inverse operation* $\div$ to $\odot$ is defined for all $a, b \in \mathbf{G}$ as follows:

$$a \div b = a \odot b^{(-1)}.$$

A nonempty set $\mathbf{G}$ is *linearly (totally) ordered* under the order relation $\leq$, if the following statements hold for all $a, b, c \in \mathbf{G}$:

- If $a \leq b$ and $b \leq a$, then $a = b$ (*antisymmetry*).
- If $a \leq b$ and $b \leq c$, then $a \leq c$ (*transitivity*).

- $a \leq b$ or $b \leq a$ (*totality*).

The *strict order* relation $<$ is defined for $a, b \in \mathbf{G}$: $a < b$ if $a \leq b$ and $a \neq b$.

Let $(\mathbf{G}, \odot)$ be an Abelian group, $\mathbf{G}$ be linearly ordered under $\leq$. $(\mathbf{G}, \odot, \leq)$ is said to be an *Abelian linearly ordered group*, *Alo-group* for short, if for all $c \in \mathbf{G}$: $a \leq b$ implies $a \odot c \leq b \odot c$.

If $\mathcal{G} = (\mathbf{G}, \odot, \leq)$ is an Alo-group, then $\mathbf{G}$ is naturally equipped with the order topology induced by $\leq$ and $\mathbf{G} \times \mathbf{G}$ is equipped with the related product topology. We say that $\mathcal{G}$ is a *continuous Alo-group* if $\odot$ is continuous on $\mathbf{G} \times \mathbf{G}$.

Because of the associative property, the operation $\odot$ can be extended by induction to n-ary operations, $n > 2$. Then, for a positive integer n, the (n)-*power* $a^{(n)}$ *of* $a \in \mathbf{G}$ is defined. We can extend the meaning of power $a^{(s)}$ to the case when s is a negative integer.

$\mathcal{G} = (\mathbf{G}, \odot, \leq)$ is *divisible* if for each positive integer n and each $a \in \mathbf{G}$ there exists the (n)*-th root of* a denoted by $a^{(1/n)}$, i.e. $\left(a^{(1/n)}\right)^{(n)} = a$. Moreover, the function $\|.\| : \mathbf{G} \to \mathbf{G}$ defined for each $a \in \mathbf{G}$ by:

$$\|a\| = \max\{a, a^{(-1)}\}$$

is called a $\mathcal{G}$-*norm*. The operation $d : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ defined by $d(a, b) = \|a \div b\|$ for all $a, b \in \mathbf{G}$ is called a $\mathcal{G}$-*distance*. It is easy to show that d satisfies the usual distance properties.

Example 1 Additive Alo-group
$\mathcal{R} = (]-\infty, +\infty[, +, \leq)$ *is a continuous Alo-group with:* $e = 0$, $a^{(-1)} = -a$, $a^{(n)} = n.a$.

Example 2 Multiplicative Alo-group
$\mathcal{R}^+ = (]0, +\infty[, \bullet, \leq)$ *is a continuous Alo-group with:* $e = 1$, $a^{(-1)} = a^{-1} = 1/a$, $a^{(n)} = a^n$. *Here, the symbol* $\bullet$ *denotes the usual multiplication.*

Example 3 Fuzzy additive Alo-group
$\mathcal{R}_a$=$(]-\infty, +\infty[, +_f, \leq)$, *see Ramik et al. (2014), is a continuous Alo-group with:* $a +_f b = a + b - 0.5$, $e = 0.5$, $a^{(-1)} = 1 - a$, $a^{(n)} = n.a - \frac{n-1}{2}$.

Example 4 Fuzzy multiplicative Alo-group
$]0,1[_m$=$(]0,1[, \bullet_f, \leq)$, *is a continuous Alo-group with:* $a \bullet_f b = \frac{ab}{ab+(1-a)(1-b)}$, $e = 0.5$, $a^{(-1)} = 1 - a$, $a^{(n)} = \frac{a^n}{a^n+(1-a)^n}$.

3 P-matrix on Alo-groups over a real interval

Let $\mathbf{G}$ be an open interval of the real line $\mathbf{R}$ and $\leq$ be the total order on $\mathbf{G}$ inherited from the usual order on $\mathbf{R}$, $\mathcal{G} = (\mathbf{G}, \odot, \leq)$ be a real Alo-group. We also assume that $\mathcal{G}$ is a divisible and continuous Alo-group. Then $\mathbf{G}$ is an open interval, see Cavallo et al. (2012).

The DM problem can be formulated as follows. Let $X = \{x_1, x_2, ..., x_n\}$ be a finite set of alternatives. These alternatives have to be classified from best to worst, using the information given by a DM in the form of pairwise comparison matrix.

The preferences over the set of alternatives X, can be represented in the following way. Let us assume that the preferences on X are described by a preference relation on X given by an $n \times n$ matrix $A = \{a_{ij}\}$, where $a_{ij} \in \mathbf{G}$ for all $i, j = 1, 2, ..., n$ indicates a preference intensity for the alternative x_i over x_j, i.e. it is interpreted as "x_i is a_{ij} times better than x_j". The elements of $A = \{a_{ij}\}$ satisfy the following reciprocity condition, see Cavallo et al. (2012).

An $n \times n$ matrix $A = \{a_{ij}\}$ is $\odot$-*reciprocal*, if:

$$a_{ij} \odot a_{ji} = e \text{ for all } i, j = 1, 2, ..., n, \tag{1}$$

or, equivalently,

$$a_{ji} = a_{ij}^{(-1)} \text{ for all } i, j = 1, 2, ..., n. \tag{2}$$

An $n \times n$ matrix $A = \{a_{ij}\}$ is $\odot$-*consistent* Cavallo et al. (2012), if:

$$a_{ik} = a_{ij} \odot a_{jk} \text{ for all } i, j, k = 1, 2, ..., n. \tag{3}$$

Here, $a_{ii} = e$ for all $i = 1, 2, ..., n$, and also (3) implies (1), i.e. an $\odot$-consistent matrix is $\odot$-reciprocal (but not vice-versa).

The following result gives a characterization of a $\odot$-consistent matrix by the vectors of weights, see Cavallo et al. (2012).

Proposition 1 *A P-matrix $A = \{a_{ij}\}$ is $\odot$-consistent if and only if there exists a vector $w = (w_1, w_2, ..., w_n)$, $w_i \in \mathbf{G}$ such that:*

$$w_i \div w_j = a_{ij} \text{ for all } i, j = 1, 2, ..., n. \tag{4}$$

If for some $i, j, k = 1, 2, ..., n$ (3) is not satisfied we say that the P-matrix $A = \{a_{ij}\}$ is *inconsistent.*

The inconsistency of A will be measured by the $\odot$-mean distance of the *ratio matrix* $W = \{w_i \div w_j\}$ to the matrix $A = \{a_{ij}\}$.

Let $A = \{a_{ij}\}, w = (w_1, ..., w_n), w_i \in \mathbf{G}$ for all $i = 1, 2, ..., n$, denote:

$$I_{\odot}(A, w) = \left(\bigodot_{1 \leq i < j \leq n} \|a_{ij} \div (w_i \div w_j)\| \right)^{(2/n(n-1))}. \tag{5}$$

Now, we define the concept of a priority vector. Consider the following optimization problem:

(P1) $\quad I_{\odot}(A, w) \longrightarrow min_e;$
subject to:

$\bigodot_{k=1}^{n} w_k = e,$
$w_i \in \mathbf{G}, i = 1, 2, ..., n.$

If an optimal solution of (P1) exists, then the *$\odot$-consistency index of A*, $I_{\odot}(A)$, is defined as:

$$I_{\odot}(A) = I_{\odot}(A, w^*), \tag{6}$$

where $w^* = (w_1^*, ..., w_n^*)$ is the optimal solution of (P1). Notice that "minimization" in (P1) is carried out with respect to the identity element e.

The optimal solution w^* of (P1) is called the *$\odot$-priority vector of A*. In (P1), $\bigodot_{k=1}^{n} w_k = e$, is a normalization condition reducing the number of the priority vectors (uniqueness), on condition that the optimal solution exists. The proof of the following theorem is evident and it is left to the reader.

Proposition 2 *A P-matrix $A = \{a_{ij}\}$ is $\odot$-consistent if and only if:*

$$I_{\odot}(A) = e.$$

4 P-matrix with missing elements

Usually, in many decision-making procedures, experts are capable of providing preference degrees for any pair of given alternatives. However, this may not be always true. A missing value can be the result of the inability of an expert to quantify the degree of preference of one alternative over another. In this case he/she may decide not to guess the preference degree between some pairs of alternatives. When an expert is not able to express a particular value a_{ij}, because he/she does not have a clear idea of how the alternative x_i is better than alternative x_j , this does not mean that he/she prefers both options with the same intensity. In order to model these situations, in the following we introduce the incomplete preference matrix. Here, we use a different approach and notation as compared to e.g. Alonso et al. (2008); on the other hand, our approach is similar to that of Ramik (2014).

We are going to define the P-matrix with missing elements. For the sake of simplicity of presentation we identify the alternatives $x_1, x_2, ..., x_n$ with integers $1, 2, ..., n$, i.e. by $X = \{1, 2, ..., n\}$ we denote the set of alternatives, $n > 1$. Moreover, let $X \times X = X^2$ be the Cartesian product of X, i.e. $X^2 = \{(i, j)|i, j \in X\}$. Let $K \subset X^2$, $K \neq X^2$ and let $\mathcal{A}$ be the preference relation on K given by the (membership) function $\mu_{\mathcal{A}} : K \to \mathbf{G}$, $\mathbf{G}$ is an Alo-group. The preference relation $\mathcal{A}$ is represented by the $n \times n$ *preference matrix* $A(K) = \{a_{ij}\}_K$ *with missing elements depending on* K as follows:

$$a_{ij} = \begin{cases} \mu_{\mathcal{A}}(i, j) & \text{if} \quad (i, j) \in K, \\ \times & \text{if} \quad (i, j) \notin K. \end{cases}$$

In what follows we shall assume that each P-matrix $A(K) = \{a_{ij}\}_K$ with missing elements is $\odot$-reciprocal, i.e.:

$$a_{ij} \odot a_{ji} = e \text{ for all } (i, j) \in K.$$

If $L \subset K$, and $L = \{(i_1, j_1), (i_2, j_2), ..., (i_q, j_q)\}$ is a set of pairs (i, j) of alternatives such that there exist a_{ij}, with $a_{ij} \in \mathbf{G}$ for all $(i, j) \in L$, then the subset L' symmetric to L, i.e. $L' = \{(j_1, i_1), (j_2, i_2), ..., (j_q, i_q)\}$ is also a subset of K, i.e. $L' \subset K$. By reciprocity, each subset K of X^2 can be represented as follows: $K = L \cup L' \cup D$, where L is the set of pairs of alternatives (i, j) of given preference degrees a_{ij} of the P-matrix $A(K)$ and D is the diagonal of this matrix, i.e. $D = \{(1, 1), (2, 2), ..., (n, n)\}$, where $a_{ii} = e$ for all $i \in X$. The reciprocity property means that the expert is able to quantify both a_{ij} and a_{ji} as well as a_{ii}. The elements a_{ij} with $(i, j) \in X^2 - K$ are called *the missing elements of the matrix* $A(K)$. Note that the missing elements of $A(K)$ are denoted by the symbol $\times$ ("ex"). On the other hand, those elements which express the preference degrees given by the experts are denoted by a_{ij}, where $(i, j) \in K$. By $\odot$-reciprocity it is sufficient that the expert quantifies only those elements a_{ij}, where $(i, j) \in L$, such that $K = L \cup L' \cup D$. In what follows we shall investigate two important particular cases: $L = \{(1, 2), (2, 3), ..., (n - 1, n)\}$, and $L = \{(1, 2), (1, 3), ..., (1, n)\}$.

Now we shall deal with the problem of finding the values of missing elements of a given P-matrix so that the extended matrix is as much $\odot$-consistent as possible. In the ideal case the extended matrix will become $\odot$-consistent.

Let $K \subset X^2$, let $A(K) = \{a_{ij}\}_K$ be a P-matrix with missing elements. The matrix $A^e(K) = \{a^e_{ij}\}_K$, called the $\odot$-*extension of* $A(K)$, is defined as follows:

$$a^e_{ij} = \begin{cases} a_{ij} & \text{if} \quad (i, j) \in K, \\ v^*_i \div v^*_j & \text{if} \quad (i, j) \notin K. \end{cases}$$

Here, $v^* = (v_1^*, v_2^*, ..., v_n^*)$ is called the *$\odot$-priority vector with respect to K*, if it is an optimal solution of the following problem:

$$\text{(P2)} \qquad d(v,K) \longrightarrow min_e \; ;$$

subject to:

$$\bigodot_{j=1}^{n} v_j = e,$$

$$v_i \in \mathbf{G} \text{ for all } i=1,2,...,n.$$

Here, $d(v,K) = \big(\bigodot_{i,j \in K} \|a_{ij} \div (v_i \div v_j)\|\big)^{(1/|K|)}$, $|K|$ denotes the cardinality of K. Note, that the $\odot$-consistency index of the matrix $A^e(K) = \{a_{ij}^e\}_K$ is defined by (6) as $I_\odot(A^e(K))$. Minimization in (P2) is carried out with respect to the identity element e.

The proof of the following proposition follows directly from Proposition 2.

Proposition 3 *$A^e(K) = \{a_{ij}^e\}_K$ is $\odot$-consistent, (i.e. $I_\odot(A^e(K)) = e$) if and only if:*

$$d(v^*, K) = e.$$

5 Special cases of preference matrices with missing elements

For a complete $n \times n$ reciprocal preference matrix we need $N = \frac{n(n-1)}{2}$ pairs of elements to be evaluated by an expert. For example, if $n = 12$, then $N = 66$, which is a considerable number of pairwise comparisons. We ask that the expert evaluates only "around n"pairwise comparisons of alternatives which seems to be a reasonable number. In this section we shall investigate two important particular cases of a fuzzy preference matrix with missing elements where the expert will evaluate only $n-1$ pairwise comparisons of alternatives. Here we generalize the approach presented in Ramik (2014). Let $K \subset X^2$ be a set of indices given by an expert, $A(K) = \{a_{ij}\}_K$ be a P-matrix with missing elements. Moreover, let $K = L \cup L' \cup D$. In fact, it is sufficient to assume that the expert will evaluate only a chain of matrix elements of L, i.e. $a_{12}, a_{23}, a_{34}, ..., a_{n-1,n}$.

5.1 Case $L = \{(1,2),(2,3),...,(n-1,n)\}$

Here, we assume that the expert evaluates $n-1$ chain elements of the P-matrix $A(K)$, i.e. $a_{12}, a_{23}, a_{34}, ..., a_{n-1,n}$. First, we investigate the $\odot$-extension of

$A(K)$. We derive the following result.

Proposition 4 *Let $L = \{(1,2),(2,3),...,(n-1,n)\}$, $a_{ij} \in \mathbf{G}$ with $a_{ij} \odot a_{ji} = e$ for all $(i,j) \in K$, $K = L \cup L' \cup D$, and $L' = \{(2,1),(3,2),...,(n,n{-}1)\}$, $D = \{(1,1),...,(n,n)\}$. Then the $\odot$-priority vector $v^* = (v_1^*, v_2^*, ..., v_n^*)$ with respect to K is given as:*

$$v_1^* = \left(\bigodot_{i=2}^{n}(a_{12} \odot ... \odot a_{i-1,i})\right)^{(1/n)}, \tag{7}$$

$$v_i^* = a_{i-1,i}^{(-1)} \odot v_{i-1}^* \text{ for } i = 2,3,...,n. \tag{8}$$

Proof
If (7) and (8) are satisfied, then:

$$v_i^* = a_{i-1,i} \odot a_{i-2,i-1} \odot ... \odot a_{1,2} \odot v_1^* \text{ for } i = 2,...,n,$$

hence for all $i = 1,2,...,n, v_i^* \in \mathbf{G}$ and:

$$\bigodot_{i=1}^{n} v_i^* = e.$$

Also,

$$a_{i-1,i} = v_{i-1}^* \div v_i^* \text{ for } i = 2,...,n.$$

Then $v = (v_1^*,...,v_1^*)$ is an optimal solution of (P2).

As a simple consequence, we obtain the following corollary.

Corollary 5 *Let $\mathcal{R} = (]-\infty,+\infty[,+,\leq)$ be an additive Alo-group, see Example 1, i.e. $\odot = +$. Then we obtain (7), (8) in the following form:*

$$v_1^* = \frac{1}{n}\sum_{i=2}^{n}(n-i+1)a_{i-1,i}, \tag{9}$$

$$v_i^* = v_{i-1}^* - a_{i-1,i} \text{ for } i = 2,3,...,n. \tag{10}$$

Example 5 *Let $\odot = +$, $L = \{(1,2),(2,3),(3,4)\}$, see Example 1. Let the chain evaluations be $a_{12} = 9, a_{23} = 8, a_{34} = 5$, with $a_{ij} + a_{ji} = 0$ for all $(i,j) \in L$, $K = L \cup L' \cup D$. Hence $A(K) = \{a_{ij}\}_K$ is the following P-matrix with missing elements:*

$$A(K) = \begin{pmatrix} 0 & 9 & \times & \times \\ -9 & 0 & 8 & \times \\ \times & -8 & 0 & 5 \\ \times & \times & -5 & 0 \end{pmatrix}.$$

By (9), (10) we obtain $+$-priority vector v^* with respect to K, particularly, $v^* = (12, 3, -5, -10)$. By (4) we obtain the following $+$-extension of $A(K)$:

$$A^e(K) = \begin{pmatrix} 0 & 9 & 17 & 22 \\ -9 & 0 & 8 & 13 \\ -17 & -8 & 0 & 5 \\ -22 & -13 & -5 & 0 \end{pmatrix},$$

where $A^e(K)$ is $+$-consistent, and $d(v, B(K)) = 0$, hence $I_+(A^e(K)) = 0$. The corresponding ranking of the alternatives is $x_1 > x_2 > x_3 > x_4$.

Also, as a simple consequence, we obtain the following corollary.

Corollary 6 *Let $\mathcal{R}^+ = (]0, +\infty[, \bullet, \leq)$ be a multiplicative Alo-group, see Example 2, i.e. $\odot = \bullet$. Then we obtain (7), (8) in the following form:*

$$P_1 = 1,\ P_i = P_{i-1}a_{i-1,i},\ \textit{for } i = 2, 3, ..., n, \tag{11}$$

$$v_1^* = \left(\prod_{i=1}^{n} P_i\right)^{\frac{1}{n}}, \tag{12}$$

$$v_i^* = \frac{v_{i-1}^*}{a_{i-1,i}}\ \textit{for } i = 2, 3, ..., n. \tag{13}$$

Example 6 *Let $\odot = \bullet$, $L = \{(1,2), (2,3), (3,4)\}$, see Example 2. Let the chain evaluations be $a_{12} = 4, a_{23} = 3, a_{34} = 2$, with $a_{ij} \bullet a_{ji} = 1$ for all $(i, j) \in L$, $K = L \cup L' \cup D$. Hence $A(K) = \{a_{ij}\}_K$ is the following P-matrix with missing elements:*

$$A(K) = \begin{pmatrix} 1 & 4 & \times & \times \\ \frac{1}{4} & 1 & 3 & \times \\ \times & \frac{1}{3} & 1 & 2 \\ \times & \times & \frac{1}{2} & 1 \end{pmatrix}.$$

By (11), (12), (13) we obtain the $\bullet$-priority vector v^* with respect to K, in case, $v^* = (5.826, 1.456, 0.485, 0.243)$. By (4) we obtain the following $\bullet$-extension of $A(K)$:

$$A^e(K) = \begin{pmatrix} 1 & 4 & 12 & 24 \\ \frac{1}{4} & 1 & 3 & 6 \\ \frac{1}{12} & \frac{1}{3} & 1 & 2 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{2} & 1 \end{pmatrix},$$

where $A^e(K)$ is $\bullet$-consistent, and $d(v, B(K)) = 1$, hence $I_\bullet(A^e(K)) = 1$. The corresponding ranking of the alternatives is $x_1 > x_2 > x_3 > x_4$.

Corollary 7 *Let $\mathcal{R}_a = (]-\infty, +\infty[, +_f, \leq)$ be a fuzzy additive Alo-group, see Example 3, i.e. $\odot = +_f$. Then we obtain (7), (8) in the following form:*

$$S_1 = 0, \;\; S_i = S_{i-1} + a_{i-1,i}, \;\; \textit{for } i = 2, 3, ..., n, \tag{14}$$

$$v_1^* = \frac{3-n}{4} + \frac{1}{n}\sum_{i=1}^{n} S_i, \tag{15}$$

$$v_i^* = v_{i-1}^* - a_{i-1,i} + 0.5 \textit{ for } i = 2, 3, ..., n. \tag{16}$$

Example 7 *Let $\odot = +_f$, $L = \{(1,2), (2,3), (3,4)\}$, see Example 3. Let the chain evaluations be $a_{12} = 0.9, a_{23} = 0.5, a_{34} = 0.3$, with $a_{ij} +_f a_{ji} = 0.5$ for all $(i,j) \in L$, $K = L \cup L' \cup D$. Hence $A(K) = \{a_{ij}\}_K$ is the following P-matrix with missing elements:*

$$A(K) = \begin{pmatrix} 0.5 & 0.9 & \times & \times \\ 0.1 & 0.5 & 0.5 & \times \\ \times & 0.5 & 0.5 & 0.3 \\ \times & \times & 0.7 & 0.5 \end{pmatrix}.$$

By (14), (15), (16) we obtain the $+_f$-priority vector v^* with respect to K, in case, $v^* = (0.75, 0.35, 0.35, 0.55)$. By (4) we obtain the following $+_f$-extension of $A(K)$:

$$A^e(K) = \begin{pmatrix} 0.5 & 0.9 & 0.9 & 0.7 \\ 0.1 & 0.5 & 0.5 & 0.3 \\ 0.1 & 0.5 & 0.5 & 0.3 \\ 0.3 & 0.7 & 0.7 & 0.5 \end{pmatrix},$$

where $A^e(K)$ is $+_f$-consistent, and $d(v, B(K)) = 0.5$, hence $I_{+_f}(A^e(K)) = = 0.5$. The corresponding ranking of the alternatives is $x_1 > x_4 > x_2 \sim x_3$.

We obtain also the following corollary.

Corollary 8 *Let $]0,1[_m = (]0,1[, \bullet_f, \leq)$ be a fuzzy multiplicative Alo-group, see Example 4, i.e. $\odot = \bullet_f$. Then for $i = 2, 3, ..., n$ we obtain (7), (8) in the following form:*

$$P_i = \frac{(1-a_{12}) \cdot ... \cdot (1-a_{i-1,i})}{(1-a_{12}) \cdot ... \cdot (1-a_{i-1,i}) + a_{12} \cdot ... \cdot a_{i-1,i}}, \tag{17}$$

$$P = \frac{P_1 \cdot ... \cdot P_n}{(1-P_1) \cdot ... \cdot (1-P_n) + P_1 \cdot ... \cdot P_n}, \tag{18}$$

$$v_1^* = \frac{(1-P)^{1/n}}{(1-P)^{1/n} + P^{1/n}}, \tag{19}$$

$$v_i^* = \frac{(1-a_{i-1,i})v_{i-1}^*}{(1-a_{i-1,i})v_{i-1}^* + a_{i-1,i}(1-v_{i-1}^*)}. \tag{20}$$

Formulas (17), (18), (19) and (20) can be easily calculated e.g. using Excel.

Example 8 *Let* $\odot = \bullet_f$*,* $L = \{(1,2),(2,3),(3,4)\}$*, see Example 4. Let the chain evaluations be* $a_{12} = 0.9, a_{23} = 0.5, a_{34} = 0.3$*, with* $a_{ij} \bullet_f a_{ji} = 0.5$ *for all* $(i,j) \in L$*,* $K = L \cup L' \cup D$*. Hence* $A(K) = \{a_{ij}\}_K$ *is the following P-matrix with missing elements:*

$$A(K) = \begin{pmatrix} 0.5 & 0.9 & \times & \times \\ 0.1 & 0.5 & 0.5 & \times \\ \times & 0.5 & 0.5 & 0.3 \\ \times & \times & 0.7 & 0.5 \end{pmatrix}.$$

By (18), (19) we obtain the $\bullet$*-priority vector* v^* *with respect to* K*, in this case,* $v^* = (0.808, 0.318, 0.318, 0.522)$*. By (4) we obtain the following* $\bullet_f$*-extension of* $A(K)$*:*

$$A^e(K) = \begin{pmatrix} 0.5 & 0.9 & 0.9 & 0.794 \\ 0.1 & 0.5 & 0.5 & 0.3 \\ 0.1 & 0.5 & 0.5 & 0.3 \\ 0.206 & 0.7 & 0.7 & 0.5 \end{pmatrix},$$

where $A^e(K)$ *is* $\bullet$*-consistent, and* $d(v, B(K)) = 0.5$*, hence* $I_{\bullet_f}(A^e(K)) = 0.5$*. The corresponding ranking of the alternatives is* $x_1 > x_4 > x_2 \sim x_3$*.*

5.2 Case $L = \{(1,2),(1,3),...,(1,n)\}$

Now we assume that the expert evaluates the pairs consisting of a given fixed element and the remaining $n-1$ elements, i.e. the P-matrix $A(K)$ is given by $a_{12}, a_{13}, ..., a_{1n}$. We investigate the extension of $A(K)$ and obtain the following result.

Proposition 9 *Let* $L = \{(1,2),(1,3),...,(1,n)\}$*,* $a_{ij} \in \mathbf{G}$ *with* $a_{ij} \odot a_{ji} = e$ *for all* $(i,j) \in K$*,* $K = L \cup L' \cup D$*, and* $L' = \{(2,1),(3,1),...,(n,1)\}$*,* $D = \{(1,1),...,(n,n)\}$*. Then the* $\odot$*-priority vector* $v^* = (v_1^*, v_2^*, ..., v_n^*)$ *with respect to* K *is given as:*

$$v_1^* = \left(\bigodot_{i=2}^{n} a_{1i} \right)^{(1/n)}, \tag{21}$$

$$v_i^* = a_{1,i}^{(-1)} \odot v_1^* \text{ for } i = 2, 3, ..., n. \tag{22}$$

Proof
If (21) and (22) are satisfied, then:

$$v_i^* = a_{1,i-1} \odot a_{1,i-2} \odot ... \odot a_{1,2} \odot v_1^* \text{ for } i = 2, ..., n,$$

hence for all $i = 1, 2, ..., n, v_i^* \in \mathbf{G}$, moreover,

$$\bigodot_{i=1}^{n} v_i^* = e,$$

and also:

$$a_{1,i-1} = v_1^* \div v_i^* \text{ for } i = 2, ..., n.$$

Then $v = (v_1^*, ..., v_1^*)$ is an optimal solution of (P2).

As a simple consequence, we obtain the following corollary.

Corollary 10 *Let $\mathcal{R} = (]-\infty, +\infty[, +, \leq)$ be an additive Alo-group, see Example 1, i.e. $\odot = +$. Then we obtain (21), (22) in the following form:*

$$v_1^* = \frac{1}{n} \sum_{i=2}^{n} a_{1,i}, \tag{23}$$

$$v_i^* = v_1^* - a_{1,i} \text{ for } i = 2, 3, ..., n. \tag{24}$$

Moreover, the extension of $A(K)$, i.e. matrix $A^e(K) = \{a_{ij}^{ac}\}_K$ is $\odot$-consistent.

Example 9 $\odot = +$, *$L = \{(1, 2), (1, 3), (1, 4)\}$, let the expert evaluations be $b_{12} = 9, b_{13} = 8, b_{14} = 5$, with $b_{ij} + b_{ji} = 0$ for all $(i, j) \in L$, let $K = L \cup L' \cup D$. Let $B(K) = \{b_{ij}\}_K$ be the following P-matrix with missing elements:*

$$B(K) = \begin{pmatrix} 0 & 9 & 8 & 5 \\ -9 & 0 & \times & \times \\ -8 & \times & 0 & \times \\ -5 & \times & \times & 0 \end{pmatrix}.$$

By (23), (24) we obtain the +-priority vector w^* with respect to K, in this case, $w^* = (5.5, -3.5, -2.5, 0.5)$. By (4) we obtain the following +-extension of $B(K)$:

$$B^e(K) = \begin{pmatrix} 0 & 9 & 8 & 5 \\ -9 & 0 & -1 & -4 \\ -8 & 1 & 0 & -3 \\ -5 & 4 & 3 & 0 \end{pmatrix},$$

where $B^e(K)$ is +-consistent, and $d(v, B(K)) = 0$, hence $I_+(B^e(K)) = 0$. The corresponding ranking of the alternatives is $x_1 > x_4 > x_3 > x_2$.

Corollary 11 *Let $\mathcal{R}^+ = (]0, +\infty[, \bullet, \leq)$ be a multiplicative Alo-group, see Example 2, i.e. $\odot = \bullet$. Then we obtain (21), (22) in the following form:*

$$v_1^* = \left(\prod_{i=2}^{n} a_{1,i}\right)^{1/n}, \tag{25}$$

$$v_i^* = \frac{v_1^*}{a_{1,i}} \text{ for } i = 2, 3, ..., n. \tag{26}$$

Moreover, the extension of $A(K)$, i.e. the matrix $A^e(K) = \{a_{ij}^{ac}\}_K$ is $\bullet$-consistent.

Example 10 *$\odot = \bullet$, $L = \{(1,2), (1,3), (1,4)\}$, see Example 2. Let the expert evaluations be $b_{12} = 4, b_{13} = 3, b_{14} = 2$, with $b_{ij} \bullet b_{ji} = 1$ for all $(i, j) \in L$, let $K = L \cup L' \cup D$. Let $B(K) = \{b_{ij}\}_K$ be the following P-matrix with missing elements:*

$$B(K) = \begin{pmatrix} 1 & 4 & 3 & 2 \\ \frac{1}{4} & 1 & \times & \times \\ \frac{1}{3} & \times & 1 & \times \\ \frac{1}{2} & \times & \times & 1 \end{pmatrix}.$$

By (25), (26) we obtain the $\bullet$-priority vector w^* with respect to K, in this case, $w^* = (2.213, 0.553, 0.738, 1.107)$. By (4) we obtain the following $\bullet$-extension of $B(K)$:

$$B^e(K) = \begin{pmatrix} 1 & 4 & 3 & 2 \\ \frac{1}{4} & 1 & \frac{3}{4} & \frac{1}{2} \\ \frac{1}{3} & \frac{4}{3} & 1 & \frac{2}{3} \\ \frac{1}{2} & 2 & \frac{3}{2} & 1 \end{pmatrix},$$

where $B^e(K)$ is $\bullet$-consistent, and $d(v, B(K)) = 1$, hence $I_\bullet(B^e(K)) = 1$. The corresponding ranking of the alternatives is $x_1 > x_2 \sim x_3 > x_4$.

Corollary 12 *Let $\mathcal{R}_a = (]-\infty, +\infty[, +_f, \leq)$ be a fuzzy additive Alo-group, see Example 3, i.e. $\odot = +_f$. Then we obtain (21), (22) in the following form:*

$$v_1^* = \frac{1}{2n} + \frac{1}{n}\sum_{i=2}^{n} a_{1,i}, \tag{27}$$

$$v_i^* = v_1^* - a_{1,i} + 0.5. \text{ for } i = 2, 3, ..., n. \tag{28}$$

Moreover, the extension of $A(K)$, i.e. matrix $A^e(K) = \{a_{ij}^{ac}\}_K$ is $+_f$-consistent.

Example 11 *$\odot = +_f$, $L = \{(1,2),(1,3),(1,4)\}$, let the expert evaluations be $b_{12} = 0.9, b_{13} = 0.5, b_{14} = 0.3$, with $b_{ij} +_f b_{ji} = 0.5$ for all $(i,j) \in L$, let $K = L \cup L' \cup D$. Let $B(K) = \{b_{ij}\}_K$ be the following P-matrix with missing elements:*

$$B(K) = \begin{pmatrix} 0.5 & 0.9 & 0.6 & 0.4 \\ 0.1 & 0.5 & \times & \times \\ 0.4 & \times & 0.5 & \times \\ 0.6 & \times & \times & 0.5 \end{pmatrix}.$$

By (27), (28) we obtain the $+_f$-priority vector w^* with respect to K, in this case, $w^* = (0.6, 0, 2, 0.5, 0.7)$. By (4) we obtain the following $+_f$-extension of $B(K)$:

$$B^e(K) = \begin{pmatrix} 0.5 & 0.9 & 0.6 & 0.4 \\ 0.1 & 0.5 & 0.2 & 0.0 \\ 0.4 & 0.8 & 0.5 & 0.3 \\ 0.6 & 1.0 & 0.7 & 0.5 \end{pmatrix},$$

where $B^e(K)$ is $+_f$-consistent, and $d(v, B(K)) = 0.5$, hence $I_{+_f}(B^e(K)) = = 0.5$. The corresponding ranking of the alternatives is $x_4 > x_1 > x_3 > x_2$.

Corollary 13 *Let $]0,1[_m = (]0,1[, \bullet_f, \leq)$ be a fuzzy multiplicative Alo-group, see Example 3, i.e. $\odot = \bullet_f$. Then for $i = 2, 3, ..., n$ we obtain (21), (22) in the following form:*

$$P_i = \frac{a_{1,i}^{1/n}}{a_{1,i}^{1/n} + (1 - a_{1,i})^{1/n}}, \tag{29}$$

$$v_1^* = \frac{P_1 \cdot ... \cdot P_n}{P_1 \cdot ... \cdot P_n + (1 - P_1) \cdot ... \cdot (1 - P_n)}, \tag{30}$$

$$v_i^* = \frac{(1 - a_{1,i}) v_1^*}{(1 - a_{1,i}) v_1^* + a_{1,i}(1 - v_1^*)}. \tag{31}$$

Moreover, the extension of $A(K)$, i.e. matrix $A^e(K) = \{a_{ij}^{ac}\}_K$ is $\bullet_f$-consistent.

Example 12 *Let $\odot = \bullet_f$, $L = \{(1,2),(1,3),(1,4)\}$, $b_{12} = 0.9$, $b_{13} = 0.6, b_{14} = 0.4$, with $b_{ij} \bullet_f b_{ji} = 0.5$ for all $(i,j) \in L$, let $K = L \cup L' \cup D$. Let $B(K) = \{b_{ij}\}_K$ be the following P-matrix with missing elements (see Example 4 and 10):*

$$B(K) = \begin{pmatrix} 0.5 & 0.9 & 0.6 & 0.4 \\ 0.1 & 0.5 & \times & \times \\ 0.4 & \times & 0.5 & \times \\ 0.6 & \times & \times & 0.5 \end{pmatrix}.$$

By (29), (30), (31) we obtain the $\bullet_f$-priority vector w^* with respect to K, in this case, $w^* = (0.634, 0.161, 0.536, 0.722)$. By (4) we obtain the following $+$-extension of $B(K)$:

$$B^e(K) = \begin{pmatrix} 0.5 & 0.9 & 0.6 & 0.4 \\ 0.1 & 0.5 & 0.143 & 0.069 \\ 0.4 & 0.857 & 0.5 & 0.308 \\ 0.6 & 0.931 & 0.692 & 0.5 \end{pmatrix},$$

where $B^e(K)$ is $\bullet_f$-consistent, and $d(v, B(K)) = 0.5$, hence $I_{\bullet_f}(B^e(K)) = = 0.5$. The corresponding ranking of the alternatives is $x_4 > x_1 > x_3 > x_2$.

6 Conclusions

In this paper we have dealt with some properties of P-matrices, namely reciprocity and consistency, with the entries from an Alo-group. We have shown how to measure the degree of consistency and also how to evaluate pairs of elements using values taken from an Alo-group if some elements are missing. Moreover, we have dealt with two particular cases of the incomplete P-matrix, and we have proposed some special methods for dealing with such cases. Finally, eight numerical examples have been presented to illustrate our approach.

Acknowledgments

This research has been supported by GACR project No. 14-02424S.

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MULTIPLE CRITERIA DECISION MAKING
Vol. 10 2015

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ON A SEQUENTIAL DECISION PROCESS WHERE OFFERS ARE DESCRIBED BY TWO TRAITS

Abstract

This article presents a model of searching for some resource, e.g. a job, whose value depends on two quantitative traits. The decision maker observes offers in a random order and must accept precisely one offer. Recall of previously observed offers is not possible. It is assumed that the value of an offer is a linear function of these two traits, which come from a bivariate normal distribution. We consider the following four strategy sets: i) the decision on whether to accept an offer is based purely on the first trait, ii) any decision is only made after observing both traits, iii) after observing the first trait, the decision maker can either immediately accept, immediately reject or observe the second trait and then decide, iv) after observing the first trait, the decision maker can either immediately reject or observe the second trait and then decide. The goal of the decision maker is to maximize his expected reward, where the reward is equal to the value of the offer selected minus the search costs. The optimal strategy from each of these four sets is derived. An example is given.

Keywords: sequential decision process, job search problem, choice based on several traits.

1 Introduction

Anyone who wishes to acquire a particular type of good must i) find offers, ii) assess the value of an offer, iii) decide whether to accept or reject a particular offer. It is assumed that offers appear in a random order. The decision maker must accept one offer and the recall of previously viewed offers is not possible. In the biology literature, this problem is often presented as the mate choice prob-

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lem (in the original version only females are choosy). In the economics literature, this problem often appears as the job search problem (in the original version only job seekers are choosy) or the problem of purchasing a given resource. Stigler (1961) was the first to consider such a model. He assumed that a client is looking for a particular type of good. The goal of the client is to acquire the resource at the lowest possible total cost, where the total cost is assumed to be the price paid for the good plus the search costs. Janetos (1980) presented a similar model within the framework of mate choice.

Classical models assume that decisions are made on the basis of a single trait, which defines the value of an offer. However, Backwell and Passmore (1996) observed that a female of the crab species *Uca annulipes* first observes the size of a male. If a male is sufficiently large, then the female observes the quality of his nest. On the basis of this, she then decides whether to lay eggs or not. Hence, the value of an offer may depend on various traits and a decision maker can collect information on each offer before making a decision. Fawcett and Johnstone (2003); Castellano and Cermelli (2011), as well as Ramsey (2012) presented models of such decision processes. Similar decision processes are also considered in the economics and psychology literature [see Analytis et al., 2014; Baucells et al., 2008; Bearden and Connolly, 2007; Hogarth and Karelaia, 2005, as well as Lim et al., 2006). Ramsey (2012) presents a model of pair formation by mutual acceptance. This model can be interpreted as a job search problem, in which a job seeker first obtains incomplete information about a job (e.g. from an advert). From the point of view of an employer, he obtains incomplete information regarding a job seeker via an application. After receiving these initial signals, if the two parties are still interested in working together, then they can meet for an interview, where both obtain additional information on the value of their prospective partner. This article considers a model in which information is obtained in two steps, but only one side is choosy.

Wiegmann et al. (2010) presented a similar model to the one considered here. They assumed that the order in which traits are observed is fixed. The decision maker incurs general search costs, as well as costs for observing individual traits. They presented the general form of the optimal strategy. In this article, a particular case of such a model, according to which the traits come from a bivariate normal distribution, is considered. The following strategy sets are considered:

i) S_1: the decision on whether to accept an offer is based on the first trait,
ii) S_2: both traits are observed and then a decision is made,
iii) S_3: after observing the first trait, the decision maker can immediately accept, immediately reject or observe the second trait and then decide,
iv) S_4: after observing the first trait, the decision maker can immediately reject or observe the second trait and then decide.

It is useful to consider various strategy sets for two reasons: i) if the gains from observing the second trait or making a decision at a particular moment are small relative to the associated costs, then strategies from the sets S_1 and S_2 can be competitive with strategies from the sets S_3 and S_4, ii) practical aspects of a given problem may mean than some strategies are infeasible. For example, someone wishing to buy a new car may initially collect information (e.g. on reliability, fuel consumption) about various models from the Internet. However, he must visit a dealer before purchasing a car. Hence, strategies from set S_3 are infeasible.

The first goal is to derive the optimal strategy from each set S_i, i =1,2,3,4. The most important results are given in Statements 1-3, which are original results regarding the form of the optimal strategy when the decision maker collects information step by step. The second goal is a description of a numerical procedure for approximating the optimal strategies from sets S_3 and S_4. This method is illustrated using an example. Chapter 2 presents the model. The form of the optimal strategies from sets S_1 and S_2 are derived in Chapter 3. Chapters 4 and 5 consider strategies from the sets S_3 and S_4, respectively. These chapters contain the most important results of this article, namely the statements regarding the form of the optimal strategies from these sets. Chapter 6 presents algorithms which approximate the optimal strategies from sets S_3 and S_4. Chapter 7 presents an example illustrating how these optimal strategies can be approximated and gives numerical results. The summary gives some possible directions for future research.

2 Model

A decision maker observes a sequence of offers whose length is not bounded. He must choose exactly one offer and recall of previously observed offers is not possible. After accepting an offer, the decision maker stops searching. The *i*-th offer appears at moment i and is described by a two-dimensional random variable (X_1, X_2), where X_j denotes the *j*-th trait of the offer. Assume that (X_1, X_2) has a two-dimensional normal distribution with expected value (0, 0) and correlation matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, where $\rho \equiv \rho(X_1, X_2)$ is the coefficient of correlation between these two traits. The value of an offer, V, is given by $V = \alpha X_1 + X_2$, where the parameter α describes the relative weight of the first trait with respect to the second. It is assumed that the first trait must be assessed before the second trait can be assessed.

From these assumptions, V has a normal distribution with mean 0 and variance $\alpha^2 + 1 + 2\rho\alpha$ and $Cov(V, X_1) = \alpha + \rho$. In addition, given that $X_1 = x_1$, the second trait has a normal distribution with mean ρx_1 and variance $1 - \rho^2$, and the value of an offer has a normal distribution with mean $u_1(x_1)$ and variance $1 - \rho^2$, where:

$$u_1(x_1) = E[V|X_1 = x_1] = x_1(\alpha + \rho). \tag{1}$$

Without loss of generality, we may assume that $\alpha > -\rho$, as when $\alpha = -\rho$, then the first trait does not give any information about the value of an offer and thus should not be taken into consideration. When $\alpha < -\rho$, then X_1 is negatively correlated with the value of an offer and thus we can treat $-X_1$ as an indicator of the offer's value.

We consider the four strategy sets $S_1, \dots, S_4$ described in the introduction. It is assumed that the cost of observing trait i is c_i, where $c_i > 0$. The cost of making a decision is d, $d \geq 0$, and the mean cost of finding an offer is $c_0, c_0 > 0$. The payoff of a searcher is equal to the value of the offer chosen minus the sum of the costs incurred. We derive the optimal strategy from each of these four sets. Let u^S denote the expected reward under the optimal strategy from the set S.

It should be noted that under the assumption that the traits come from any two-dimensional normal distribution and the observation and decision costs are linear, then the corresponding search problem can be reduced to the one described above by using the appropriate standardisation procedure.

3 Optimal strategies from the sets S_1 and S_2

Assume that the searcher bases his decision purely on the first trait, X_1, i.e. the strategy belongs to S_1. The observation and decision costs incurred at each moment are $c_0 + c_1 + d$. Since these costs are additive, the optimal strategy is stationary (i.e. the optimal decision of the searcher is independent of the moment). After observing the first trait, the searcher should accept an offer if and only if its expected value is greater than the expected reward from future search. It follows that:

$$u^{S_1} = E[\max\{u^{S_1}, u_1(X_1)\}] - c_0 - c_1 - d. \tag{2}$$

There is no analytic solution to Equation (2), which is of the form $u^{S_1} = h(u^{S_1})$. Differentiating, $0 < h'(u) = 1 - F(u) < 1$, where F is the distribution function of the standard normal distribution. Hence, h is a contraction mapping. It follows that there is exactly one solution to this equation, which can be approximated using the following iterative process: i) $u_1 = 0$, ii) $u_{k+1} = h(u_k)$. Thus $\lim_{k\to\infty} u_k = u^{S_1}$.

Since $u_1(x_1)$ is increasing in x_1, it follows from Equation (2) that the optimal strategy is of the following form: accept an offer as long as its value is at least x_c, where $u_1(x_c) = u^{S_1}$, i.e. $x_c = \frac{u^{S_1}}{\alpha+\rho}$.

Now assume that the searcher assesses both traits before making a decision, i.e. his strategy is from the set S_2. In this case, at each moment the search costs incurred are $c_0 + c_1 + c_2 + d$. The searcher should accept an offer if and only if its value is greater than the expected reward from future search. It follows that:

$$u^{S_2} = E[\max\{u^{S_2}, V\}] - c_0 - c_1 - c_2 - d. \tag{3}$$

Equation (3) can be solved in an analogous way to Equation (2). An offer should be accepted if and only if $V > u^{S_2}$.

4 Optimal strategy from the set S_3

After observing the first trait, the searcher can reject an offer, accept it or observe the second trait. Let $u_3^*(x_1)$ denote the optimal expected reward when the searcher observes the second trait and $X_1 = x_1$. The optimal strategy satisfies the following conditions:

a) After observing the first trait, the searcher should immediately accept an offer if and only if $u_1(x_1) \geq \max\ [u^{S_3}, u_3^*(x_1)]$, i.e. when the expected value of an offer is greater than both the expected reward from search and the optimal expected reward from observing the second trait. Similarly, the searcher should observe the second trait if and only if $u_3^*(x_1) \geq \max\ [u^{S_3}, u_1(x_1)]$, i.e. when the expected reward from observing the second trait is greater than both the expected reward from future search and the expected value of the offer. Otherwise, an offer should be immediately rejected.

b) After observing the second trait, the searcher should accept an offer if and only if $V \geq u^{S_3}$, i.e. when the value of the offer is greater than the expected reward from future search.

The expected reward from observing the second trait when $X_1 = x_1,\ u_3^*(x_1)$, is given by:

$$u_3^*(x_1) = E[\max\{u^{S_3}, V\} | X_1 = x_1] - c_2 - d,$$

$$u_3^*(x_1) = u_1(x_1) + E[\max\{u^{S_3} - u_1(x_1), \sqrt{1-\rho^2}Z\}] - c_2 - d, \tag{4}$$

where *Z* has the standard normal distribution. From Equation (4) and Criterion a), given above, it follows that the searcher should observe the second trait rather than accept an offer immediately after observing the first trait if and only if:

$$E[\max\ \{u^{S_3} - u_1(x_1), \sqrt{1-\rho^2}\, Z\}] - c_2 - d \geq 0.$$

Statement 1 describes the form of the optimal strategy, including a necessary and sufficient condition for the second trait to be observed with a positive probability.

Statement 1: *When:*

$$c_2 + d \geq\ E[\max\ \{0, \sqrt{1-\rho^2}Z\}] = \sqrt{\frac{1-\rho^2}{2\pi}}, \tag{5}$$

then the optimal strategy is of the following form: accept an offer if and only if $X_1 \geq x_c$*, where* x_c *is the threshold used under the optimal strategy from the set* S_1*. In this case,* $u^{S_3} = u^{S_1} = u_1(x_c)$*. Otherwise, the optimal strategy is of the following form: there exist three constants* $x^{*,1}$*,* $x^{*,2}$ *and* u^{S_3}*, such that* $x^{*,1} < x^{*,2}$ *and:*

a) when $x_1 < x^{*,1}$, a searcher should immediately reject an offer after observing the first trait,

b) when $x_1 \geq x^{*,2}$, a searcher should immediately accept an offer after observing the first trait,

c) when $x^{*,1} \leq x_1 < x^{*,2}$, a searcher should observe the second trait; after observing the second trait, the searcher should accept the offer if and only if $V \geq u^{S_3}$,

d) the constants $x^{*,1}$, $x^{*,2}$ and u^{S_3} satisfy the following conditions: i) u^{S_3} is the optimal expected reward, ii) when $x_1 = x^{*,1}$, the expected reward from rejecting an offer is equal to the expected reward from observing the second trait, iii) when $x_1 = x^{*,2}$ the expected reward from accepting the offer is equal to the expected reward from observing the second trait.

From these conditions, it follows that $x^{*,1}$, $x^{*,2}$ and u^{S_3} satisfy the following system of equations:

$$u^{S_3} = u^{S_3}F(x^{*,1}) + \int_{x^{*,1}}^{\infty} \max[u_1(x), u_3^*(x)]f(x)dx - c_0 - c_1 - d, \quad (6)$$

$$u^{S_3} = E[\max \ \{u^{S_3}, u_1(x^{1,*}) + \sqrt{1-\rho^2}Z\}] - c_2 - d, \quad (7)$$

$$u_1(x^{*,2}) = E[\max\{u^{S_3}, u_1(x^{2,*}) + \sqrt{1-\rho^2}\, Z\}] - c_2 - d, \quad (8)$$

where f and F denote the density function and the cumulative distribution function, respectively, of the standard normal distribution.

The proof of Statement 1 is given in the Appendix.

The form of the optimal strategy is rather intuitive. When the value of the first trait is particularly low or high, then an offer should be immediately rejected or accepted, as appropriate. However, it is worthwhile observing the second trait when the value of the first trait is neither particularly low nor particularly high, the costs of observing the second trait, c_2, and of making a decision, d, are low and the second trait contains a large amount of information about the value of an offer given the value of the first trait, i.e. $|\rho|$ is small. It should be noted that the condition determining when it is optimal to observe the second trait is independent of c_0, c_1 and α. This results from the properties of the multivariate normal distribution, in particular from the fact that $Var(X_2|X_1 = x_1)$ does not depend on x_1. On the other hand, the qualitative form of the optimal strategy would be similar under more general assumptions regarding the joint distribution of the two traits.

Statement 2, presented below, shows that a simple substitution can transform Equations (7) and (8) into a single equation, which is independent of Equation (6).

Statement 2: *Independently of the value u^{S_3}, the solution to Equations (7) and (8) satisfies $u_1(x^{*,1}) = u^{S_3} - s$ and $u_1(x^{*,2}) = u^{S_3} + s$, where s is the solution of the following equation:*

$$0 = E[\max\{-s, \sqrt{1-\rho^2}\,Z\}] - c_2 - d. \tag{9}$$

From Statement 2, it follows that Equation (6) can be written in the form:

$$U^{S_3} = U^{S_3}F(U^{S_3} - s) + \int_{U^{S_3}-s}^{\infty} \max\ [u_1(x), u_3^*(x)]f(x)dx - c_0 - c_1 - d, \tag{10}$$

where $u_3^*(x)$ is given by Equation (4). Hence, s can be derived from Equation (9), and then the only unknown in Equation (10) is U^{S_3}. The proof of Statement 2 is given in the Appendix.

5 The optimal strategy from the set S_4

After observing the first trait, the searcher must either reject an offer or observe the second trait. Hence, the optimal strategy must satisfy the following conditions:

i) after observing the first trait, the searcher should observe the second trait if and only if the expected payoff from observing the second trait is greater than the expected reward from future search,
ii) after observing the second trait the searcher should accept an offer if and only if the value of an offer is greater than the expected reward from future search.

Let $u_4^*(x_1)$ denote the expected reward from observing the second trait, when the value of the first trait is x_1. It follows that:

$$u_4^*(x_1) = E[\max\{u^{S_4}, u_1(x_1)\}] - a_2 - d. \tag{11}$$

From the optimality criteria, it follows that the searcher should observe the second trait if and only if:

$$E[\max\{u^{S_4}, u_1(x_1) + \sqrt{1-\rho^2}Z\}] - a_2 - d \geq u^{S_4}.$$

The next statement follows from the equation defining the optimal expected reward when the first trait is being observed, together with the fact that the left hand side of the above equation is increasing in x_1.

Statement 3: *Under the optimal strategy from the set S_4, the searcher observes the second trait if and only if $x \geq x^{3,*}$, where $u_1(x^{*,3}) = u^{S_4} - s$ and s satisfies Equation (9). After observing the second trait, the searcher should accept an offer when its value is at least u^{S_4}. The thresholds $x^{3,*}$ and u^{S_4} satisfy the following pair of equations:*

$$E[max\{u^{S_4}, u_1(x^{3,*}) + \sqrt{1-\rho^2}Z\}] = u^{S_4} + c_2 + d\,, \tag{12}$$

$$u^{S_4} = u^{S_4}F(x^{3,*}) + \int_{x^{3,*}}^{\infty} u_4^*(x)f(x)dx - c_0 - c_1 - d\,. \tag{13}$$

It should be noted that after subtracting u^{S_4} from both sides of Equation (12), we obtain an equation of analogous form to Equation (9). The proof of Statement 3 is analogous to the proof of Statement 1 and is thus omitted.

6 A procedure for determining the optimal strategies from S_3 and S_4

When determining the optimal strategy from either of these sets, we first solve the following equation of the form $s = g(s)$, which is equivalent to Equation (9):

$$s = s + E[\max\{-s, \sqrt{1-\rho^2}Z\}] - c_2 - d.$$

It is possible to solve this equation numerically using the following iterative process:

$$s_1 = 0;\ s_{n+1} = g(s_n).$$

Since $0 < g'(s) = 1 - F(-s) < 1$, this iteration is based on a contraction mapping and there exists exactly one solution of this equation, $s = \lim_{n\to\infty} s_n$.

Now we derive the optimal strategy from the set S_3. Setting $u_1(x^{*,1}) = u^{S_3} - s$, $u_1(x^{*,2}) = u^{S_3} + s$ and $u_1(x_1) = x_1(\alpha+\rho)$, we obtain, $x^{*,1} = \frac{u^{S_3}-s}{\alpha+\rho}$ and $x^{*,2} = \frac{u^{S_3}+s}{\alpha+\rho}$. From Equation (6) it follows that:

$$u^{S_3} = u^{S_3}F(x^{*,1}) + \int_{x^{*,1}}^{x^{*,2}} u_3^*(x)f(x)dx + \int_{x^{*,2}}^{\infty} u_1(x_1)f(x)dx - c_0 - c_1 - d,$$

where $u_3^*(x)$ is defined by Equation (4). This equation is of the form $u^{S_3} = h(u^{S_3})$ and thus can be solved using the iterative procedure: $u_0 = 0$; $u_{n+1} = h(u_n)$. The numerical results obtained by the author suggest that the function h is a contraction mapping, but no proof of this hypothesis could be found.

Now we derive the optimal strategy from the set S_4. Substituting $x^{*,3} = \frac{u^{S_4}-s}{\alpha+\rho}$ into Equation (13), we obtain:

$$u^{S_4} = u^{S_4}F\left(\frac{u^{S_4}-s}{\alpha+\rho}\right) + \int_{\frac{u^{S_4}-s}{\alpha+\rho}}^{\infty} u_4^*(x)f(x)dx - c_0 - c_1 - d,$$

where $u_4^*(x)$ is defined by Equation (11). This equation can also be solved using an iterative numerical procedure. This procedure is illustrated in the following section.

7 Example

We now consider the realization of such a search problem where $\alpha = 1$, i.e. the traits have equal weights, $\rho = 0.5$ (the coefficient of correlation between the traits), $c_0 = 0.1$ (search costs), $c_1 = c_2 = 0.05$ (observation costs), $d = 0$ (costs for making a decision).

7.1 Optimal strategy from the set S_1

From Equation (1), the expected value of an offer when $X_1 = x_1$ is given by $u_1(x_1) = = x_1(\alpha+\rho) = 1.5x_1$. Given the form of the optimal strategy, it follows that the searcher should accept an offer when $x_1 \geq \frac{2u^{S_1}}{3}$. From Equation (2), it follows that:

$$u^{S_1} = E[\max\{u^{S_1}, 1.5Z\}] - c_0 - c_1 - d,$$

$$u^{S_1} = u^{S_1}F\left(\frac{2u^{S_1}}{3}\right) + \frac{1.5}{\sqrt{2\pi}}\exp\left(\frac{-2[u^{S_1}]^2}{9}\right) - 0.15.$$

This equation was solved using an iterative procedure. The optimal expected reward is approximately $u^{S_1} \approx 1.3535$. The searcher should accept an offer if and only if $x_1 \geq \frac{2u^{S_1}}{3} \approx 0.9023$.

7.2 The optimal strategy from set S_2

Now we derive the optimal strategy in the case when both traits are observed automatically. The variance of the value of an offer, σ_V^2, is given by:

$$\sigma_V^2 = \alpha^2 + 1 + 2\rho\alpha = 3.$$

From Equation (3), we obtain:

$$u^{S_2} = u^{S_2}F\left(\frac{u^{S_2}}{\sqrt{3}}\right) + \sqrt{\frac{3}{2\pi}}\exp\left[\frac{-(u^{S_2})^2}{6}\right] - 0.2.$$

Solving this equation numerically, we obtain that the optimal expected reward is approximately $u^{S_2} \approx 1.4250$. Under the optimal strategy, the searcher accepts an offer if and only if the its value satisfies $V \geq u^{S_2} \approx 1.4250$.

7.3 Optimal strategy from the set S_3

From Condition (5), it follows that the optimal strategy is based purely on the first trait when $a_2 + d \geq \sqrt{\frac{1-\rho^2}{2\pi}} \approx 0.3455$. Since this condition is not satisfied, the optimal strategy is thus described by a set of three parameters: $x^{1,*}, x^{2,*}$ and u^{S_3} (see Statement 1). First we derive s, where:

$$s = u_1(x^{*,2}) - u^{S_3} = u^{S_3} - u_1(x^{*,1}).$$

From Equation (9), we obtain:

$$s = s + E[\max\{-s, \sqrt{1-\rho^2}Z\}] - c_2 - d,$$

$$s = sF\left(\frac{s}{\sqrt{0.75}}\right) + \sqrt{\frac{0.75}{2\pi}}\exp\left(\frac{-s^2}{1.5}\right) - 0.05.$$

Solving this equation by an iterative procedure, we obtain $s \approx 1.0271$.

Now we derive the optimal expected reward. We have:

$$u_1(x^{*,1}) = u^{S_3} - s \Rightarrow x^{*,1} = \frac{u^{S_3} - s}{1.5}, \quad (14)$$

$$u_1(x^{*,2}) = u^{S_3} + s \Rightarrow x^{*,2} = \frac{u^{S_3} + s}{1.5}. \quad (15)$$

From Equation (6), we obtain:

$$u^{S_3} = u^{S_3}F(x^{1,*}) + \int_{x^{1,*}}^{x^{2,*}} u_3^*(x)f(x)dx + \int_{x^{2,*}}^{\infty} u_1(x)f(x)dx - 0.15. \qquad (16)$$

Solving this equation by an iterative procedure, we obtain $u^{S_3} \approx 1.5730$. It should be noted that the first integral was approximated using the trapezium rule based on 1000 subintervals of equal length. From Equations (14) and (15), it follows that $x^{1,*} \approx 0.3639, x^{*,2} \approx 1.7334$. Hence, the optimal strategy is of the following form:

a) If $x_1 < 0.3639$, an offer should be immediately rejected.
b) If $x_1 \geq 1.7334$, an offer should be immediately accepted.
c) If $0.3639 \leq x_1 < 1.7334$, the searcher should observe the second trait. After observing the second trait, an offer should be accepted if and only if $x_1 + x_2 \geq 1.5730$.

7.4 Optimal strategy from the set S_4

From Statement 3, we have $u_1(x^{*,3}) = x^{*,3}(1+\rho) = u^{S_4} - s$, where s was derived in Section 7.3. Hence, $x^{3,*} = \frac{u^{S_4}-s}{1+\rho}$. From Equation (13), it follows that:

$$u^{S_4} = u^{S_4}F(x^{*,3}) + \int_{x^{*,3}}^{\infty} u_4^*(x)f(x)dx - c_0 - 0.15,$$

where $u_4^*(x)$ is given by Equation (12). The solution to this equation, $u^{S_4} \approx$ ≈ 1.5641, was derived using a similar iterative approach to the one used to solve Equation (16). It follows that $x^{*,3} \approx 0.3580$. Hence, the optimal strategy from the set S_4 is of the form:

a) Reject an offer immediately if and only if $x_1 \leq 0.3580$.
b) Otherwise, observe the second trait and based on both traits accept the offer if and only if $x_1 + x_2 \geq 1.5641$.

8 Summary

This article has presented the form of optimal strategies in decision problems where the value of an offer depends on two quantitative traits which come from the bivariate normal distribution. These results can be fairly easily generalized to a larger number of traits, since the form of the conditional distribution of a single trait given the values of the traits that have already been seen is analogous to the conditional distribution of the second trait given the value of the first trait in the model presented above. In particular, the variances of these conditional distributions do not depend on the values of the traits already observed. In this case, it is possible to derive the appropriate threshold (relative to the optimal value) when k traits have yet to be observed by recursion. It is more difficult to derive the optimal strategy when the joint distribution of the traits is not normal.

It was also assumed that the order in which the traits are observed is fixed and offers appear in a random order. In many practical problems of this form, the searcher can choose the order in which objects are seen. For example, when someone wishes to buy a car, then they can choose the order in which models are observed according to the mark of a car. In this case, it is necessary to find the optimal order in which to observe offers. This problem has been considered to some degree in the biology literature (Fawcett and Johnstone, 2003). Hogarth and Karelaia (2005) consider this problem from a psychological point of view. Analytis et al. (2014) consider a problem in which the searcher can choose the order in which offers are observed using *a priori* information about the expected value of each particular offer. Considering an analogous model to the one considered here, but where the assumptions regarding the order in which traits and/or offers are observed and regarding the joint distribution of the traits are relaxed, would seem to be a fruitful area for future research.

Appendix

Proof of Statement 1: Assume that the optimal strategy from the set S_3 is of the following form: accept the first offer such that the value of the first trait is at least x_c, where $u_1(x_c) = u^{S_3}$. It follows from this assumption that $u^{S_3} = u^{S_1}$ and when the value of the first trait is x_c, the searcher prefers to immediately accept this first offer rather that observe the second trait. From Equation (4), we obtain:

$$u_1(x_c) + E[\max\{u^{S_3} - u_1(x_c), \sqrt{1-\rho^2}\, Z\}] - c_2 - d \le u_1(x_c).$$

Hence, $c_2 + d \ge E[\max \ \{0, \sqrt{1-\rho^2} Z\}]$. In addition:

$$E[\max\{0, \sqrt{1-\rho^2} Z\}] = \sqrt{1-\rho^2} \int_0^{\infty} \frac{x e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = \sqrt{\frac{1-\rho^2}{2\pi}}.$$

It follows that Inequality (5) gives a necessary condition for the optimal strategy to be based on purely one trait. In order to show that it is a sufficient condition, we need to show that when Inequality (5) is satisfied, then i) the searcher prefers to immediately accept an offer rather than observe the second trait whenever $x_1 > x_c$, and ii) the searcher prefers to immediately reject an offer when $x_1 < x_c$. Let:

$$g(x) = E[\max\{x, \sqrt{1-\rho^2}\, Z\}] - c_2 - d.$$

Differentiating, we obtain $0 < g'(x) < 1$. Let $x_1 > x_c$. The searcher prefers to immediately accept an offer rather than immediately reject it. From Condition (4), it follows that the searcher prefers to immediately accept an offer rather than observe the second trait when:

$$E[\max\{u^{S_3} - u_1(x_1), \sqrt{1-p^2}\,Z\}] - c_2 - d \leq 0 \Rightarrow g(k) \leq 0,$$

where $k < 0$, since $u_1(x_1) > u^{S_3}$. This inequality holds since $g'(x) > 0$ and, by assumption, $g(0) \leq 0$. Hence, when $x_1 > x_c$, the searcher prefers to immediately accept an offer rather than observe the second trait.

Now assume $x_1 < x_c$. In this case, the searcher prefers to immediately reject an offer rather than immediately accept it. The expected reward from future search is equal to u^{S_3}. From Equation (4), the expected reward from observing the second trait is equal to:

$$u_3^*(x_1) = E[\max\{u^{S_3}, u_1(x_1) + \sqrt{1-\rho^2}\,Z\}] - c_2 - d.$$

Since u_1 is an increasing function, it follows that u_3^* is also an increasing. By assumption, $u_3^*(x_c) < u^{S_3}$. It follows that for $x_1 < x_c$, $u_3^*(x_1) < u^{S_3}$. Hence, when $x_1 < x_c$, the searcher prefers to immediately reject an offer rather than observe the second trait. Hence, Inequality (5) gives a necessary and sufficient condition for the optimal strategy to be based purely on the first trait.

Now assume that $g(0) > 0$, i.e. Condition (5) is not satisfied. It follows that when $x_1 = x_c$, the searcher prefers to observe the second trait rather than accept an offer at once, i.e. $u^{S_3} > u^{S_1}$. Generally, the searcher prefers to observe the second trait rather than immediately accept the first offer at once when $u_3^*(x_1) >$ $> u_1(x_1)$. It follows that:

$$E\,[\max\{u^{S_3} - u_1(x_1), \sqrt{1-\rho^2}\,Z\}] - c_2 - d > 0. \tag{Z.1}$$

The left hand side of this inequality is increasing in u^{S_3} for fixed x_1 and decreasing in x_1 for fixed u^{S_3}. In addition, when $x_1 \to \infty$, the left hand side of this equation tends to $-c_2 - d$. Hence, for each $u^{S_3} > u_1(x_c)$, there exists exactly one constant $x^{*,2}$, where $x^{*,2} > x_c$ such that:

$$E[\max\{u^{S_3} - u_1(x^{*,2}), \sqrt{1-\rho^2}Z\}] - c_2 - d = 0.$$

Adding $u_1(x^{*,2})$ to both sides, we obtain Equation (8).

It should be noted that when $x_1 = x^{*,2}$, the searcher is indifferent between immediately accepting an offer and observing the second trait. From Inequality (Z.1), it follows that the searcher prefers to immediately accept an offer than to observe the second trait if and only if $x_1 > x^{*,2}$. In addition, since $g(0) > 0$ and $g'(0) > 0$, when $x^1 \geq x^{*,2}$, $u^{S_3} - u_1(x^{*,2}) < 0$, and thus the searcher prefers to immediately accept an offer rather than immediately rejecting it. Hence, the searcher should immediately accept an offer when $x_1 = x^{*,2} > x_c$.

Assume that the searcher is indifferent between observing the second trait and immediately rejecting the offer when $x_1 = x^{*,1}$. We have $x_3^*(x^{*,1}) = u^{S_3}$, i.e. Equation (7) is satisfied. From the form of the function g, it follows that $x^{*,1} < x^{*,2}$. Since u_1 is an increasing function, for $x_1 > x^{*,1}$ we obtain:

$$E[\max\{u^{S_3}, u_1(x_1) + \sqrt{1-\rho^2}\, Z\}] - c_2 - d > u^{S_3}.$$

It follows that for $x_1 > x^{*,1}$, the searcher prefers to observe the second trait rather than immediately reject the offer. Hence, when $x^{*,1} \leq x_1 < x^{*,2}$, the searcher should observe the second trait.

Using an analogous argument, when $x_1 < x^{*,1}$, the searcher prefers to reject an offer at once rather than observe the second trait. It was shown above that when $x_1 < x^{*,2}$, the searcher prefers to observe the second trait rather than immediately accept an offer. Thus when $x_1 < x^{*,1}$, the searcher should immediately reject an offer. It follows that the optimal strategy is of the form described in Statement 1.

In order to derive the optimal strategy, apart from Equations (7) and (8), we require another equation. Since u^{S_3} is the optimal expected reward and the first trait has a standard normal distribution, we obtain:

$$u^{S_3} = \int_{-\infty}^{\infty} \max[\, u^{S_3}, u_1(x), u_3^*(x)] f(x) dx - c_0 - c_1 - d.$$

Using the fact that under the optimal strategy, the searcher should immediately reject an offer when $x_1 \leq x^{*,1}$, we obtain Equation (6).

Proof of Statement 2: It is sufficient to show that Equations (7) and (8) are equivalent to Equation (9). Let $s = u_1(x^{*,2}) - u^{S_3}$. From Equation (8), it follows that:

$$u_1(x^{2,*}) = E[\max\left\{u_1(x^{2,*}) - s, u_1(x^{2,*}) + \sqrt{1-\rho^2}Z\right\}] - c_2 - d,$$

$$0 = E[\max\ \{-s, \sqrt{1-\rho^2}Z\}] - c_2 - d.$$

Let $s = u^{S_3} - u_1(x^{*,1})$. From Equation (7), we obtain:

$$u_1(x^{1,*}) + s = E[\max\{u_1(x^{1,*}) + s, u_1(x^{1,*}) + \sqrt{1-\rho^2}Z\}] - c_2 - d,$$

$$0 = E[\max\{0, -s + \sqrt{1-\rho^2}Z\}] - c_2 - d,$$

$$0 = E[\sqrt{1-\rho^2}Z + \max\{-\sqrt{1-\rho^2}Z, -s\}] - c_2 - d.$$

From the symmetry of the standard normal distribution, it follows that:

$$0 = E[\max\{-s, \sqrt{1-\rho^2}Z\}] - c_2 - d.$$

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M U L T I P L E   C R I T E R I A   D E C I S I O N   M A K I N G
Vol. 10 2015

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SINGLE GOOD EXCHANGE MODEL WITH CHANGEABLE PREFERENCES GIVEN AS A TWO-SIDED MATCHING

Abstract

Markets are usually considered as strongly efficient – each investor is said to have the same information at the same time. But due to incomplete, false or vague information on the market, significant data have become an expensive good. Thus, the accessibility to it may vary.

In the following paper a behavioural approach to decision-making is presented. An investor's decision to enter a trade is based on multiple criteria such as knowledge, personal experience, investing history and individual characteristics. All those factors are reflected in individual investor's preference toward a short or long position in a trade of good.

In the paper we present two exchange models of an arbitrary good, where information about the market is reflected in investors' preferences. A two-sided matching approach for choosing contract sides is given. Simulations of market dynamics, including asymmetry and changeability of information, are performed and a possible equilibrium is discussed. The main idea of this paper is to research possible states of market equilibrium on the basis of behavioural factors and describe its usefulness for modelling market dynamics.

Keywords: two-sided matching, exchange model, game theory.

1 Introduction

The main problem researched in this paper is the influence of asymmetric information on investors' decisions, which is reflected in contracts made on the market. The behavioural factors represented in investing preferences are included in the model beside the economic laws.

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We assume here that differences in accessibility to data do exist. The reasoning is that information is quite an expensive good. With money, investor may get access to good market brokers, faster and better equipment, business partners with more experience. All of this may result in investors' different knowledge about the market, which influence their decisions and reactions to the same market factors.

By the knowledge of the market we understand not only raw data, but also methods of processing and conclusions based on it. The term also includes personal characteristics that allow investors to successfully operate on the market. Examples may be connections, back office, risk aversion and education.

The main motivation for the research conducted is the assumption of market effectiveness in most of existing models. Furthermore, many models do not include behavioural factors or the possibility that the information will change over time or will be updated.

In this article, simulations of the market are performed, using the Visual Basic for Applications language. The sets of initial preferences are created randomly. The preferences are changing with every iteration step, on the basis of the investing history. Each step corresponds to the time required to finalize a contract. The simulation ends if a certain equilibrium is reached. This will be explained further in the paper.

There are many models of market exchange, encompassing, for example, vague information or behavioural aspects of decision (Kunreuther, Pauly, 1985). In some of them the market is considered quantitatively, in others the behavioural aspects are formulated in terms of fuzzy numbers (Piasecki, Witoch, 2014). There is also a game-theoretical approach in which the market is represented by a game, with investors being players (Shapley, Shubik 1969).

2 Theoretical assumptions

The main idea of this paper is to represent the market as a simple exchange model, based on Two-Sided Matching theory. In this approach we consider only four investors, willing to take short or long position in a contract, without possibility of not investing. The short side of a contract sells a particular good and the investor on the long side buys it. Each of the investors has a set of preferences toward a bargain with the remaining ones. These are based on his or her market knowledge, personal characteristics and experience, and may change over time.

In general, in Two-Sided Matching problems there are two disjoint sets U and W. Each agent from U and W submits a list of acceptable participants from the other set, which may be ranked in order of preference. We say that any two participants and $n_j \in W$, where $i, j \in \{1,2, \dots, N\}$, find each other acceptable if both

m_i and n_j rank each other on their respective preference lists. A matching M is a set of disjoint pairs (m, n) such that $m \in U, n \in W$, where for each pair m and n find each other acceptable, and M satisfies certain assumptions, that is specific capacity constraints (Abraham, 2003). We will denote the pair (m, n) from a particular matching M, by $M(m, n)$.

A matching M is called unstable if there are pairs $(m_i, n_j) \in M$ and $(m_s, n_t) \in M$, $i, j, s, t \in N \backslash \{\infty\}$, such that m_i prefers n_t to n_j and n_j prefers m_s to m_i (Gale, Shapley, 1962). The pair $(m_i, n_j) \in M$ is called a blocking pair.

A matching M is stable if it cannot be improved upon by any individual or any pair of agents (Roth, Sotomayor, 1992).

Let us now focus on the definition of preferences. We will denote by P the set of all preference lists, $\mathrm{P} = \{P(m_1), \dots, P(m_t), P(n_1), P(n_2), \dots, P(n_t)\}$, one for each agent from each side of the contract. A specific market is denoted by the triple $(U, W; P)$. We will write $m_i >_{n_k} m_j$ to mean that n_k strictly prefers m_i to m_j. Analogously, $m_i \geq_{n_k} m_j$ means that n_k prefers m_i at least as well as m_j. Similarly with $n_i >_{m_k} n_j$ and $n_i \geq_{m_k} n_j$ (Roth, Sotomayor, 1992).

Also, if m_i is the best possible partner for n_j, then n_j is the worst possible partner for m_i (Biro, 2007), which means that in the market model, agents forming a contract have opposite interest over the possible outcome. It follows that the matchings do not treat both sides of the trade equally. The side that is first to pick the partner is the favoured one. In the remainder of the paper we will assume that the favoured side is the short side. The interpretation is that if an agent sells goods on the market it means that he has already entered the market and has a knowledge about it or has means and knowledge to enter it with a product to sell, and that knowledge gives him an advantage over the agent on the long side.

Apart from Game Theory, in the paper we also consider aspects of market equilibrium and its stability. In the dynamic concept of market we have that partial equilibrium is reached if for a certain good in a particular moment there exist a vector of prices that global demand equals global supply. Global equilibrium exists when this situation arises for every good on the market (Arrow, Hurwicz, 1958; Malaga, 2012).

Regarding the asymmetry of information, we will assume that there is a certain amount of shared information, interpreted as official information about an agent, that is accessible to every other agent. However, preferences of a particular entrepreneur are based not only on that information, but also on his individual interpretation of other data, given to him, for example, by different brokers.

The information possessed is reflected directly in the agent's preferences regarding other agents. If the data about target companies is vast and useful, the entrepreneur has a chance of using it to achieve an income in a trade with another agent. That is, if information is unambiguous and complete, the agent it refers to will be more preferred by the entrepreneur possessing the information. Otherwise, if the information is vague, trade with the agent is risky and less preferred.

In addition, we may deduce that if the preferences differ from one agent to another, there may be some kind of instability in the company, which is reflected in increased risk, there is an information chaos on the market or the entrepreneur is investing based strictly on his behavioural decisions. On the other hand, if the preferences are similar, the situation may suggest perfect information on the market, stability of the companies and lack of behavioural factors in decision making.

While studying numerical examples we may encounter two scenarios. First, the case may end in a cycle: that is, the preferences of each agent will be the same as at some point in the past. Taking into account that the probability of a certain side winning will be computed on the basis of preferences, the repetition of preferences clearly indicate a cycle of investment decisions on the market. We will interpret this as the information being explicit and complete. That is, all the investors had opportunities to make decisions based on fair information and the market appears to be stable – we may predict what will happen next, and the investment risk has decreased.

The other case is when all the probabilities of a certain side winning take the same values. There would be the same chance for every agent to win, which we interpret as an information chaos. Because everything may happen, the risk of the investment is high. The market is not stable, we cannot assume that at some moment in the future a cycle will appear and an opportunity to predict future market conditions will arise. The computations will be performed until one of these situations occurs.

3 Matching models

The simplest exchange model includes four agents willing to buy or sell a contract for some kind of asset. In this case investors do not specify which position – long or short – they want to take. From the information given they choose an agent they want to trade with, and they choose to buy or sell depending on the chance of achieving a revenue.

The first set of investors' preferences reflects the situation on the market. From those, the probability of each entrepreneur's winning is calculated as follows:

Procedure 1
Weights are assigned to the preferred matchings – if an agent was first on the investor's preference list he gets weight 3, if second weight 2, else 1.

For every matching, the weighted number of investors willing to trade with the given agent is calculated (let us call it the sum of revenues and denote by sr). That is, if we denote by w_{ij} the weight of a matching between agents W_i on the long side and W_j on the short side, we have:

$$sr_j = \sum_{i=1}^{n} w_{ij}$$

The probability of agent's W_j winning is computed as follows:

$$p_j = \frac{\sum_{i=1}^{n} w_{ij}}{\sum_{k=1}^{n} \sum_{i=1}^{n} w_{ik}} = \frac{sr_j}{\sum_{k=1}^{n} sr_k}$$

where *n* is the number of agents on the market.

The next step is to choose pairs for the trades. In this situation we will assume that the trades favour the short side. Following that, the first two agents selling the asset are determined by taking the maximal value of the winning probability. The interpretation may be the following: a new market is being created, and only trustworthy (and well informed) companies are allowed to sell their assets. In order to choose long sides of the trades, we take the short side with the maximal probability and assign to them the first agent from their preference list who is not on the short side of the other contract. The other investor takes the long side of the second contract.

While modelling the formation of the contracts we assume that the agents change their investing positions, that is, if an agent was selling the contract then in the next iteration he will be buying and vice versa. Thus, we take the agents from the long position in the previous iteration and assign them to the short position. Long positions in the contract are then calculated as in the first case.

With each contract executed, the preferences of the investors change. If an agent was on the winning side, in the next contract his preferences are exactly the same as in the previous one. If he was on the losing side, the agent he has lost to will now be last on his preference list. We interpret it that the investor has lost and he assumes that his information about the market was wrong or incomplete. He now wishes to invest in the agent he thought was second-best to trade with. The agent he had lost with is treated as suspicious and falls to the last place of his preference list. The iterations are performed until we encounter a cycle or all the probabilities have the same value.

The second model of an exchange presented here includes separate preferences and blocking pairs. We assume now that the agents have two sets of preferences: one for taking the short side in a trade and another one for the long side. By a blocking pair we mean a pair for which there exists another matching with a higher revenue than the one considered.

Procedure 2

First, the tables of preferences are created for short (S) and long (L) sides with the following probabilities:

$$p_j^L = \frac{\sum_{i=1}^{n} w_{ij}{}^S}{\sum_{k=1}^{n}\sum_{i=1}^{n} w_{ik}{}^S} = \frac{sr_j{}^S}{\sum_{k=1}^{n} sr_k{}^S}$$

$$p_j^S = \frac{\sum_{i=1}^{n} w_{ij}{}^L}{\sum_{k=1}^{n}\sum_{i=1}^{n} w_{ik}{}^L} = \frac{sr_j{}^L}{\sum_{k=1}^{n} sr_k{}^L}$$

A table containing the sums of pairs $w_{ij}^S + w_{ij}^L$ is created and the maximal sum is found $\underset{i,j}{max}{}^1\,(w_{ij}^S + w_{ij}^L)$.

If the indices for the maximal value are k and l, we find second-to-maximal value, that is $\underset{i\neq k, j\neq l}{max}{}^1\,(w_{ij}^S + w_{ij}^L)$.

To find a blocking pair, another table of revenues is created.

Using the second table, we calculate $\underset{i,j}{max}{}^2\,(w_{ij}^S + w_{ij}^L)$ and, assuming that the indices for the last value were s and t, we set $\underset{i\neq s, j\neq t}{max}{}^2\,(w_{ij}^S + w_{ij}^L)$.

If we have:

$$w_{kl}^S + w_{kl}^L + \underset{i\neq k, j\neq l}{max}{}^1\,(w_{ij}^S + w_{ij}^L) \geq w_{st}^S + w_{st}^L + \underset{i\neq s, j\neq t}{max}{}^2\,(w_{ij}^S + w_{ij}^L)$$

then we choose $w_{kl}^S + w_{kl}^L$ and $\underset{i,j}{max}{}^1\,(w_{ij}^S + w_{ij}^L)$ as the revenues characterizing the first and second contracts, respectively. Otherwise we choose the values $w_{st}^S + w_{st}^L$ and $\underset{i,j}{max}{}^2\,(w_{ij}^S + w_{ij}^L)$.

To sum up, the introduction of blocking pairs we ensures that the contracts created on the market were optimal. That is, if the first two contracts were to be set up, and the total revenue of some other matching was higher, then the other contracts based on the other matching would be eventually formed. The pairs introducing higher revenue to the second matching are called the blocking pairs.

4 Empirical examples

The results of the computations for the exchange models are given in the Appendix. From the initial random preferences (Iteration 0) we may conclude that the information on the market is quite clear and complete. The argument may be that the data regarding agent 3 must be satisfying, since two of three agents want to trade with him, and the last one has agent 3 second of his preference list. Similarly, agent 4 must not be a good partner for business, because two of three agents prefer him least. Thus, we can conclude that the initial market information level was high.

The table of preference shows the computation of the sum of revenues (sr_j) and the probability of winning (p_j). The 'Contracts' table shows formed contracts and their winners, who are determined based on the probability of achieving success.

When moving to net iteration, investors change contract sides. Those who won retain their previous preference while for those who lost, the last trade partner falls to the last place on their preference list. The procedure then continues.

In Iteration 8, the preferences are exactly the same as those in Iteration 4, which means that we have encountered a cycle. An interesting issue is that of possible connection between high information level (similar initial preferences) and cycle generation.

For the model with blocking pairs 38 iterations were necessary to obtain a cycle. From the preference table we can obtain detailed information about W1 and vague information about other agents. That is, the information about the market is not perfect.

Now, we have both short and long preferences for every investor. In the preference table we compute the sums of revenues from a possible contract for each investor (${sr_j}^S$, ${sr_j}^L$) and probability of winning when taking a side in a contract (p_j^S, p_j^L). In the table 'Contracts, we designate initial trades to be made, check if there exist a blocking pair with higher revenue under some other matching and create final contracts. As a last step we indicate winners and change preferences of those who lost.

It took 38 iterations to find a cycle, while in other observed models, even with same initial preferences, fewer were necessary. Therefore it is possible that the blocking pairs influence the market and make it harder to find a cycle, hence a kind of stability.

5 Conclusion

We consider the market as stable when during a cycle, because we are able to predict what will happen in the next moment (here represented as an iteration step). We associate this situation with complete or nearly perfect initial information about the market, which allows investors to act only on rational premises and optimization.

On the other hand, we find the market to be in chaos, if the probabilities of each investor to win in a contract are equal, everything is possible, hence the information about the market must be incomplete or vague. That means that no investor has information that will give him an advantage over other investors, reflected in his probability of winning.

The innovative aspect of this paper is use of two-sided matching theory in market simulation. Furthermore, the assumption of general market information being encompassed in changeable investors' preferences which are the only incentive for a decision has not been yet fully explained. The idea of market equilibrium given as a cyclic set of preferences and market instability as a set of equal probabilities for all decision alternatives have not yet been researched, either.

One of the disadvantages of this two models is that there is no way to indicate which information or behaviour influences the investors' preferences. The information is taken as a whole, showing only the general state of the market. Moreover, the simulations are computed ceteris paribus – no other factors than the preferences change. No price factor or market broker's fee is taken into account.

Another problem is the size of the model – the simple version includes only four entrepreneurs and needs to be generalized for an arbitrary number of them. Furthermore, the model does not reflect the type of the instrument being traded, although its type greatly influences the way the contracts are being made, which requires creating submodels.

The advantages, on the other hand, are that the given model includes behavioural aspects of decision-making and different information existing on a market. Also, the model is quite universal for different types of goods and trades and also allows more sides of a contract to be introduced. What is more, the model allows to simulate tones of the market and thus, future decisions that will be made by the investors relying on their market knowledge.

As for the further research, the main idea is to bring the model closer to the reality of market dynamics, to make it possible to predict future markets behaviours. To do so, we need to introduce more trading agents, include trading fee, allow investors to exit the market and let new investors enter it. All of these modifications are possible, but require more complex computation techniques.

We may consider a third party entering the market (e.g. a broker facilitating the conclusion of a contract). In this case Three Sided Matching theory can be used (Biro, McDermid, 2010; Eriksson, Sjostrand, Strimling, 2006). The third party might be a solution to the problem of equal probabilities in some contracts. Preferences of the third party may be represented by a fee level. If the fifth agent enters the market, we may need to introduce the procedure for the possibility of leaving the market because the contracts require an even number of agents.

Even though it is a possible to generalize the model for n agents, a few problems arise. First of all, the tables of preferences for a large number of agents are immense, because for both short and long position they have the size $n \times n$. Second, the more contracts there are, the longer the blocking pairs procedure is. With each two new agents, another step in the blocking pairs procedure is necessary. Also, when facing a possibility of a draw, the more agents tie, the less the model reflects the information about the market.

Appendix

i=0

Preferences

w1:	3	2	4
w2:	3	1	4
w3:	2	4	1
w4:	1	3	2

table of preference					
	1	2	3	4	
1	x	2	3	1	
2	2	x	3	1	
3	1	3	x	2	
4	3	1	2	x	
sr	6	6	8	4	24
p	0,25	0,25	0,33	0,17	

Contracts

Position:	Short		Long	winner	loser
Agent no.	3	vs	4	3	4
p	0,33		0,17		
Agent no.	2	vs	1	1	2
p	0,25		0,25		

i=1

Preferences

w1:	3	2	4
w2:	3	4	1
w3:	2	4	1
w4:	1	2	3

table of preference					
	1	2	3	4	
1	x	2	3	1	
2	1	x	3	2	
3	1	3	x	2	
4	3	2	1	x	
sr	5	7	7	5	24
p	0,21	0,29	0,29	0,21	

Contracts

Position:	Short		Long	winner	loser
Agent no.	4	vs	2	2	4
p	0,21		0,29		
Agent no.	1	vs	3	3	1
p	0,21		0,29		

i=2

Preferences

w1:	2	4	3
w2:	3	4	1
w3:	2	4	1
w4:	1	3	2

table of preference					
	1	2	3	4	
1	x	3	1	2	
2	1	x	3	2	
3	1	3	x	2	
4	3	1	2	x	
sr	5	7	6	6	24
p	0,21	0,29	0,25	0,25	

Contracts

Position:	Short		Long	winner	loser
Agent no.	2	vs	4	2	4
p	0,29		0,25		
Agent no.	3	vs	1	3	1
p	0,25		0,21		

i=4

Preferences

w1:	4	3	2
w2:	3	4	1
w3:	2	1	4
w4:	1	3	2

table of preference					
	1	2	3	4	
1	x	1	2	3	
2	1	x	3	2	
3	2	3	x	1	
4	3	1	2	x	
sum	6	5	7	6	24
prob.	0,25	0,21	0,29	0,25	

Contracts

Position:	Short		Long	winner	loser
Agent no.	3	vs	1	3	1
prob.	0,29		0,25		
Agent no.	2	vs	4	4	2
prob.	0,21		0,25		

i=8

Preferences

w1:	4	3	2
w2:	3	4	1
w3:	2	1	4
w4:	1	3	2

table of preference					
	1	2	3	4	
1	x	1	2	3	
2	1	x	3	2	
3	2	3	x	1	
4	3	1	2	x	
sum	6	5	7	6	24
prob.	0,25	0,21	0,29	0,25	

Contracts

Position:	Short		Long	winner	loser
Agent no.	3	vs	1	3	1
prob.	0,29		0,25		
Agent no.	2	vs	4	4	2
prob.	0,21		0,25		

Figure 1. Tables of preferences and contracts for a simple exchange model, iterations 0-8

i=0			
SHORT	**Preferences**		
w1:	3	2	4
w2:	4	1	3
w3:	2	1	4
w4:	3	1	2

LONG	**Preferences**		
w1:	4	3	2
w2:	3	4	1
w3:	2	4	1
w4:	2	3	1

preference table

short\lon	1		2		3		4		sum
1	0	0	2	1	3	1	1	1	
2	2	1	0	0	1	3	3	3	
3	2	2	3	3	0	0	1	2	
4	2	3	1	2	3	2	0	0	
sum S	3		7		7		7		24
sum L		6		6		7		5	24
probability S	0,1		0,3		0,3		0,3		
probability L		0,25		0,3		0		0,21	

Contracts

Table of revenues for each possible pairing

short\long	1	2	3	4
1	0	3	4	2
2	3	0	4	6
3	4	6	0	3
4	5	3	5	0
max revenue for first pair	short		long	winner
	2		4	2
Sum of revenues				10

Table of revenues for each possible pairing after removing first pair

	1	2	3	4
1	0	0	4	0
2	0	0	0	0
3	4	0	0	0
4	0	0	0	0

remaining max revenue	short	long	winner
	1	3	3

Revenues for choosing first blocking pair

	1	2	3	4
1	0	3	0	2
2	3	0	4	0
3	4	6	0	3
4	5	3	5	0
max revenue for alternative first pair	short	long	winner	
	3	2	3	
Summed revenues of			11	

Revenues for choosing second blocking pair

	1	2	3	4
1	0	0	0	2
2	0	0	0	0
3	0	0	0	0
4	5	0	0	0
Second blocking pair	short	long	winner	
	4	1	1	

Final pairing

contract	1st	2nd
winner	3	1
position	short	long
loser	2	4
position	long	short

Figure 2. Tables of preferences and contracts for a simple exchange model with blocking pairs, iteration 0

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MULTIPLE CRITERIA DECISION MAKING
Vol. 10 2015

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MCDM APPLICATIONS OF NEAR OPTIMAL SOLUTIONS IN DYNAMIC PROGRAMMING

Abstract

One of the methods of scalarization of a multi-criteria problem is the application of a quasi-hierarchy, determined by the decision maker. In discrete problems, to apply this method it is necessary to have an algorithm which generates the optimal solution and the consecutive solutions, contained within the tolerance interval determined by the decision maker. This paper presents algorithms generating the consecutive realizations for a multi-stage deterministic decision-making process as well as an algorithm generating the consecutive strategies for a multi-stage stochastic decision-making process. Algorithms using these solutions in a multi-criteria quasi-hierarchical process are also proposed.

Keywords: multiple objective dynamic programming, quasi-hierarchy, *i*-th process realization, *i*-th strategy, optimality equations.

1 Introduction

In this paper we shall deal with discrete one- and multi-criteria decision-making problems, divided into a finite number of stages. Their characteristic feature is that for each individual stage of the problem, finite sets of feasible states are known, and for each state, the finite set of admissible decisions is also known.

In deterministic processes, the transition from one state to another in the consecutive stages is determined by transition functions, whose arguments are: the state of the process at the beginning of the given stage and the decision made. In stochastic processes we assume that we know the probabilities of the transition, depending on the state of the process at the beginning of the given stage, and of the decision made.

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A realization of the process in the deterministic case consists of a sequence of states and decisions, which transfer the process from an admissible start state to an end state, taking into account the relationships resulting from the transition function. In single-criterion problems we are interested in the optimal realization, that is, a realization maximizing the given multi-stage criterion function. This function is a composition (usually an additive one) of stage criterion functions. In multi-criteria problems we are interested in finding the set of non-dominated process realizations (which is usually very large).

In the stochastic case, a strategy is a function mapping each admissible state to a given decision. In single-criterion problems we are interested in the optimal strategy, that is, a strategy which maximizes the expected value of the multi-stage criterion function. In multi-criteria problems we are interested in finding the set of non-dominated strategies (which, as in the deterministic case, is usually very large).

When solving the problem of finding the optimal strategy of the process, we apply Bellman's optimality principle. Many applications of dynamic programming can be found already in early books in operations research, for instance, in Wager (1975). Multi-criteria decision-making processes were discussed by Trzaskalik, in Trzaskalik (1990, 1998) and in other papers. Extensions and applications for multi-criteria processes can be found, for instance, in Nowak, Trzaskalik (2014, 2013); Trzaskalik, Do Thien Hoa (1999); Trzaskalik, Sitarz (2007, 2009).

While in the single-criterion case usually only one optimal realization of the process exists, in the multi-criteria case the number of non-dominated realizations can be considerable. The search for the set of all efficient realizations can be difficult or even impossible. For that reason, various methods of scalarization of the multi-criteria problem are used.

One of the scalarization methods is the use of a hierarchy of criteria determined by the decision maker. This means that the decision maker is able to formulate a hierarchy of criteria so that the most important criterion is assigned the number 1; the number 2 is reserved for the second-most important criterion, and so on. We assume that all the criteria considered in the problem can be numbered in this way.

We solve the hierarchical problem sequentially. First we find the set of solutions which are optimal with respect to the most important criterion. Out of this set, we select the subset of solutions optimal with respect to the criterion number 2. We continue this procedure until we determine the subset of solutions which are optimal with respect to the least important criterion.

The hierarchical approach has a certain essential shortcoming. It turns out that very often the subset of solutions obtained when an important criterion in the hierarchy is considered has only one element. As a result, the selection of the solution with respect to less important criteria is determined and these criteria do not play an essential role in the process of determining the final solution. For that reason, a quasi-hierarchical approach is often applied. It consists in taking into account, once the (single-criterion) problem has been solved with respect to the most important criterion, not only the best solution, but also those solutions which are close to the optimal solution and contained within the tolerance interval determined in advance by the decision maker. Among the solutions found this way we find the best solution with respect to the second criterion and in the next step we take into account this solution as well as those solutions which are close to the optimal solution and contained within the tolerance interval with respect to the second criterion fixed in advance by the decision maker. This procedure is continued until the least important criterion.

In the application of the quasi-hierarchical procedure the possibility of generating not only the optimal solution, but also near optimal solutions, plays a key role. The solutions considered with respect to the consecutive criteria should be ordered so as to place the optimal solution first, the solution having the second value, second, etc. The ordering of solutions with respect to the first (most important) criterion is of particular importance. The consecutive solutions should be generated as long as they are contained within the tolerance intervals determined by the decision maker.

The problems of generating near optimal solutions in dynamic programming and related fields were taken up already in the past. Elmaghraby (1970) described a solution of the problem of seeking the *k*-th path between two arbitrary nodes in a graph. The search for the consecutive values in the multi-stage deterministic process was described in Trzaskalik (1990). But the problem of generating the consecutive realizations of a process has not been exhaustively described there. The problem of finding near optimal strategies in a decision tree and an application of the algorithm proposed to the quasi-hierarchical approach have been proposed by Nowak (2014), who has observed that the search for near optimal strategies can begin with a strategy differing from the optimal strategy by the decision in one state only. This approach, as applied to multi-criteria stochastic dynamic programming, was developed in Trzaskalik (2015).

The aim of this paper is to describe a method of finding the consecutive solutions in the stochastic and deterministic cases of single-criterion dynamic programming, and to apply this approach to finding solutions of multi-criteria quasi-hierarchical problems.

This paper consists of an introduction, two main sections, final remarks and two appendices. In the second section, which follows the introduction, we will describe deterministic discrete decision-making processes. We will show how to find the consecutive values of the criterion function and to generate the consecutive realizations of a process, on the basis of optimality equations. The algorithm obtained will be used in the quasi-hierarchical procedure proposed. In the third section we will describe stochastic processes. As in the deterministic case, we will show how to find the consecutive expected values of the criteria function and how to generate the consecutive strategies, on the basis of optimality equations. Next, we will present the quasi-hierarchical procedure for the stochastic case. Final remarks conclude the paper. Because of the importance and the degree of complexity of the algorithm generating the *i*-th realization of a process and the *i*-th strategy, complete solutions of these examples are in the appendices.

2 Deterministic case

2.1 *i*-th optimal value and *i*-th process realization

We will use the following notation (Trzaskalik, 1998, 2015):
T – number of stages of the decision process under consideration,
y_t – state of the process at the beginning of stage t ($t = 1,\ldots,T$),
$\mathbf{Y}_t$ – finite set of process states at stage t,
$\mathbf{Y}_{T+1}$ – finite set of process states at the end of the process,
x_t – feasible decision at stage t,
$\mathbf{X}_t(y_t)$ – finite set of decisions feasible at stage t, when the process was in state $y_t \in \mathbf{Y}_t$ at the beginning of this stage,
d_t – process realization in the stage t; we have:

$$d_t = (y_t, x_t) \tag{1}$$

$\mathbf{D}_t$ – set of process realizations in stage t,
$\mathbf{\Omega}_t(y_t, x_t)$ – transition function; we have:

$$y_{t+1} = \mathbf{\Omega}_t(y_t, x_t) \tag{2}$$

d – process realization; we have:

$$d = ((y_1, x_1), (y_2, x_2), \ldots, (y_T, x_T)) \tag{3}$$

where:

$$y_1 \in \mathbf{Y}_1, x_1 \in \mathbf{X}_1(y_1)$$
$$y_2 = \mathbf{\Omega}_1(y_1, x_1)\ x_2 \in \mathbf{X}_2(y_2)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y_T = \mathbf{\Omega}_{T-1}(y_{T-1}, x_{T-1})\ x_T \in \mathbf{X}_T(y_T)$$
$$y_{T+1} = \mathbf{\Omega}_t(y_T, x_T)$$

$\mathbf{D}$ – set of all process realizations,
$d_{\overline{t,T}}(y_t)$ – shortened realization, starting from y_t and encompassing stages from t to T; we have:

$$d_{\overline{t,T}}(y_t) = [(y_t, x_t), (y_{t+1}, x_{t+1}), \ldots, (y_T, x_T)] \tag{4}$$

$\mathbf{D}_{\overline{t,T}}(y_t)$ – set of all shortened realizations, starting from y_t and encompassing stages from t to T,

$F_t(d_t)$ – stage criterion function,

$F(d)$ – criterion function evaluating process realization d; we have:

$$F_t(d) = \sum_{t=1}^{T} F_t(d_t) \tag{5}$$

The finite set $\mathbf{D}$ of process realizations can be divided into M classes in such a way that:

$$\mathbf{D} = \mathbf{D}^1 \cup \mathbf{D}^2 \cup \ldots \mathbf{D}^{\mathrm{M}} \tag{6}$$

where:

$$\mathbf{D}^i \cap \mathbf{D}^j \text{ for } i \neq j \tag{7}$$

$$\forall_{i=1,\ldots,M} \forall_{d^j, d^k \in \mathbf{D}^i} F(d^j) = F(d^k) \tag{8}$$

$$\forall_{i<j} \forall_{\{x^i\} \in \{\mathbf{X}^i\}} \forall_{\{x^j\} \in \{\mathbf{X}^j\}} G\{x^i\} > G\{x^j\} \tag{9}$$

Let $d^1 \in \mathbf{D}^1$, $d^2 \in \mathbf{D}^2$, …, $d^M \in \mathbf{D}^M$ and $F(\mathbf{D})=\{F(d^1), \ldots, F(d^M)\}$. i-th process value is defined as G^i. We have:

$$G^i = F(d^i) \tag{10}$$

Each realization from the set $\mathbf{D}^i$ is named i-th process realization. We will use notation:

$$\max_i F(\mathbf{D}) = F^i(d) \tag{11}$$

The way of determining i-th process value and i-th process realization is described below.

Algorithm 1

1. Starting from $i = 1$ for each $y_T \in \mathbf{Y}_T$ we calculate the i-th value:
$$G_T^i(y_T) = \max_i F_T(y_T, x_T) \tag{12}$$
and find the set of shortened process realizations $\mathbf{D}_{\overline{T,T}}(y_T)$, for which this value is attained.
2. Starting from $i = 1$ for stage t, $t \in \overline{T-1,1}$ and each $y_t \in \mathbf{Y}_t$ we calculate the i-th value:
$$G_t^i(y_t) = \max_{x_t \in \mathbf{X}(y_t)}{}_i \left\{F_t(y_t, x_t) + G_{t+1}^j(\Omega_t(y_t, x_t)) : j = 1, \ldots, i\right\} \tag{13}$$
and find the set of shortened process realizations $\mathbf{D}_t(y_t)$, for which this value is attained.
3. The i-th process value is calculated from the formula:
$$G^i = \max_i \left\{G_1^j(y_1) : j = 1, \ldots, i, y_1 \in \mathbf{Y}_1\right\} \tag{14}$$
4. The set of all i-th process realizations is calculated from the formula:
$$\mathbf{D}^i = \bigcup_{y_1 \in \mathbf{Y}_1} \left\{\mathbf{D}^j(y_1) : G^j(y_1) = G^i, j = 1, \ldots, i\right\} \tag{15}$$

Example 1

We consider a three-stage deterministic decision process. The sets of states for the consecutive stages are as follows:

$$\mathbf{Y}_1 = \{1,2\} \qquad \mathbf{Y}_2 = \{3,4\} \qquad \mathbf{Y}_3 = \{5,6\}$$

We have the following set of final states of the process:

$$\mathbf{Y}_4 = \{7,8\}$$

The sets of feasible decisions are as follows:

$$\mathbf{X}_1(1) = \{A, B\} \qquad \mathbf{X}_2(3) = \{E, F\} \qquad \mathbf{X}_3(5) = \{I, J\}$$
$$\mathbf{X}_1(2) = \{C, D\} \qquad \mathbf{X}_2(4) = \{G, H\} \qquad \mathbf{X}_3(6) = \{K,L\}$$

The graph of the process is given in Figure 1.

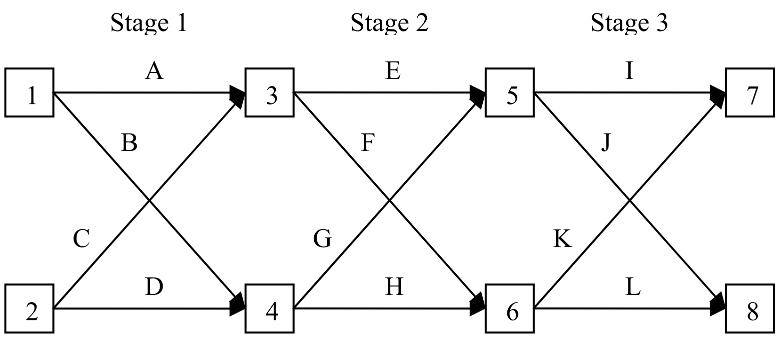

Figure 1. Graph of the process

The values of stage criteria are given in Table 1.

Table 1: Numerical values

Stage	(y_t, x_t)	$F^1(\cdot)$	$F^2(\cdot)$	$F^3(\cdot)$	Stage	(y_t, x_t)	$F_1(\cdot)$	$F_2(\cdot)$	$F_3(\cdot)$
1	(1, A)	6	120	13	2	(4, G)	6	140	16
1	(1, B)	8	110	11	2	(4, H)	4	128	20
1	(2, C)	5	115	14	3	(5, I)	4	102	16
1	(2, D)	9	117	12	3	(5, J)	3	107	15
2	(3, E)	5	132	15	3	(6, K)	5	103	12
2	(3, F)	3	135	14	3	(6, L)	2	101	10

For clarity and due to small size of this illustrative problem, the existing realizations can be written down and numbered from 1 to 16. This numbering is presented in Table 2.

Table 2: List of process realizations

No	Realization	No	Realization	No	Realization	No	Realization
1	(1,A,3,E,5,I)	5	(1,B,4,G,5,I)	9	(2,C,3,E,5,I)	13	(2,D,4,G,5,I)
2	(1,A,3,E,5,J	6	(1,B,4,G,5,I)	10	(2,C,3,E,5,J)	14	(2,D,4,G,5,J)
3	(1,A,3,F,6,K)	7	(1,B,4,H,6,IK	11	(2,C,3,F,6,K)	15	(2,D,4,H,6,K)
4	(1,A,3,F,6,L)	8	(1,B,4,H,6,L)	12	(2,C,3,F,6,L)	16	(2,D,4,H,6,L)

Applying Algorithm 1 for the criterion F^1 we obtain:

$$G^1 = 19,\ \mathbf{D}^1 = \{d^{13}\}$$

$$G^2 = 18,\ \mathbf{D}^2 = \{d^5, d^{14}, d^{15}\}$$

Detailed calculations can be found in Appendix 1.

2.2 MCDM quasi-hierarchical application

We will use the following notation:
K – number of considered criteria,
$F^k(d_t)$ – k-th stage criterion function ($k = 1, \ldots, K$),
$F^k(d)$ − k-th multistage criterion function evaluating process realization d,
ε_k – tolerance limit for k-th multistage criterion function.

We assume that the decision maker in his/her final selection applied the quasi-hierarchical approach. For this reason the criteria have been numbered appropriately, starting with the most important criterion, which is assigned the number 1.

Algorithm 2

1. Using Algorithm 1, find the optimal value $G^1(d)$ for the most important criterion F^1.
2. Ask the decision maker to determine ε^1 for the first criterion.
3. Using Algorithm 1, create the set:
$$\mathbf{D}^{(1)} = \{d \in \mathbf{D}: F^1((d) \geq G^1 - \varepsilon^1\} \tag{16}$$
containing these realizations of the process which are contained within the tolerance interval $[G^1 - \varepsilon^1, G^1]$, determined by the DM for the most important criterion.
4. Set $k = 2$.
5. Determine the optimal realization $d^{(k)}$ in $D^{(k-1)}$, with respect to the k-th criterion:
$$F^k(d^{(k)}) = \max F^k(d){:}\ d \in \mathbf{D}^{(k-1)}\} \tag{17}$$
6. Ask the DM to determine ε^k for the k-th criterion.
7. Create the set of realizations $\mathbf{D}^{(k)}$:
$$\mathbf{D}^{(k)} = \{d \in \mathbf{D}^{(k-1)}: F^k(d) \geq F^k(d^{(k)}) - \varepsilon^1\} \tag{18}$$
8. Set $k = k + 1$.
9. If $k \leq K$, go to Step 5.
10. Ask the DM to select the final realization from $\mathbf{D}^{(K)}$.
11. End of procedure.

The algorithm proposed will be illustrated by a numerical example.

Example 2

Now we regard the considered process as a three-criteria hierarchical process, in which the most important is the first criterion, the second-most important is the second criterion, and the least important is the third criterion. Numerical values of stage criteria are given in Table 1.

The determination of the final process realization using the quasi-hierarchical procedure described in **Algorithm 2** is performed as follows:

1. Using Algorithm 1 find the optimal value $G^1 = 19$ for the most important criterion (see Example 1).
2. Ask the DM to determine ε^1 for the first criterion. The DM set $\varepsilon^1 = 2$.
3. Using Algorithm 1, find the set:
$$\mathbf{D}^{(1)} = \{d \in \mathbf{D}: F^1((d) \geq 17\} = \{d^{13}, d^5, d^{14}, d^{15}, d^6, d^7\}$$
4. Set $k = 2$.
5. Determine the optimal realization in $\mathbf{D}^{(1)}$ with respect to the second criterion. To do this, we calculate:
$$F^2(d^{13}) = 359 \; F^2(d^5) = 352 \; F^2(d^{14}) = 364$$
$$F^2(d^{15}) = 348 \quad F^2(d^6) = 357 \; F^2(d^7) = 341$$
From among the values calculated choose the largest one. We have:
$$F^2(d^{(2)}) = F(d^{14}) = 364$$
6. Ask the DM to determine ε^2 for the second criterion. The DM set $\varepsilon^2 = 8$.
7. Create the set $\mathbf{D}^{(2)}$:
$$\mathbf{D}^{(2)} = \{d \in \mathbf{D}^{(1)}: F^2(d) \geq 356\} = \{d^{14}, d^{13}, d^6\}$$
8. Set $k = 3$.
9. Since $k \leq 3$, go to Step 5.
5. Determine the optimal realization in the set $\mathbf{D}^{(2)}$ with respect to the third criterion. To do this, we calculate:
$$F^3(d^{14}) = 43 \quad F^3(d^{13}) = 44 \; F^3(d^6) = 43$$
6. Ask the DM to determine ε^3 for the third criterion. The DM set $\varepsilon^3 = 1$.
7. Create the set $\mathbf{D}^{(3)}$:
$$\mathbf{D}^{(3)} = \{d \in \mathbf{D}^{(2)}: F^2(d) \geq 44\} = \{d^{14}, d^{13}, d^6\}$$
8. Set $k = 4$.
9. Since $k > 3$, go to Step 10.
10. Suggest the selection of the final realization from $\mathbf{D}^3$ to the DM. This selection can be aided by the values of the multi-stage criteria for the following process realizations:
$$F^1(d^{14}) = 18 \quad F^2(d^{14}) = 364 \; F^3(d^{14}) = 43$$
$$F^1(d^{15}) = 19 \quad F^2(d^{15}) = 359 \; F^3(d^{15}) = 44$$
$$F^1(d^6) = 17 \quad F^2(d^6) = 357 \; F^3(d^6) = 43$$
The DM prefers realization d^{14}.
11. End of procedure.

3 Stochastic case

3.1 *i*-th expected value and *i*-th process strategy

We will use additional notation:

$F_t(y_{t+1}| y_t, x_t)$ – value of stage criterion at stage t for the transition from state y_t to state y_{t+1}, when the decision taken was $x_t \in \mathbf{X}_t(y_t)$,

$P_t(y_{t+1}| y_t, x_t)$ – probability of the transition at stage t from state y_t to state y_{t+1}, when the decision taken was $x_t \in \mathbf{X}_t(y_t)$; the following holds:

$$\forall_{t\in\overline{1,T}} \forall_{y_t \in \mathbf{Y}_t} \forall_{x_t \in \mathbf{X}_t(y_y)} \sum_{y_{t+1}\in \mathbf{Y}_{t+1}} P_t(y_{t+1} \mid y_t, x_t) = 1 \tag{19}$$

$\{x(y_1)\}$ – strategy starting from the state y_1 – a function assigning to y_1 and each state $y_t \in \mathbf{Y}_t$ ($t = 2, \ldots, T$) exactly one decision $x_t \in \mathbf{X}_t(y_t)$,

$\{\mathbf{X}(y_1)\}$ − set of all the strategies $\{x(y_1)\}$,

$\{\mathbf{X}\}$ – the set of all strategies of the process under consideration; we have:

$$\{\mathbf{X}\} = \bigcup_{y_1 \in \mathbf{Y}_1} \{\mathbf{X}(y_1)\} \tag{20}$$

$\{x\} \in \{\mathbf{X}\}$ – a strategy starting from any state $y_1 \in \mathbf{Y}_1$,

$G\{x\}$ – expected value for strategy $\{x\}$:

$$G\{x^*\} = \max_{\{x\}\in\{\mathbf{X}\}} G\{x\} \tag{21}$$

$\{x_{\overline{t,T}}(y_t)\}$ – shortened strategy, starting from y_t and encompassing stages from t to T,

$\{\mathbf{X}_{\overline{t,T}}(y_t)\}$ – set of all shortened strategies, starting from y_t and encompassing stages from t to T.

Let us consider strategy $\{\bar{x}(y_1)\} \in \{\mathbf{X}(y_1)\}$ starting from any state y_1. The expected value for that strategy is calculated as follows:

Algorithm 3

1. For each state $y_T \in \mathbf{Y}_T$ calculate:

$$G_T(y_T, \{\bar{x}_{\overline{T,T}}\}) = \sum_{y_{T+1}\in \mathbf{Y}_{T+1}} F_T(y_{T+1} \mid y_T, \bar{x}_T) P_T(y_{T+1} \mid y_T, \bar{x}_T) \tag{22}$$

2. For each stage t, $t \in \overline{T-1,1}$ calculate the expected value:

$$G_t(y_t, \{\bar{x}_{\overline{t,T}}\}) = \sum_{y_{t+1}\in \mathbf{Y}_{t+1}} (F_t(y_{t+1} \mid y_t, \bar{x}_t) + G_{t+1}(y_{t+1}, \{\bar{x}_{\overline{t+1,T}}\})) P_t(y_{t+1} \mid y_t, \bar{x}_t) \tag{23}$$

The expected value G of the strategy $\{\bar{x}(y_1)\} \in \{\mathbf{X}(y_1)\}$ is equal to $G_1(y_1, \{x_{\overline{1,T}}(y_1)\})$

The finite set of all strategies $\{\mathbf{X}\}$ can be divided into M classes so, that:

$$\{\mathbf{X}\} = \{\mathbf{X}^1\} \cup \{\mathbf{X}^2\} \cup \ldots \cup \{\mathbf{X}^M\} \tag{24}$$

where:

$$\{\mathbf{X}^i\} \cap \{\mathbf{X}^j\} \quad \text{for } i \neq j \tag{25}$$

$$\forall_{i=1,\ldots,M} \forall_{\{x^k\},\{x^l\} \in \{\mathbf{X}^i\}} G\{x^k\} = G\{x^l\} \tag{26}$$

$$\forall_{i<j} \forall_{\{x^i\} \in \{\mathbf{X}^i\}} \forall_{\{x^j\} \in \{\mathbf{X}^j\}} G\{x^i\} > G\{x^j\} \tag{27}$$

$$G^i = F(d^i)$$

Let $\{x^1\} \in \{\mathbf{X}^1\}$, $\{x^2\} \in \{\mathbf{X}^2\}$, …, $\{x^M\} \in \{\mathbf{X}^M\}$ and $G\{\mathbf{X}\}=\{G\{x^1\}, \ldots, G\{x^M\}\}$. The i-th expected value is defined as G^i. We have:

$$G^i = G\{x^i\} \tag{28}$$

Each strategy from the set $\{\mathbf{X}^i\}$ is called an i-th strategy.

The method of determining the i-th expected value and the i-th optimal strategy is described below.

Algorithm 4

1. Starting from $i = 1$ for each $y_T \in \mathbf{Y}_T$ calculate the i-th expected value:
$$G_T^i(y_T) = \max_{x_T \in \mathbf{X}_T(y_T)}{}_i \sum_{y_{T+1} \in \mathbf{Y}_{T+1}} F_T(y_{T+1} \mid y_T, x_T) \cdot P_T(y_{T+1} \mid y_T, x_T) \tag{29}$$
and find the set of shortened strategies $\{\mathbf{X}_{\overline{T,T}}(y_T)\}$, for which this value is reached.
2. Starting from $i = 1$ for stage t, $t \in \overline{T-1,1}$ and each $y_t \in \mathbf{Y}_t$ calculate the i-th expected value:
$$G_t^i(y_t) = \max_{x_t \in \mathbf{X}_t(y_t)}{}_i \left\{ \sum_{y_{T+1} \in \mathbf{Y}_{T+1}} \left[F_t(y_{t+1} \mid y_t, x_t) + G_{t+1}^j(y_{t+1})\right] \cdot P_T(y_{T+1} \mid y_T, x_T) : j = 1, \ldots, i \right\} \tag{30}$$
and find the set of shortened strategies $\{\mathbf{X}_{\overline{t,T}}(y_t)\}$, for which this value is reached.
3. The i-th process value is calculated from the formula:
$$G^i = \max{}_i \left\{ G_1^j(y_1) : j = 1, \ldots, i, y_1 \in \mathbf{Y}_1 \right\} \tag{31}$$
4. The set of all i-th strategies is calculated from the formula:
$$\left\{\mathbf{X}^i\right\} = \bigcup_{y_1 \in \mathbf{Y}_1} \left\{ \left\{ \mathbf{X}^j(y_1) \right\} : G^j(y_1) = G^i, j = 1, \ldots, i \right\} \tag{32}$$

Example 3

We consider a three-stage stochastic decision process. The sets of states for the consecutive stages are as follows:

$$\mathbf{Y}_1 = \{1,2\} \qquad \mathbf{Y}_2 = \{3,4\} \qquad \mathbf{Y}_3 = \{5,6\}$$

We have the following set of final states of the process:

$$\mathbf{Y}_4 = \{7,8\}$$

The sets of feasible decisions are as follows:

$$\mathbf{X}_1(1) = \{A, B\} \quad \mathbf{X}_2(3) = \{E, F\} \quad \mathbf{X}_3(5) = \{I, J\}$$
$$\mathbf{X}_1(2) = \{C, D\} \quad \mathbf{X}_2(4) = \{G, H\} \quad \mathbf{X}_3(6) = \{K,L\}$$

The graph of the process is given in Figure 2. Rectangles denote states of the process in the consecutive stages, circles − random nodes.

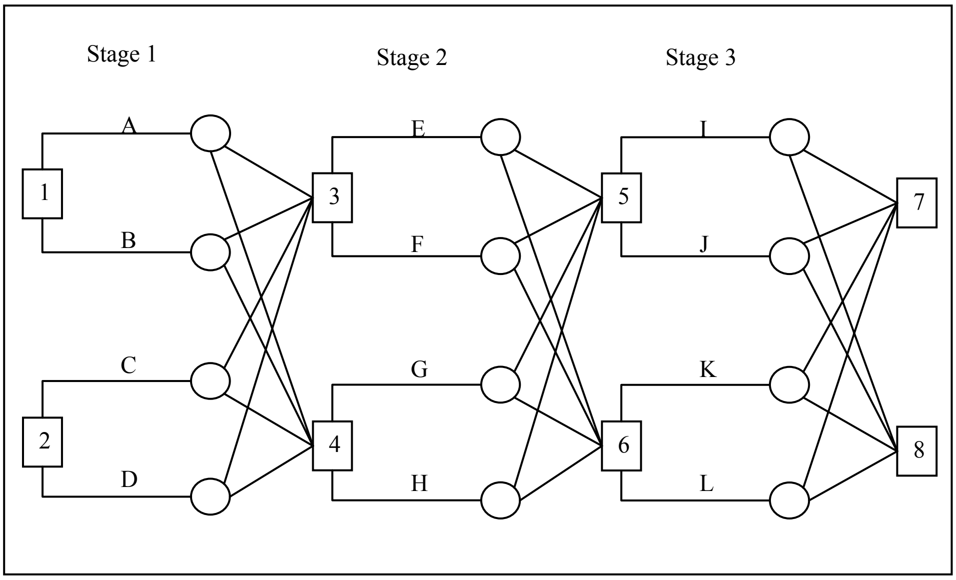

Figure 2. Graph of the process

The possible stage realizations of the process, probabilities of their occurrence, as well as the values of the stage criteria functions are shown in Table 3.

Table 3: Numerical values

Stage	$(y_{t+1}\vert y_t, x_t)$	$P(\cdot)$	$F^1(\cdot)$	$F^2(\cdot)$	$F^3(\cdot)$	Stage	$(y_{t+1}\vert y_t, x_t)$	$P(\cdot)$	$F^1(\cdot)$	$F^2(\cdot)$	$F^3(\cdot)$
1	(3| 1,A)	0.4	6	15	22	2	(5| 4,G)	0.6	5	15	20
1	(4| 1,A)	0.6	8	17	14	2	(6| 4,G)	0.4	6	18	13
1	(3| 1,B)	0.7	6	15	22	2	(5| 4,H)	0.8	5	15	20
1	(4| 1,B)	0.3	8	17	14	2	(6| 4,H)	0.2	6	18	13
1	(3| 2,C)	0.5	6	15	22	3	(7| 5,I)	0.8	5	30	12
1	(4| 2,C)	0.5	8	17	14	3	(8| 5,I)	0.2	1	12	15
1	(3| 2,D)	0.8	6	15	22	3	(7| 5,J)	0.3	5	30	12
1	(4| 2,D)	0.2	8	17	14	3	(8| 5,J)	0.7	1	12	15
2	(5| 3,E)	0.5	5	15	20	3	(7| 6,K)	0.2	5	30	12
2	(6| 3,E)	0.5	6	18	13	3	(8| 6,K)	0.8	1	12	15
2	(5| 3,F)	0.3	5	15	20	3	(7| 6,L)	0.9	5	30	12
2	(6| 3,F)	0.7	6	18	13	3	(8| 6,L)	0.1	1	12	15

For clarity and due to small size of this illustrative problem, the existing strategies can be written down and numbered from 1 to 64. This numbering is presented in Table 4.

Table 4: List of strategies

No	Decision	No	Decision	No	Decision	No	Decision
1	(A,_,E,G,I,K)	17	(B,_,E,G,I,K)	33	(_,C,E,G,I,K)	49	(_,D,E,G,I,K)
2	(A,_,E,G,I,L)	18	(B,_,E,G,I,L)	34	(_,C,E,G,I,K)	50	(_,D,E,G,I,L)
3	(A,_,E,G,J,K)	19	(B,_,E,G,J,K)	35	(_,C,E,G,J,K)	51	(_,D,E,G,J,K)
4	(A,_,E,G,J,L)	20	(B,_,E,G,J,L)	36	(_,C,E,G,J,L)	52	(_,D,E,G,J,L)
5	(A,_,E,H,I,K)	21	(B,_,E,H,I,K)	37	(_,C,E,H,I,K)	53	(_,D,E,H,I,K)
6	(A,_,E,H,I,L)	22	(B,_,E,H,I,L)	38	(_,C,E,H,I,L)	54	(_,D,E,H,I,L)
7	(A,_,E,H,J,K)	23	(B,_,E,H.J,K)	39	(_,C,E,H,J,K)	55	(_,D,E,H,J,K)
8	(A,_,E,H,J,L)	24	(B,_,E,H,J,L)	40	(_,C,E,H,J,L)	56	(_,D,E,H,J,L)
9	(A,_,F,G,I,K)	25	(B,_,F,G,I,K)	41	(_,C,F,G,I,K)	57	(_,D,F,G,I,K)
10	(A,_,F,G,I,L)	26	(B,_,F,G,I,L)	42	(_,C,F,G,I,L)	58	(_,D,F,G,I,L)
11	(A,_,F,G,J,K)	27	(B,_,F,G,J,K)	43	(_,C,F,G,J,K)	59	(_,D,F,G,J,K)
12	(A,_,F,G,J,L)	28	(B,_,F,G,J,L)	44	(_,C,F,G,J,L)	60	(_,D,F,G,J,L)
13	(A,_,F,H,I,K)	29	(B,_,F,H,I,K)	45	(_,C,F,H,I,K)	61	(_,D,F,H,I,K)
14	(A,_,F,H,I,L)	30	(B,_,F,H,I,L)	46	(_,C,F,H,I,L)	62	(_,D,F,H,I,L)
15	(A,_,F,H,J,K)	31	(A,D,F,H,J,K)	47	(_,C,F,H,J,K)	63	(_,D,F,H,J,K)
16	(A,_,F,H,J,L)	32	(A,D,F,H,J,L)	48	(_,C,F,H,J,L)	64	(_,D,F,H,J,L)

Applying Algorithm 3 for the criterion F^1 we obtain:

$$G^1 = 17.128, \quad \{\mathbf{X}^1\} = \{x^{10}\}$$

$$G^2 = 17.016, \quad \{\mathbf{X}^2\} = \{x^2\}$$

Detailed calculations can be found in Appendix 2.

3.2 MCDM quasi-hierarchical application

We assume again that the decision maker, in his/her final selection, applies the quasi-hierarchical approach. For this reason the criteria have been numbered appropriately, starting with the most important one, which is assigned the number 1.

Algorithm 5

1. Using Algorithm 4 find the expected optimal value G^1 for the most important criterion F^1.
2. Ask the DM to determine ε^1 for the first criterion.
3. Using Algorithm 1, create the set:

$$\{\mathbf{X}^{(1)}\} = \{\{x\} \in \{\mathbf{X}\}: G^1\{x\} \geq G^1 - \varepsilon^1\} \tag{33}$$

   which contains, for the most important criterion (number 1) and for each initial state $y_1 \in \mathbf{Y}_1$, the strategies which are contained within the tolerance interval $[G^1 - \varepsilon^1, G^1]$, given by the DM.

4. Set $k = 2$.
5. Determine strategy $\{x^{(k)}\}$ in the set $\{\mathbf{X}^{(k-1)}\}$ which is optimal with respect to the k-th criterion:
$$G^k\left\{x^{(k)}\right\} = \max\left\{G^k\left\{x^k\right\}:\{x\} \in \left\{\mathbf{X}^{(k-1)}\right\}\right\} \tag{34}$$
6. Ask the DM to determine ε^k for the k-th criterion.
7. Create the set of strategies $\{\mathbf{X}^{(k)}\}$:
$$\{\mathbf{X}^{(k)}\} = \{\{x\} \in \{\mathbf{X}^{(k-1)}: G^k\{x\} \geq G^k\{x^{(k)}\} - \varepsilon^k\} \tag{35}$$
8. Set $k = k + 1$.
9. If $k \leq K$, go to Step 5.
10. Ask the DM to select a strategy from the set $\{\mathbf{X}^{(K)}\}$.
11. End of procedure.

The algorithm proposed will be illustrated by a numerical example.

Example 4

Now we regard the considered process as a three-criteria hierarchical process, in which the most important is the first criterion, the second-most important is the second criterion, and the least important is the third criterion. Numerical values of stage criteria are given in Table 1.

The determination of the final strategy using the quasi-hierarchical procedure described in **Algorithm 5** is performed as follows:

1. Using Algorithm 4 find the expected optimal value $G^1 = 17.128$ (see Example 3) for the most important criterion.
2. Ask the DM to determine ε^1 for the first criterion. The DM set $\varepsilon^1 = 0.342$.
3. Using Algorithm 3, find the set:
$$\{\mathbf{X}^{(1)}\} = \{\{x\} \in \{\mathbf{X}\}: G^1\{x\} \geq 16.585\} =$$
$$= \{\{x^{10}\}, \{x^2\}, \{x^{42}\}, \{x^{14}\}, \{x^6\}, \{x^{34}\}, \{x^{46}\}\}$$
4. Set $k = 2$.
5. Determine the strategy $\{x^{(2)}\}$ in $\{\mathbf{X}^{(1)}\}$ which is optimal with respect to the second criterion. To do this, we calculate:
$G^2\{x^{10}\} = 48.94$, $G^2\{x^2\} = 48.376$, $G^2\{x^{42}\} = 55.104$, $G^2\{x^{14}\} = 49.08$
$G^2\{x^6\} = 48.568$, $G^2\{x^{34}\} = 54.168$, $G^2\{x^{46}\} = 56.08$
From among the values found select the largest one. We have:
$$G^2\{x^{(2)}\} = G^2\{x^{46}\} = 56.08$$
6. Ask the DM to determine ε^2 for the second criterion. The DM set $\varepsilon^2 = 5$.
7. Create the set of strategies $\{\mathbf{X}^{(2)}\}$:
$$\{\mathbf{X}^{(2)}\} = \{\{x\} \in \{\mathbf{X}^{(1)}\} : G^2\{x\} \geq 51.08\} = \{\{x^{42}\}, \{x^{34}\}, \{x^{46}\}\}$$
8. Set $k = 3$.
9. Since $k \leq 3$, go to Step 5.

5. Determine the strategy $\{x^{(3)}\}$ in $\{\mathbf{X}^{(2)}\}$ which is optimal with respect to the third criterion. To do this, calculate:
$$G^3\{x^{42}\} = 46{,}585,\ \ G^3\{x^{34}\} = 47.315,\ \ G^3\{x^{46}\} = 47.315$$
From among the values found select the largest one. We have:
$$G^3\{x^{(3)}\} = G^2\{x^{34}\} = G^2\{x^{46}\} = 47.315$$
6. Ask the DM to determine ε^3 for the third criterion. The DM set $\varepsilon^3 = 1$.
7. Create the set of strategies $\{\mathbf{X}^{(3)}\}$:
$$\{\mathbf{X}^{(3)}\} = \{\{x\}\in\{\mathbf{X}^{(2)}\} : G^3\{x\} \geq 46{,}315\} = \{\{x^{42}\}, \{x^{34}\}, \{x^{46}\}\}$$
8. Set $k = 4$.
9. Since $k > 3$, go to Step 10.
10. Suggest to the DM the selection of the final strategy from $\{\mathbf{X}^{(3)}\}$. This selection can be aided by the expected values of the multi-stage criteria which are:
$$G^1\{x^{42}\} = 16.97,\quad G^2\{x^{42}\} = 55.104,\ \ G^3\{x^{42}\} = 46{,}585$$
$$G^1\{x^{34}\} = 16.83,\quad G^2\{x^{34}\} = 54.168,\ \ G^3\{x^{34}\} = 47.315$$
$$G^1\{x^{46}\} = 16.83,\quad G^2\{x^{46}\} = 56.08,\quad G^3\{x^{46}\} = 47.315$$
The DM prefers strategy $\{x^{46}\}$.
11. End of procedure.

4 Final remarks

The algorithms presented in this paper, generating the *i*-th realization of a process in the deterministic case and the *i*-th strategy in the stochastic case have both advantages and disadvantages. An advantage of both is that they make it possible to generate the consecutive realizations and strategies, respectively. The decision maker can determine whether the number of the solutions generated is appropriate with regard to the given tolerance interval. If this number is too small or too large, the decision maker can increase or decrease this interval, respectively.

One can also observe certain disadvantages of the quasi-hierarchical approach. The first one is the increasing complexity of the generation of the consecutive solutions and the need for more resource-intensive calculations. The second one is more general and concerns the quasi-hierarchical procedure. An important assumption in all scalarization procedures is that the final solution obtained should be an efficient solution. The quasi-hierarchical procedure does not guarantee this. In the deterministic case it is possible to test the efficiency of the solution obtained and, if this solution is not efficient, to generate efficient solutions better than the solution tested. For the stochastic case, such a procedure has not yet been worked out, which suggest a direction for future research.

Acknowledgements

This research was supported by National Science Center, decision No. DEC 2013/11/B/HS4/01471.

Appendix 1

Stage $T = 3$

According to formula (12) we obtain:

State $y_3 = 6$

$G_3^1(6) = \max_1 \{F_3^1(6,\text{K}), F_3^1(6,\text{L})\} = \max_1 \{5,2\} = 5$ $\mathbf{D}_{33}^1(6) = (6, \text{K})$

$G_3^2(6) = \max_1 \{F_3^1(6,\text{K}), F_3^1(6,\text{L})\} = \max_1 \{5,2\} = 2$ $\mathbf{D}_{33}^2(6) = (6, \text{L})$

State $y_3 = 5$

$G_3^1(5) = \max_1 \{F_3^1(5,\text{I}), F_3^1(5,\text{J})\} = \max_1 (4, 3\} = 4$ $\mathbf{D}_{33}^1(5) = (5, \text{I})$

$G_3^1(5) = \max_2 \{F_3^1(5,\text{I}), F_3^1(5,\text{J})\} = \max_2 (4, 3\} = 3$ $\mathbf{D}_{33}^2(5) = (5, \text{J})$

Stage $t = 2$

According to formula (13) we obtain:

State $y_2 = 4$

$G_2^1(4) = \max_1\{[F_2^1(4,\text{G}) + G_3^1(5)], F_2^1(4,\text{H}) + G_3^1(6)] =$
$= \max_1 \{6 + 4, 4 + 5\} = 10$ $\mathbf{D}_{2,3}^1(4) = [(4,\text{G}),(5,\text{I})]$

$G_2^2(4) = \max_2 \{F_2^1(4, \text{G}) + G_3^j(5), F_2^1(4, \text{H}) + G_3^k(6): j, k = 1, 2\} =$
$= \max_2\{[F_2^1(4,\text{G}) + G_3^1(5)], [F_2^1(4,\text{G}) + G_3^2(5)], [F_2^1(4,\text{H}) + G_3^1(6)], F_2^1(4,\text{H}) + G_3^2(6)]\} =$
$= \max_2\{6 + 4, 6 + 3, 4 + 5, 4 + 2\} = \max_2\{10, 9, 9, 6\} = 9$
$\mathbf{D}_{2,3}^2(4) = \{[(4,\text{G}), (5,\text{J})], [(4,\text{H}), (6,\text{K})]\}$

State $y_2 = 3$

$G_2^1(3) = \max_1\{[F_2^1(3,\text{E}) + G_3^1(5)], F_2^1(4,\text{F}) + G_3^1(6)] = \max_1 \{5 + 4, 3 + 4\} = \max_1 \{9, 7\} = 9$
$\mathbf{D}_{2,3}^1(3) = [(3, \text{E}), (5,\text{I})]$

$G_2^2(3) = \max_2 \{F_2^1(3, \text{E}) + G_3^j(5), F_2^1(4, \text{H}) + G_3^k(6): j, k = 1, 2\} =$
$= \max_2\{[F_2^1(3,\text{E}) + G_3^1(5)], [F_2^1(3,\text{E}) + G_3^2(5)], [F_2^1(3,\text{F}) + G_3^1(6)], F_2^1(3,\text{F}) + G_3^2(6)]\} =$
$= \max_2 \{5 + 4, 5 + 3, 3 + 5, 3 + 2\} = \max_2\{9, 8, 8, 7\}$
$\mathbf{D}_{2,3}^2(3) = \{[(3,\text{E}), (5,\text{J})], [(3,\text{F}), (6,\text{K})]\}$

Stage 1

According to formula (13) we obtain:

State $y_1 = 2$

$G_2^1(2) = \max_1\{[F_2^1(2,\text{C}) + G_3^1(3)], F_2^1(2,\text{D}) + G_3^1(4)] = \max_1 \{5 + 9, 9 + 10\} =$
$= \max_1 \{14, 19\} = 19$
$\mathbf{D}_{1,3}^1(2) = [(2,\text{D}),(3, \text{E}), (5,\text{I})]$

$G_2^2(2) = \max_2 \{F_2^1(2, \text{C}) + G_3^j(3), F_2^1(2, \text{D}) + G_3^k(4): j, k = 1, 2\} =$
$= \max_2 \{[F_1^1(2,\text{C}) + G_2^1(3)], [F_1^1(2,\text{C}) + G_2^2(3)], [F_1^1(2,\text{D}) + G_2^1(4)], F_1^1(2,4) + G_2^2(4)]\} =$
$= \max_2 \{5+9, 5+8, 9+10, 9+9\} = \max_2\{14, 13, 19, 18\} = 18$
$\mathbf{D}_{2,3}^2(2) = \{[(2,\text{D}), (4,\text{G}), (5,\text{J})], [(2,\text{D}), (4,\text{H}), (6,\text{K})]\}$

State $y_1 = 1$

$G_2^1(1) = \max_1\{[F_1^1(1,A) + G_2^1(3)], F_1^1(1,B) + G_2^1(4)] = \max_1 \{6 + 9, 8 + 10\} =$
$= \max_1 \{15, 18\} = 18$

$\mathbf{D}_{1,3}^1(1) = [(1,B),(4, G), (5,I)]$

$G_2^2(1) = \max_2 \{F_1^1(1, A) + G_2^j(3), F_1^1(1, B) + G_2^k(4): j, k = 1, 2\} =$
$= \max_2 \{[F_1^1(2,C) + G_2^1(3)], [F_1^1(2,C) + G_2^2(3)], [F_1^1(2,D) + G_2^1(4)], F_1^1(2,4) + G_2^2(4)]\} =$
$= \max_2 \{6 + 9, 6 + 8, 8 + 10, 8 + 9\} = \max_2\{15, 14, 18, 17\} = 17$

$\mathbf{D}_{1,3}^2(1) = \{[(1,B), (4,G), (5,J)], [(1,B), (4,H), (6,K)]\}$

1st process value and 1st process realization:
$G_1^1 = \max_1 \{G_1^1(1), G_1^1(2)\} = \max_1\{18, 19\} = 19$
We have: $x_1^* = 1$ and $d^1 = d_{1,3}^1(2) = [(2,D),(3, E), (5,I)]$.

2nd process value and 2nd process realization:
$G_1^2 = \max_2 \{G_1^1(1), G_1^2(1), G_1^1(2), G_1^2(2)\} = \max_2\{19, 18, 18, 17\} = 18$
$\mathbf{D}^2 = \{\mathbf{D}_{1,3}^1(1), \mathbf{D}_{2,3}^2(2)\} = \{[(2,D), (4,G), (5,J)], [(2,D), (4,H), (6,K)], [(1,B),(4, G), (5,I)]\}$

Appendix 2

Stage T = 3

According to formula (29) we obtain:

State $y_3 = 6$

$G_3^1(6) = \max_1 \{F_3^1(7|6,K)\cdot P_3(7|6,K) + F_3^1(8|6,K)\cdot P_3(8|6,K),$
$F_3^1(7|6,L)\cdot P_3(7|6,L) + F_3^1(8|6,L)\cdot P_3(8|6,L)\} =$
$= \max_1 \cdot\{(5\cdot 0.2 + 1\cdot 0.8), (5\cdot 0.9 + 1\cdot 0.1)\} = \max_1 \{1.8, 4.6\} = 4.6$

$\{X_{3,3}^1(6)\} = \{\_, L\}$

$G_3^2(6) = \max_2 \{F_3^1(7|6,K)\cdot P_3(7|6,L) + F_3^1(8|6,K)\cdot P_3(8|6,K),$
$F_3^1(7|6,L)\cdot P_3(7|6,L) + F_3^1(8|6,L)\cdot P_3(8|6,L)\} =$
$= \max_2 \{(5\cdot 0.2 + 1\cdot 0.8), (5\cdot 0.9 + 1\cdot 0.1)\} = \max_2 \{1.8, 4.6\} = 4.6$

$\{X_{3,3}^2(6)\} = \{\_, K\}$

State $y_3 = 5$

$G_3^1(5) = \max_1 \{F_3^1(7|5,I)\cdot P_3(7|5,I) + F_3^1(8|5,I)\cdot P_3(8|5,I),$
$F_3^1(7|5,J)\cdot P_3(7|5,J) + F_3^1(8|5,J)\cdot P_3(8|5,J)\} =$
$= \max_2 \cdot\{(5\cdot 0.8 + 1\cdot 0.2), (5\cdot 0.3 + 1\cdot 0.7)\} = \max_2 \{4.2, 2.2\} = 4.2$

$\{X_{3,3}^1(5)\} = \{I, \_\}$

$G_3^2(5) = \max_2 \{F_3^1(7|5,I)\cdot P_3(7|5,I) + F_3^1(8|5,I)\cdot P_3(8|5,I),$
$F_3^1(7|5,J)\cdot P_3(7|5,J) + F_3^1(8|5,J)\cdot P_3(8|5,J)\} =$
$= \max_2 \cdot\{(5\cdot 0.8 + 1\cdot 0.2), (5\cdot 0.3 + 1\cdot 0.7)\} = \max_2 \{4.2, 2.2\} = 2.2$

$\{X_{3,3}^2(6)\} = \{J, \_\}$

Stage $t = 2$

According to formula (30) we obtain:

State $y_2 = 4$

$G_2^1(4) = \max_1 \{[F_2^1(5|4,G) + G_3^1(5)]\cdot P_2(5|4,G) + [F_2^1(6|4,G) + G_3^1(6)]\cdot P_2(6|4,G),$
$F_2^1(5|4,H) + G_3^1(5)]\cdot P_2(5|4,H) + [F_2^1(6|4,H) + G_3^1(5)]\cdot P_2(6|4,H)\} =$
$= \max_1 \{(5 + 4.2)\cdot 0.6 + (6 + 4.6)\cdot 0.4, (5 + 4.2)\cdot 0.8 + (6 + 4.6)\cdot 0.2\} = \max_1 \{9.76, 9.48\} = 9.76$

$\{X_{2,3}^1(4)\} = \{\_,G, I, L\}$

$G_2^2(4) = \max_2\{[F_2^1(5|4,G) + G_3^1(5)]\cdot P_2(5|4,G) + [F_2^1(6|4,G) + G_3^1(5)]\cdot P_2(5|4,G),$
$[F_2^1(5|4,G) + G_3^2(5)]\cdot P_2(5|4,E) + [F_2^1(6|4,G) + G_3^1(5)]\cdot P_2(5|4,G),$
$[F_2^1(5|4,G) + G_3^1(5)]\cdot P_2(5|4,E) + [F_2^1(6|4,G) + G_3^2(5)]\cdot P_2(5|4,G),$
$[F_2^1(5|4,G) + G_3^2(5)]\cdot P_2(5|4,E) + [F_2^1(6|4,G) + G_3^2(5)]\cdot P_2(5|4,G),$
$[F_2^1(5|4,H) + G_3^2(5)]\cdot P_2(5|4,H) + [F_2^1(6|4,H) + G_3^1(5)]\cdot P_2(5|4,H),$
$[F_2^1(5|4,H) + G_3^1(5)]\cdot P_2(5|4,H) + [F_2^1(6|4,H) + G_3^1(5)]\cdot P_2(5|4,H),$
$[F_2^1(5|4,H) + G_3^2(5)]\cdot P_2(5|4,H) + [F_2^1(6|4,H) + G_3^2(5)]\cdot P_2(5|4,H),$
$[F_2^1(5|4,H) + G_3^1(5)]\cdot P_2(5|4,H) + [F_2^1(6|4,H) + G_3^2(5)]\cdot P_2(5|4,H)\} =$
$= \max_2\{(5 + 4.2)\cdot 0.6 + (6 + 4.6)\cdot 0.4, (5 + 2.2)\cdot 0.6 + (6 + 4.6)\cdot 0.4,$
$(5 + 4.2)\cdot 0.6 + (6 + 1.8)\cdot 0.4, (5 + 2.2)\cdot 0.6 + (6 + 1.8)\cdot 0.4,$
$(5 + 4.2)\cdot 0.8 + (6 + 4.6)\cdot 0.2, (5 + 2.2)\cdot 0.8 + (6 + 4.6)\cdot 0.2,$
$(5 + 4.2)\cdot 0.8 + (6 + 1.8)\cdot 0.2, (5 + 2.2)\cdot 0.8 + (6 + 1.8)\cdot 0.2 =$
$= \max_2\{9.76, 8.56, 7.64, 7.44, 9.48, 7.88, 8.76, 7.32\} = 9.48$

$\{X_{2,3}^2(4)\} = \{\_,H, I, L\}$

State $y_2 = 3$

$G_2^1(3) = \max_1\{[F_2^1(5|3,E) + G_3^1(5)]P_2(5|3,E) + [F_2^1(6|3,E) + G_3^1(5)]\cdot P_2(5|3,E),$
$[F_2^1(5|3,F) + G_3^1(5)]\cdot P_2(5|3,F) + [F_2^1(6|3,F) + G_3^1(5)]\cdot P_2(5|3,F)\} =$
$= \max_1\{(5 + 4.2)\cdot 0.5 + (6 + 4.6)\cdot 0.5, (5 + 4.2)\cdot 0.3 + (6 + 4.6)\cdot 0.7\} = \max_1\{9.9, 10.18\} = 10.18$

$\{X_{2,3}^1(3)\} = \{F, \_, I, L\}$

$G_3^2(3) = \max_2\{[F_2^1(5|3,E) + G_3^1(5)]\cdot P_2(5|3,E) + [F_2^1(5|4,E) + G_3^1(5)]\cdot P_2(5|4,E),$
$[F_2^1(5|3,E) + G_3^2(5)]\cdot P_2(5|3,E) + [F_2^1(5|4,E) + G_3^1(5)]\cdot P_2(5|4,E),$
$[F_2^1(5|3,E) + G_3^1(5)]\cdot P_2(5|3,E) + [F_2^1(5|4,E) + G_3^2(5)]\cdot P_2(5|4,E),$
$[F_2^1(5|3,E) + G_3^2(5)]\cdot P_2(5|3,E) + [F_2^1(5|4,E) + G_3^2(5)]\cdot P_2(5|4,E),$
$[F_2^1(5|3,F) + G_3^2(5)]\cdot P_2(5|3,F) + [F_2^1(5|4,F) + G_3^1(5)]\cdot P_2(5|4,F),$
$[F_2^1(5|3,F) + G_3^1(5)]\cdot P_2(5|3,F) + [F_2^1(5|4,F) + G_3^1(5)]\cdot P_2(5|4,F),$
$[F_2^1(5|3,F) + G_3^2(5)]\cdot P_2(5|3,F) + [F_2^1(5|4,F) + G_3^2(5)]\cdot P_2(5|4,F),$
$[F_2^1(5|3,F) + G_3^1(5)]\cdot P_2(5|3,F) + [F_2^1(5|4,F) + G_3^2(5)]\cdot P_2(5|4,F)\}=$
$= \max_2\{(5 + 4.2)\cdot 0.5 + (6 + 4.6)\cdot 0.5, (5 + 2.2)\cdot 0.5 + (6 + 4.6)\cdot 0.5,$
$(5 + 4.2)\cdot 0.5 + (6 + 1.8)\cdot 0.5, (5 + 2.2)\cdot 0.5 + (6 + 1.8)\cdot 0.5,$
$(5 + 4.2)\cdot 0.3 + (6 + 4.6)\cdot 0.7, (5 + 2.2)\cdot 0.3 + (6 + 4.6)\cdot 0.7,$
$(5 + 4.2)\cdot 0.3 + (6 + 1.8)\cdot 0.7, (5 + 2.2)\cdot 0.3 + (6 + 1.8)\cdot 0.7 =$
$= \max_2\{9.9, 8.93, 8.5, 7.5, 10.18, 9.58, 8.22, 7.62\} = 9.9$

$\{X_{2,3}^2(3)\} = \{E, \_, I, L\}$

Stage 1

According to formula (30) we obtain:

State $y_1 = 2$

$G_2^1(2) = \max_1\{[F_1^1(3|2,C)\cdot P_3(3|2,C) + G_3^1(3)]\cdot P_2(3|2,C) + [F_2^1(4|2,C)\cdot P_3(4|2,C) + G_3^1(4)]\cdot P_2(4|2,C),$
$[F_2^1(5|2,D)\cdot P_3(5|2,D)+G_3^1(5)]\cdot P_2(5|2,D) + [F_2^1(6|2,D)\cdot P_3(5|2,D) + G_3^1(5)]\cdot P_2(5|2,D)\} =$
$= \max_1\{(6 + 10.18)\cdot 0.5 + (8 + 9.76)\cdot 0.5, (6 + 10.18)\cdot 0.8 + (8 + 9.76)\cdot 0.2\} =$
$= \max_1\{16.97, 16.496\} = 16.97$

$\{X_{1,3}^1(2)\} = \{\_,C, F, G, I, L\}$

$G_1^2(2) = \max_2\{[F_1^1(3|2,C) + G_2^1(3)]\cdot P_1(3|2,C) + [F_1^1(4|2,C) + G_2^1(4)]\cdot P_1(4|4,C),$
$[F_1^1(3|2,C) + G_2^2(3)]\cdot P_1(3|2,C) + [F_1^1(4|2,C) + G_2^1(4)]\cdot P_1(4|4,C),$
$[F_1^1(3|2,C) + G_2^1(3)]\cdot P_1(3|2,C) + [F_1^1(4|2,C) + G_2^1(4)]\cdot P_1(4|4,C),$
$[F_1^1(3|2,C) + G_2^2(3)]\cdot P_1(3|2,C) + [F_1^1(4|2,C) + G_2^1(4)]\cdot P_1(4|4,C),$
$[F_1^1(3|2,D) + G_2^1(3)]\cdot P_1(3|2,D) + [F_1^1(4|2,D) + G_2^1(4)]\cdot P_1(4|4,D),$
$[F_1^1(3|2,D) + G_2^2(3)]\cdot P_1(3|2,D) + [F_1^1(4|2,D) + G_2^1(4)]\cdot P_1(4|4,D),$
$[F_1^1(3|2,D) + G_2^1(3)]\cdot P_1(3|2,D) + [F_1^1(4|2,D) + G_2^1(4)]\cdot P_1(4|4,D),$
$[F_1^1(3|2,D) + G_2^2(3)]\cdot P_1(3|2,D) + [F_1^1(4|2,D) + G_2^1(4)]\cdot P_1(4|4,D)\} =$
$= \max_2\{(6 + 10.18)\cdot 0.5 + (8 + 9.76)\cdot 0.5, (6 + 9.9)\cdot 0.5 + (8 + 9.76)\cdot 0.5,$
$(6 + 10.18)\cdot 0.5 + (8 + 9.48)\cdot 0.5, (6 + 9.9)\cdot 0.5 + (8 + 9.48)\cdot 0.5,$
$(6 + 10.18)\cdot 0.8 + (8 + 9.76)\cdot 0.2, (6 + 9.9)\cdot 0.8 + (8 + 9.76)\cdot 0.2$
$(6 + 10.18)\cdot 0.8 + (8 + 9.48)\cdot 0.2, (6 + 9.9)\cdot 0.8 + (8 + 9.48)\cdot 0.2\} =$
$= \max_2\{16.97, 16.83, 16.83, 16.69, 16.496, 16.152, 16.44, 16.216\} = 16.83$
$\{X_{1,3}^2(2)\} = \{\{ \_,C, F, H, I, L\}, \{ \_,C, E, H, I, L\},\}$

State $y_1 = 1$

$G_1^1(1) = \max_1\{[F_1^1(3|1,A) + G_2^1(3)]\cdot P_1(3|2,C) + [F_1^1(4|1,A) + G_2^1(4)]\cdot P_1(4|1,A),$
$[F_1^1(3|1,B) + G_3^1(3)]\cdot P_1(3|1,B) + [F_1^1(4|1,B) + G_2^1(4)]\cdot P_1(4|1,B)\} =$
$= \max_1\{(6 + 10.18)\cdot 0.4 + (8 + 9.76)\cdot 0.6, (6 + 10.18)\cdot 0.7 + (8 + 9.76)\cdot 0.3\} =$
$= \max_1\{17.128, 16.654\} = 17.128$
$\{X_{1,3}^1(1)\} = \{A, \_,F, G, I, L\}$

$G_1^2(1) = \max_2\{[F_1^1(3|1,A) + G_2^1(3)]\cdot P_1(3|1,A) + [F_1^1(4|1,A) + G_2^1(4)]\cdot P_1(4|1,A),$
$[F_1^1(3|1,A) + G_2^2(3)]\cdot P_1(3|1,A) + [F_1^1(4|1,A) + G_2^1(4)]\cdot P_1(4|1,A),$
$[F_1^1(3|1,A) + G_2^1(3)]\cdot P_1(3|1,A) + [F_1^1(4|1,A) + G_2^2(4)]\cdot P_1(4|1,A),$
$[F_1^1(3|1,A) + G_2^2(3)]\cdot P_1(3|1,A) + [F_1^1(4|1,A) + G_2^2(4)]\cdot P_1(4|1,A),$
$[F_1^1(3|1,B) + G_2^1(3)]\cdot P_1(3|1,B) + [F_1^1(4|1,B) + G_2^1(4)]\cdot P_1(4|1,B),$
$[F_1^1(3|1,B) + G_2^1(3)]\cdot P_1(3|1,B) + [F_1^1(4|1,B) + G_2^1(4)]\cdot P_1(4|1,B),$
$[F_1^1(3|1,B) + G_2^1(3)]\cdot P_1(3|1,B) + [F_1^1(4|1,B) + G_2^1(4)]\cdot P_1(4|1,B),$
$[F_1^1(3|1,B) + G_2^1(3)]\cdot P_1(3|1,B) + [F_1^1(4|1,B) + G_2^1(4)]\cdot P_1(4|1,B)\} =$
$= \max_2\{(6 + 10.18)\cdot 0.4 + (8 + 9.76)\cdot 0.6, (6 + 9.9)\cdot 0.4 + (8 + 9.76)\cdot 0.6,$
$(6 + 10.18)\cdot 0.4 + (8 + 9.48)\cdot 0.6, (6 + 9.9)\cdot 0.4 + (8 + 9.48)\cdot 0.6,$
$(6 + 10.18)\cdot 0.7 + (8 + 9.76)\cdot 0.3, (6 + 9.9)\cdot 0.7 + (8 + 9.76)\cdot 0.3,$
$(6 + 10.18)\cdot 0.7 + (8 + 9.48)\cdot 0.3, (6 + 9.9)\cdot 0.7 + (8 + 9.48)\cdot 0.3 =$
$= \max_2\{17.128, 17.016, 16.96, 16.848, 16.654, 16.458, 16.546, 16.374\} = 17.016$
$\{X_{1,3}^2(1)\} = \{A, \_, E, G, I, L\}$

According to formula (31) we have:

$G^1 = \max_1 \{16.97, 17.128\} = 17.128$
$G^2 = \max_2 \{16.97, 16.83, 17{,}128, 17.016\} = 17.016$

According to formula (32) we have:

$\{X^1\} = \{A, \_,F, G, I, L\}$
$\{X^2\} = \{A, \_, E, G, I, L\}$

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MULTIPLE CRITERIA DECISION MAKING
Vol. 10 2015

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THE CONCEPT OF RISK DOMINANCE IN MADM WITH NO INTER-CRITERIA INFORMATION

Abstract

This paper deals with the analysis of a multiple attribute decision making problem with no inter-criteria information. The problem is studied as a multiplayer, non-cooperative coordination game. Each equilibrium in the game corresponds to a decision variant. To choose a variant the general theory of equilibrium selection in games is used. The relation of risk dominance, introduced by Harsanyi and Selten (1988), is applied. In the method proposed a key element is to determine the reference point (status quo situation) – the least desirable situation with respect to each criterion separately. The method proposed supports decision making as regards selection and ordering.

Keywords: risk dominance, inter-criteria information, MADM.

1 Introduction

In this paper we analyze the issues in which both the set of decision variants and the set of criteria are finite. Therefore, we deal here with Multiple Attribute Decision Making (MADM) problems (Hwang and Yoon, 1981, p. 4). Such decision making problems are treated in this paper as multiple criteria decision making (MCDM) problems with a finite set of feasible solutions. Additionally, no information on inter-criteria preferences is known – the decision-maker does not want or cannot determine them.

The multiple attribute decision making problem will be treated as a game. Multiple attribute problems as games have been formulated as two-person zero-sum games (Kofler, 1967) and in the form of games with nature have been used

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for solving problems with no information on preferences (Hwang and Yoon, 1981). An analysis of the multiple attribute decision making problem as a multiplayer non-cooperative game with non-zero sum was presented in the paper by Madani and Lund (2011) and earlier in the papers by Wolny (2007, 2008). The starting point for building a model in the form of a multiplayer game is to identify the correspondences between the elements of the multiple attribute problem and the game.

In general, the player is identified with the decision-maker who considers the problem from the point of view of one criterion (player-criterion). The single strategy of the player consists in the choice of a decision variant (strategy--variant). The payoff of the player is the estimate of the decision variant with respect to a given criterion. Therefore, the game is an abstraction analyzed "in the decision-maker's mind". The essence of the problem (that is, the selection of one variant) is the choice by all players of a strategy connected with the same decision variant. In order to determine the game fully it is necessary to establish the payoffs in the situation when the players-criteria choose the strategies corresponding to different decision variants, taking into account the consequences of this action (Wolny, 2013). Analysis of the game defined in this way may involve cooperation among the players, in which case the key element is to determine the tradeoff for the player's payoff. It may also deal with the situation when there is no cooperation (incomparability of the estimates of variants with respect to criteria). The approach in the first case involves the aggregation of estimates and requires additional information needed to determine the tradeoffs for the payoffs or to construct the characteristic function of the game. This last issue was raised in the paper by Wolny (2007). At the same time it should be taken into account that methods based on the scalarization of the problem and on various notions of aggregation have been developed for many years in many theoretical and practical areas (Brans and Vincke, 1985; Greco et al., 2005; Nowak, 2008; Trzaskalik, 2014a; Trzaskalik, 2014b), and many methods and ideas for solving such problems have been suggested in the literature. In the second approach, based on lack of cooperation, it is assumed that the estimates of decision variants with respect to different criteria are not comparable. This is especially important in the situations when the preferences of the decision-maker are not revealed (lack of inter-criteria information).

The investigated game may be approached in two ways:
– the game is played only once (between player-criteria) with perfect information of strategies and payoffs,
– the game is played in many stages until a stable solution (equilibrium) is reached, also with perfect information.

This paper is focused on the analysis of the non-cooperative game only, that is, on the analysis of problems with no information on the relationships between the criteria. Furthermore, only the game played once is considered[1]. It is assumed that the estimates for the individual criteria are expressed on an interval scale at least and that they reflect the utility of the variants considered only as regards each criterion separately – the payoff for the player-criterion in each situation reflects the utility of the variants for the player (the higher the payoff, the higher the utility). However, there are no assumptions or information on the possibility of determining the collective utility for all players-criteria in a particular situation in the game.

The notion of risk dominance was introduced by Harsanyi and Selten (1988) in order to choose equilibrium in the game. These authors propose to choose the risk dominance equilibrium in the situation when there is no payoff dominance equilibrium. The relation of risk dominance will be presented further in the paper on the example of a two-criteria problem.

The main objective of the paper is to present the possibilities of using risk dominance for multiple criteria decision making support with no information on inter-criteria preferences. The starting point is to present the multiple criteria problem in the form of a multiplayer, non-cooperative non-zero sum game in which each equilibrium from the set of pure strategies corresponds to a decision variant. This is a typical coordination game with the problem of equilibrium selection. The application of the risk dominance relation will be presented on a numerical example.

2 Multiple criteria problem as a game

Let a multiple criteria decision problem of the following form be given:

$$\max_{x \in X} F(x) = \max_{x \in X}\left[f_1(x), f_2(x), ..., f_k(x)\right], \tag{1}$$

where X is a finite set of feasible decision variants, $X = \{x_1, x_2, \dots, x_n\}$, x is an element of this set, f_j is the j-th criterion-function defined on the set X ($j = 1,2,\dots,k$), $F(x)$ is a vector grouping together all the objective functions, $f_j(x)$ is the estimate of the decision variant with respect to the j-th criterion. Furthermore, all estimates of the decision variants with respect to all the criteria are given. The solution of the problem of vector optimization (1) is the set of effective solutions.

[1] The game played in many stages is discussed in the papers by Madani and Lund (2011); Wolny (2013).

Using the correspondence between the multiple criteria problem and the game presented in the introduction, we can transform the problem (1) into a *k*-person non-cooperative non-zero sum game in the standard form:

$$G = (\Phi, H) \tag{2}$$

where $\Phi = X^k$ is the set of all possible situations in the game while H is the function of the players' payoffs defined on Φ. Each situation in the game is uniquely defined by the vector of pure strategies chosen by each player. Elements of the set Φ are vectors $\phi = (x_{i1}^1, x_{i2}^2, \ldots, x_{ik}^k), x_{ij}^j \in X$, whose components are the strategies of the individual players chosen in the given situation – the *ij*-th strategy is chosen by the *j*-th player (*i, j* = 1,2,…,*n*). The situation in which all players select a strategy related to the same *i*-th decision variant is:

$$\phi_i = (x_i^1, x_i^2, \ldots, x_i^n), x_i^1 = x_i^2 = \ldots = x_i^n \, . \tag{3}$$

The motivation of the players-criteria to achieve a situation ϕ_i, i.e. a unique determination of the decision variant, is the reference point. Its payoffs reflect the situation in which the players-criteria achieve a situation different from ϕ_i. Achieving coordination between the players in order to reach the situation ϕ_i is possible if the analyzed game is a coordination game (Wolny, 2008). The situation described as the reference point (status quo) should generate the lowest possible payoffs for the players; this will create motivation to achieve any situation ϕ_i, i.e., the choice of the same variant by all players-criteria. In view of this, the payoff function will have the following form:

$$H(\phi) = \begin{cases} (f_1(x_i), f_2(x_i), \ldots, f_k(x_i)) & \text{in situation } \varphi_i, \\ (\min\limits_{i=1,2,..n} f_1(x_i), \min\limits_{i=1,2,..n} f_2(x_i), \ldots, \min\limits_{i=1,2,..n} f_k(x_i)) & \text{in other situation.} \end{cases} \tag{4}$$

The model of the multiple criteria problem in the form of game (2) with the payoff function (4) is a coordination game with *n* equilibriums in the set of pure strategies. The determination of an equilibrium is equivalent to the choice of a decision variant.

In a situation when domination occurs with respect to the payoffs, rational players, having perfect information about the payoffs, will use the strategies implying risk dominance equilibrium, though the risk is tied to subjective probability.

3 Utility of risk dominance

The concept of risk dominance will be presented using the example of a two-player game with two non-payoff-dominant strategies, which will be then compared. Furthermore, we assume that in the multiple criteria decision problem there are at least three strategies-variants; for simplicity, only the set of effective solutions is considered. The comparison of a pair of strategies can be presented as a game in normal form using the following matrix:

$$\begin{bmatrix} (f_1(x_1), f_2(x_1) & (\min_i f_1(x_i), \min_i f_2(x_i)) \\ (\min_i f_1(x_i), \min_i f_2(x_i)) & (f_1(x_2), f_2(x_2) \end{bmatrix} \tag{5}$$

and in order to meet the condition of non-payoff-dominance the following conditions have to be met simultaneously:

$$\begin{matrix} f_1(x_1) > f_1(x_2) \\ f_2(x_1) < f_2(x_2) \end{matrix}. \tag{6}$$

In other words, the first strategy-variant (x_1) is better than the second one (x_2) for the first player-criterion (f_1), while for the second player-criterion (f_2) the converse is true: x_2 is better than x_1.

Player-criterion f_1 will select his better strategy if the expected value of his payoff when using this strategy is greater than that resulting from the application of strategy x_2, that is:

$$(1-q)\cdot f_1(x_1) + q\cdot \min_i f_1(x_i) > (1-q)\cdot \min_i f_1(x_i) + q\cdot f_1(x_2), \tag{7}$$

where q is the probability of player-criterion f_2 applying his better strategy--variant (x_2). As a consequence, the first player will select the first strategy if the following condition is met:

$$q < \frac{f_1(x_1) - \min_i f_1(x_i)}{f_1(x_1) + f_1(x_2) - 2\cdot \min_i f_1(x_i)}, \tag{8}$$

which means that he will expect the probability of the second player using his better strategy to be lower than:

$$q_0 = \frac{f_1(x_1) - \min_i f_1(x_i)}{f_1(x_1) + f_1(x_2) - 2\cdot \min_i f_1(x_i)}. \tag{9}$$

Similarly, player f_2 will select his better strategy-variant x_2 if the expected value of the payoff resulting from x_2 is greater than the payoff from using x_1, that is:

$$p\cdot \min_i f_2(x_i) + (1-p)\cdot f_2(x_2) > p\cdot f_2(x_1) + (1-p)\cdot \min_i f_2(x_i), \tag{10}$$

where p is the probability of player-criterion f_1 using his better strategy-variant (x_1). Consequently, the second player, similarly to the first player, will select his better strategy, that is x_2, if the following condition is met:

$$p < \frac{f_2(x_2) - \min_i f_2(x_i)}{f_2(x_1) + f_2(x_2) - 2\cdot \min_i f_2(x_i)}, \tag{10}$$

so he will expect the probability of the first player using his better strategy to be lower than:

$$p_0 = \frac{f_2(x_2) - \min_i f_2(x_i)}{f_2(x_1) + f_2(x_2) - 2\cdot \min_i f_2(x_i)}. \tag{11}$$

The players' expectations are subjective, but both of them have perfect information about the payoffs. Therefore, if they approach the game in a similar way, they will both select the variant which is better for the first player, if the first player has stronger indications to select his better strategy than the second one has to select his own better strategy, that is:

$$p_0 < q_0\,, \tag{12}$$

therefore borderline, subjective probability causing the first player to select his own better strategy is greater than the borderline, subjective probability causing the second player to select his better strategy. In this case strategy-variant x_1 is risk dominant over variant x_2, and therefore x_1 will be preferred over x_2.

When $p_0 > q_0$, variant x_2 is preferable over variant x_1 and for $p_0 = q_0$ both variants are equivalent or impossible to compare.

It can be observed that when only two decision variants are considered they are always equivalent in terms of the suggested approach. This is a consequence of adopting a minimal estimate of the decision variant as the reference point: in the case of two non-dominant variants we compare the best one and the worst one with respect to each criterion, taking into account that the best variant with respect to one criterion is the worst one with respect to the other criterion. The goal of considering such a situation is to show that the comparison of two variants such that for one of them the estimate with respect to a given criterion is minimal, will generate a borderline value of the probability equal to one – with respect to this criterion the better variant will never be risk dominant[2]. To sum up, the reference point is of significant importance in forming the relationships of risk dominance.

In the case of more than two criteria when the variants are compared pairwise the criteria are gathered into two concordant coalitions (groups). Each coalition prefers a different decision variant[3]. Each coalition is represented by a player who has the strongest indications to select a variant which is better for the coalition. The choice of the variant is made among the players representing consistent coalitions playing a game.

The suggested approach will be illustrated using a simple numerical example.

4 Numerical example

In this problem nine decision variants are being considered with respect to three criteria. All criteria are maximized. The estimates of the decision variants are presented in Table 1.

[2] In this situation the equilibrium in the game is related to the variant with the minimal estimate, with respect to the player-criterion, is a weak equilibrium, because whatever other strategy--variant he chooses he obtains the same result regardless of the action of other players.

[3] In the case of payoff-dominance one coalition is created.

Table 1: The assessments of decision variants

Decision variants	f_1	f_2	f_3
x_1	411	55252	19
x_2	469	58251	11
x_3	297	82739	29
x_4	1581	89022	20
x_5	1092	99118	22
x_6	966	78119	25
x_7	650	84084	38
x_8	414	68300	10
x_9	737	85071	39

The problem will be treated as a game. It can be observed that variant--strategy x_9 payoff-dominates variants x_8, x_7, x_3, x_2 and x_1. Payoff dominance relationships existing between all the variants are presented in Table 2.

Table 2: Payoff dominance

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_1		0	0	0	0	0	0	0	0
x_2	0		0	0	0	0	0	0	0
x_3	0	0		0	0	0	0	0	0
x_4	1	1	0		0	0	0	1	0
x_5	1	1	0	0		0	0	1	0
x_6	1	1	0	0	0		0	1	0
x_7	0	0	0	0	0	0		1	0
x_8	0	0	0	0	0	0	0		0
x_9	1	1	1	0	0	0	1	1	

'0' means that a relation does not exist, '1' that it exists between the variant in the row and the variant in the column of the table: e.g., x_4 payoff-dominates x_1.

The use of payoff dominance does not allow a single variant to be chosen in this case. According to the suggested approach, in further analysis risk dominance will be used.

For the pair of variants (x_4, x_9), x_4 is a better variant for criteria f_1 and f_2, while x_9 is better for f_3. The borderline probability values, expressed by formulas (9) and (11) and condition (12), make it possible to determine the relationship of risk dominance for this pair of variants. Those values for the consecutive criteria are: for f_1 – 0.745, for f_2 – 0.531, for f_3 – 0.744. Therefore, player-criterion f_1 has the strongest indications to select his better strategy (variant). It implies that x_4 risk dominates x_9.

The remaining relationships of risk dominance existing between the variants are presented in Table 3.

Table 3: Risk dominance

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_1		0	0	0	0	0	0	0	0
x_2	1		1	0	0	0	0	1	0
x_3	0	0		0	0	0	0	0	0
x_4	1	1	1		1	1	1	1	1
x_5	1	1	1	0		**1**	**0**	1	0
x_6	1	1	1	0	**0**		**1**	1	0
x_7	1	1	1	0	**1**	**0**		1	0
x_8	0	0	0	0	0	0	0		0
x_9	1	1	1	0	1	1	1	1	

On the basis of the information in table 3[4] it may be stated that the best decision variant in the sense of the suggested approach is x_4.

Analyzing the relationships for variants x_5, x_6, x_7 we can observe that this relation is not transitive, because it is impossible to determine the preferences between those variants. In general, risk dominance may not sort the set of decision variants[5].

In the final sorting (Table 4) the variants for which the relation is not transitive are on the same preference level.

Table 4: Ranking of decision variants

Rank	Decision variants		
1	x_4		
2	x_9		
3	x_5	x_6	x_7
4	x_2		
5	x_1	x_3	x_8

5 Summary

In this paper we have proposed a game-theoretic approach to the discrete multiple criteria problems with no information on inter-criteria preferences. The multiple criteria problem is treated as a multiplayer (k-person), non-cooperative non-zero sum game.

4 When there is a relationship of payoff dominance, there is also a relationship of risk dominance. In general, this regularity does not occur in game theory (Harsanyi and Selten, 1988).

5 For two-criteria problems it was shown that risk dominance may sort the set of decision variants (Wolny, 2014).

The determination of the status quo situation (Wolny, 2013) which corresponds to the least desirable situation is the key element of the suggested approach[6]. The solution to this problem depends to a large extent on the selected reference point. Therefore, it is recommended to acquire information on the values related to status quo from the decision-maker. If the status quo situation cannot be explicitly determined it is suggested that the lowest possible values of the maximized criterion-functions be adopted. As a result, the variants with the lowest estimate with respect to any criterion are discriminated against. The equilibrium in the game corresponding to such a variant is weak. The player-criterion achieves the least possible payoff, similarly to any other situation. For this reason he has no 'motivation' to achieve the equilibrium, other than the indications from other players-criteria.

The application of risk dominance to solving the multiple criteria problem is based on the comparison of the probabilities expressing the strength of the indications for the selection of a given decision variant. The suggested approach originates in the construction of the model of the multiple criteria problem in the form of non-cooperative non-zero sum game. The choice of the equilibrium is based on the general theory of equilibrium selection in games.

An important feature of the suggested method is that the estimates of the decision variants do not have to be normalized. The presentation of the multiple criteria problem as a game can assist in the interaction with the decision-maker and make the structuring of the problem possible, particularly when it is not possible to acquire information on inter-criteria preferences.

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