Abstract

In the paper we consider a bi-criteria version of the Stochastic Generalized Transportation Problem, where one goal is the minimization of the expected total cost, and the second one is the minimization of the risk. We present a model and a solution method for this problem.

Keywords: Stochastic Generalized Transportation Problem, Bi-criteria Stochastic Generalized Transportation Problem, expected cost, variance of the cost, Equalization Method, branch and bound.

1 Introduction

The Generalized Transportation Problem (GTP) and its generalizations can be used in many real-life applications, in particular in modeling of transportation of perishable products, see e.g. Nagurney et al. (2013). One can look at the GTP as a special kind of the Generalized Minimum Cost Flow Problem or as a generalization of the ordinary Transportation Problem. The generalized flows, as well as some solution methods, can be found e.g. in Ahuja et al. (1993). The generalized flows were also studied by Glover et al. (1972), Goldberg et al. (1988), and Wayne (2002), among others. The particular case of the GTP was studied in particular by Balas (1966), Balas and Ivanescu (1964), and Lourie (1964).

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and Kawa (2012) considered the two-stage GTP and its application in the supply chain in which complaints are involved.

The transportation of perishable goods is not the only application of generalized flows. In Ahuja et al. (1993) several others have been discussed. In particular, they may be used in the modeling of conversions of physical entities in financial, mineral and energy networks or machine loading. Nagurney et al. (2013) discuss, in turn, the application of generalized flows in the modeling of selected kinds of logistic chains, in particular in the distribution process of medical materials, food, pharmaceuticals and clothes.

Very often (also in the above mentioned papers) it is assumed that the demand is fixed. In fact, it is usually impossible to predict \textit{a priori} the exact values of demand. However, in many cases it is possible to estimate its probability distribution.

The Stochastic Generalized Transportation Problem (SGTP) is the generalized version of the GTP, where one assumes that the values of demand are given as random variables. At least two approaches can be applied to transform this kind of problem into an equivalent, deterministic form. One could assume that the probability of satisfying the demand constraints has to be not less than some fixed value. This, together with the demand distribution, allows to transform the constraints (and hence the problem) into a deterministic form. However, in the case of transportation problems, another approach is more common. In this approach we remove the demand constraints and use them to introduce a new cost function, including the expected extra cost, increasing with the discrepancy between the actual value of the demand and the size of delivery. This approach has been used in such classic papers as Williams (1963), Cooper and LeBlanc (1977), but also in more recent ones, such as Holmberg and Jörnsten (1984), Holmberg (1995), Qi (1985, 1987) and Anholcer (2012, 2015). It is also worth mentioning that this approach is related to the classical Newsvendor Problem which has been known at least from the moment of the publication of Edgeworth (1888), and then analyzed and generalized by numerous authors, see e.g. Khouja et al. (1996), Şen and Zhang (1999), Chen and Chuang (2000), Yang et al. (2007), Goto (2013) (in fact, the Newsvendor Problem can be considered as an instance of the Stochastic Transportation Problem with one source and one destination).

A more general version of the Nonlinear Transportation Problem (where any convex costs at the destination points are applicable) was discussed by Anholcer (2005, 2008a, 2008b), Sikora (1993) and Sikora et al. (1991), among others. In those papers the Equalization Method was considered and it was proved to be convergent in Anholcer (2005, 2008a). The convergence of the general versions for the Nonlinear and Stochastic GTP was also proved by Anholcer (2012, 2015).

In all the above papers only the expected costs were taken under consideration. It can be useful, however, to involve also the risk, measured by variance.
This makes the problem bi-criterial. The problem of stochastic programming involving both expected cost and variance has been recently studied by Li et al. (2014) who transformed this problem into a quasi-linear form and applied it to the Transportation Problem. A version of the bi-criteria SGTP, this time with expected cost and time criteria, has been studied by Anholcer (2013). Also Nagurney et al. (2013) studied the generalized flows where two criteria (expected cost and risk) were involved (the authors assumed that the risk can be represented by a function convex with respect to the flow, which is, however, not always true; see below). Bi- and Multi-criteria Transportation Problems were discussed also e.g. by Aneja and Nair (1979), Gupta and Gupta (1983), Shi (1995), Li (2000), Basu and Acharya (2002), Khurana and Arora (2011), Kesavarz and Khorram (2011) and Kumar et al. (2012). The (linear) Generalized Transportation Problem in the multi-criteria version was studied by Gen et al. (1999), among others.

In this paper we present a method for finding efficient solutions of the Bi-criteria Stochastic Generalized Transportation Problem with two criteria: expected cost and variance. In Section 2 the problem is formulated. In Sections 3 and 4 the algorithm, together with its theoretical justification, is presented. Section 5 contains an illustrative example. The results of computational experiments are presented in Section 6. Section 7 contains final remarks.

2 Problem formulation

In the Generalized Transportation Problem, the goal is to minimize the transportation costs of a uniform good delivered from \( m \) supply points to \( n \) destination points. The amount of the transported good changes during the transportation process. More precisely, the amount delivered to demand point \( j \) from supply point \( i \) is equal to \( r_{ij} x_{ij} \), where \( x_{ij} \) is the amount of the good that leaves supply point \( i \) and \( r_{ij} \) is the reduction ratio. The unit transportation costs \( c_{ij} \) are constant, the demand \( b_j \) of every demand point \( j \) has to be satisfied and the supply \( a_i \) of any supply point \( i \) cannot be exceeded. The model looks as follows:

\[
\min \left\{ f(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \right\}, \\
\text{s. t.} \sum_{i=1}^{m} r_{ij} x_{ij} = b_j, j = 1, ..., n, \\
\sum_{j=1}^{n} x_{ij} \leq a_i, i = 1, ..., m, \\
x_{ij} \geq 0, i = 1, ..., m, j = 1, ..., n.
\]
In the Stochastic GTP (SGTP), the demands \( b_j \) are independent continuous random variables \( X_j \) with density functions \( \varphi_j \). We will assume that for every \( j = 1, \ldots, n \) and for every \( x > 0 \),
\[
\varphi_j(x) > 0.
\]
The unit surplus cost \( s_j^{(1)} \) and the unit shortage cost \( s_j^{(2)} \) are defined for every destination point \( j \). This implies that the expected extra cost at the destination \( j \) is equal to:
\[
f_j(x_j) = s_j^{(1)} \int_0^{x_j} (x_j - t) \varphi_j(t) dt + s_j^{(2)} \int_{x_j}^{\infty} (t - x_j) \varphi_j(t) dt.
\]
Using elementary transformations and integrating by parts, we obtain that:
\[
f_j(x_j) = s_j^{(2)} \int_0^{x_j} t \varphi_j(t) dt - x_j \int_0^{x_j} \varphi_j(t) dt + (s_j^{(1)} + s_j^{(2)}) \int_0^{x_j} \varphi_j(t) dt = \]
\[
= s_j^{(2)} \left( E(X_j) - x_j \right) + (s_j^{(1)} + s_j^{(2)}) \int_0^{x_j} \Phi_j(t) dt,
\]
where \( \Phi_j \) is the cumulative distribution function of the demand at destination \( j \) (the last equality uses the fact that \( \Phi_j(0) = 0 \)).

Finally, the SGTP takes the form:
\[
\min \left\{ f(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{j=1}^{n} f_j(x_j) \right\},
\]
s. t.
\[
\sum_{i=1}^{m} r_{ij} x_{ij} = x_j, j = 1, \ldots, n,
\]
\[
\sum_{j=1}^{n} x_{ij} \leq a_i, i = 1, \ldots, m,
\]
\[
x_{ij} \geq 0, i = 1, \ldots, m, j = 1, \ldots, n.
\]
The first derivative of the expected cost function has the form:
\[
f'_j(x_j) = -s_j^{(2)} + (s_j^{(1)} + s_j^{(2)}) \Phi_j(t),
\]
while the second derivative is equal to:
\[
f''_j(x_j) = (s_j^{(1)} + s_j^{(2)}) \varphi_j(t).
\]
This means that each function \( f_j \) is twice differentiable and strictly convex on the interval where \( \varphi_j(t) > 0 \). This allows to use the corresponding version of the Equalization Method (Anholcer, 2012 and 2015) to solve this problem.
Of course it may happen that a Decision Maker considers the transportation costs, shortage costs and surplus costs as not equally important. In such a situation one could use three criteria instead of one, or even when using one objective, one could still introduce weights, reflecting the Decision Maker’s preferences. However, this would not change the structure or the general form of the resulting weighting problem, discussed in Section 3 (Observation 1).

The second criterion of interest is variance. The formula for variance for destination $j$ is:

$$g_j(x_j) = p_j(x_j) - q_j(x_j),$$

where:

$$p_j(x_j) = \left(s_j^{(1)}\right)^2 \int_0^{x_j} (x_j - t)^2 \varphi_j(t) dt + \left(s_j^{(2)}\right)^2 \int_{x_j}^{\infty} (x_j - t)^2 \varphi_j(t) dt$$

and:

$$q_j(x_j) = \left(f_j(x_j)\right)^2.$$

One can see that:

$$p_j'(x_j) = 2 \left(s_j^{(1)}\right)^2 \int_0^{x_j} (x_j - t) \varphi_j(t) dt + 2 \left(s_j^{(2)}\right)^2 \int_{x_j}^{\infty} (x_j - t) \varphi_j(t) dt$$

and:

$$p_j''(x_j) = 2 \left(s_j^{(1)}\right)^2 \int_0^{x_j} \varphi_j(t) dt + 2 \left(s_j^{(2)}\right)^2 \int_{x_j}^{\infty} \varphi_j(t) dt.$$

Moreover:

$$q_j'(x_j) = 2f_j(x_j)f_j'(x_j)$$

and:

$$q_j''(x_j) = 2\left[f_j'(x_j)\right]^2 + 2f_j(x_j)f_j''(x_j).$$

This means that each of the functions $g_j(x_j)$ is a twice differentiable DC-function. Namely, it is the difference of two convex functions, which are strictly convex if $\varphi_j(x_j) > 0$. However, in general, the functions $g_j$ do not need to be convex.

As the demands are independent random variables, the variance of total extra cost is equal to the sum of the variances at the destination points. Thus the bi-criteria problem (BSGTP) takes the form:

$$\min \left\{ f(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j(x_j) \right\},$$

$$\min \left\{ g(x) = \sum_{j=1}^{n} g_j(x_j) \right\},$$
s. t.
\[\sum_{i=1}^{m} r_{ij} x_{ij} = x_j, j = 1, ..., n,\]
\[\sum_{j=1}^{n} x_{ij} \leq a_i, i = 1, ..., m,\]
\[x_{ij} \geq 0, i = 1, ..., m, j = 1, ..., n.\]

Usually the two objective functions have different minima. Our goal is to find a solution method that finds the efficient (Pareto-optimal) solutions.

3 Algorithm – the main idea

Let \( S \) denote the set of all feasible solutions of the BSGTP. The problem may be rewritten as:
\[
\min f(x), \quad \min g(x),
\]
s. t.
\[x \in S.\]

The following observation is a corollary from the well-known result about the efficiency of the solution to the weighting problem (see e.g. Miettinen, 1998, p. 78, Theorem 3.1.2).

**Observation 1**

If \( x^* \) is, for some \( \lambda > 0 \), an optimal solution to the problem:
\[
\min h(x) = f(x) + \lambda g(x)
\]
s. t.
\[x \in S,\]
then it is a Pareto-optimal solution of the BSGTP.

Minimizing \( h(x) \) on \( S \) always leads to an efficient solution. The problem obtains then the form of a GTP with a nonlinear objective function. The function \( h(x) \) is not necessarily convex, but it is a separable function in which each summand is a DC-function. Thus one can use a branch-and-bound method to determine an exact solution. We will discuss such a method in the next section.

4 Algorithm – the details

The method that we are going to present uses the ideas discussed by Falk and Soland (1969), as well as by Holmberg and Tuy (1999). Assume that the variable \( x_j \) is bounded from below and from above: \( l_j \leq x_j \leq u_j \). Since the function \( q_j(x_j) \) is convex, we have:
\[ q_j(x_j) \leq r_j(x_j; l_j, u_j) \]
for \( l_j \leq x_j \leq u_j \), where:
\[ r_j(x_j; l_j, u_j) = q_j(l_j) + \frac{x_j - l_j}{u_j - l_j}(q_j(u_j) - q_j(l_j)) \]
is a linear function such that:
\[ r_j(l_j; l_j, u_j) = q_j(l_j) \]
and:
\[ r_j(u_j; l_j, u_j) = q_j(u_j). \]
This means that for each index \( j \) we have:
\[ g_j(x_j) = p_j(x_j) - q_j(x_j) \geq p_j(x_j) - r_j(x_j; l_j, u_j) = g_j^*(x_j; l_j; u_j). \]
One can see that \( g_j^*(x_j; l_j; u_j) \) is a lower estimate of \( g_j(x_j) \) on the interval \([l_j, u_j]\). Let \( l \) be the vector of the lower bounds and \( u \) the vector of the upper bounds. Let:
\[ h^*(x; l; u) = \sum_{j=1}^{n} (f_j(x_j) + \lambda g_j^*(x_j; l_j; u_j)). \]
Of course, \( h^*(x; l; u) \) is a lower estimate of \( h(x) \) on the generalized rectangle defined by the inequalities \( l \leq x \leq u \). This means that the new problem:
\[
\begin{align*}
\min h^*(x; l; u) \\
\text{s. t.} \\
x \in S,
\end{align*}
\]
has the form of an SGTP and can be solved using the Equalization Method (see Anholcer, 2012 and 2015). Note that no additional constraints are introduced, so the set of feasible solutions does not change.

The rule of branching is as follows. After solving the problem with function \( h^*(x; l; u) \), we check whether the solution is satisfactory for some predefined accuracy level \( \varepsilon \). If it is not, we choose \( j \) for which the difference \( r_j(x_j; l_j; u_j) - q_j(x_j) \) is the largest and define two child problems by setting \( l_j := x_j \) and \( u_j := x_j \), respectively, for the new problems (recall that we do not change the set of feasible solutions; those values are used only to find the formula of the lower estimate function).

Finally, we can write the algorithm as follows (\( U_h \) and \( U_x \) denote the upper bound on the optimal value of the objective and the point at which this value is reached, respectively; for a given node \( v \) of the solution tree, \( L(v) \) and \( P(v) \) denote the lower bound on the optimal value of the objective and the corresponding convex problem).
Algorithm 1: The Branch and Bound Method for BSGTP

Input: initial problem, the value of \( \lambda > 0 \), accuracy level \( \varepsilon \).
Output: Pareto-optimal solution \( x^* \).

1. Initial solution. Let the initial bounds for each \( x_j \) be:
   \[
   l_j = 0, u_j = \sum_{i=1}^{m} r_{ij} a_i.
   \]
   Solve (using the Equalization Method) the corresponding problem \( P(v_0) \):
   \[
   \min h^*(x; l; u)
   \]
   s. t. \( x \in S \).
   Assume that the obtained optimum is \( x^* \). Set \( u_j^* = x_j^* \), \( u_j^* = \sum_{i=1}^{m} r_{ij} a_i \), where \( v_0 \) is the root of the solution tree.
   Go to step 2.

2. Checking the optimality. Find an active node \( v^* \), for which \( L(v) \) has the minimum value. If:
   \[
   |U_h - L(v^*)| < \varepsilon,
   \]
   then STOP. The solution \( U_x \) is satisfactory. Otherwise go to step 3.

3. Branching and bounding. Consider the problem \( P(v^*) \). Let \( j^* \) be an index \( j \) for which the difference \( r_j(x_j; l_j; u_j) - q_j(x_j) \) is the largest. Remove the node \( v^* \) from the set of active nodes. Add two new active nodes \( v' \) and \( v'' \) and define the corresponding convex problems. To obtain \( P(v') \), set \( u_j^* = x_j^* \) in \( P(v^*) \). To obtain \( P(v'') \), set \( l_j = x_j^* \) in \( P(v^*) \). Let us denote the new bounding vectors by \( l', u', l'', u'' \), respectively.
   Solve \( P(v') \) and \( P(v'') \) using the Equalization Method. Assume that the obtained optima are \( x' \) and \( x'' \), respectively. If \( U_h > h(x') \), then set \( U_x = x' \) and \( U_h = h(x) \). If \( U_h > h(x) \), then set \( U_x = x'' \) and \( U_h = h(x') \). Set \( L(v') = h^*(x'; l'; u') \). If \( U_h > h(x') \), then set \( U_x = x'' \) and \( U_h = h(x'') \). Set \( L(v'') = h^*(x''; l''; u'') \).
   Close all the active nodes \( v \) for which \( L(v) > U_h - \varepsilon \).
   Go back to step 2.

5 Illustrative example

Let us analyze a simple example that illustrates the algorithm. Assume that there are two supply points with the supply equal to \( a_1 = a_2 = 15 \) and three destinations, with uniform demand distribution given by the density functions:

\[
\varphi_1(x_1) = \begin{cases} 
\frac{1}{10}, & x \in [0,10], \\
0, & x \notin [0,10], 
\end{cases}
\]
\[ \varphi_2(x_2) = \begin{cases} \frac{1}{12}, & x \in [0,12], \\ 0, & x \notin [0,12], \end{cases} \]

\[ \varphi_3(x_3) = \begin{cases} \frac{1}{14}, & x \in [0,14], \\ 0, & x \notin [0,14]. \end{cases} \]

The unit transportation costs \( c_{ij} \), the reduction ratios \( r_{ij} \), the surplus costs \( s_j^{(1)} \) and the shortage costs \( s_j^{(2)} \) are given in the Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( j = 1 )</th>
<th>( j = 2 )</th>
<th>( j = 3 )</th>
<th>( k = 1 )</th>
<th>( k = 2 )</th>
<th>( k = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_{1j} )</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>( r_{1j} )</td>
<td>0.92</td>
<td>0.95</td>
</tr>
<tr>
<td>( c_{2j} )</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>( r_{2j} )</td>
<td>0.91</td>
<td>0.87</td>
</tr>
<tr>
<td>( s_j^{(1)} )</td>
<td>1</td>
<td>4</td>
<td>5</td>
<td>( s_j^{(2)} )</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

Assume that we are interested in finding the solution for \( \lambda = 0.5 \) and \( \varepsilon = 0.01 \). The functions of expected costs are given by:

\[ f_1(x_1) = \begin{cases} \frac{1}{4}x^2 - 4x + 20, & x \in [0,10], \\ x - 5, & x > 10, \end{cases} \]

\[ f_2(x_2) = \begin{cases} \frac{5}{12}x^2 - 6x + 36, & x \in [0,12], \\ 4x - 24, & x > 12, \end{cases} \]

\[ f_3(x_3) = \begin{cases} \frac{15}{28}x^2 - 10x + 70, & x \in [0,14], \\ 5x - 35, & x > 14. \end{cases} \]

The functions \( p_j \) have the form:

\[ p_1(x_1) = \begin{cases} \frac{1}{2}x^3 + 16x^2 - 160x + \frac{1600}{3}, & x \in [0,10], \\ x^2 - 10x + \frac{100}{3}, & x > 10, \end{cases} \]

\[ p_2(x_2) = \begin{cases} -\frac{5}{9}x^3 + 36x^2 - 432x + 1728, & x \in [0,12], \\ 16x^2 - 192x + 768, & x > 12, \end{cases} \]

\[ p_3(x_3) = \begin{cases} -\frac{25}{14}x^3 + 100x^2 - 1400x + \frac{19600}{3}, & x \in [0,14], \\ 25x^2 - 350x + \frac{4900}{3}, & x > 14. \end{cases} \]
The first bounds on the variables (corresponding to the node $v_0$) are defined by $0 \leq x_1 \leq 27.45$, $0 \leq x_2 \leq 27.3$ and $0 \leq x_3 \leq 27.75$. The respective linear estimates of $q_j$ are equal to:

\[
\begin{align*}
r_1(x_1) &= 400 + \frac{x_1 - 0}{27.45 - 0} (504.0025 - 400) \\
r_2(x_2) &= 1296 + \frac{x_2 - 0}{27.3 - 0} (7259.04 - 1296) \\
r_3(x_3) &= 4900 + \frac{x_3 - 0}{27.75 - 0} (10764.0625 - 4900)
\end{align*}
\]

The solution of the problem $P(v_0)$ is as follows:

<table>
<thead>
<tr>
<th>Value</th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
<th>Value</th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{ij}$ &amp; 0.00 &amp; 3.02 &amp; 11.98</td>
<td>$p_j(x_j)$ &amp; 74.03 &amp; 628.64 &amp; 878.24</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_{ij}$ &amp; 5.40 &amp; 9.60 &amp; 0.00</td>
<td>$q_j(x_j)$ &amp; 40.67 &amp; 446.96 &amp; 628.89</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_j$ &amp; 4.92 &amp; 11.22 &amp; 11.14</td>
<td>$r_j(x_j)$ &amp; 418.63 &amp; 3747.27 &amp; 7253.62</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_j(x_j)$ &amp; 6.38 &amp; 21.14 &amp; 25.08</td>
<td>$r_j(x_j) - q_j(x_j)$ &amp; 377.96 &amp; 3300.30 &amp; 6624.73</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The objectives of the initial problem and of the convex problem are $h(x^*) = 338.215$ and $h^*(x^*) = -4813.279$. This means that $L(v_0) = -4813.279$ and $U_h = U(v_0) = 338.215$. Since $v_0$ is the only (active) node and $|U_h - L(v_0)| > \varepsilon$, we perform branching with respect to the variable $x_3$ (the maximum difference $r_j(x_j) - q_j(x_j)$ is $r_3(x_3) - q_3(x_3)$). Since $x_3 = 11.138$, the new nodes $v_1$ and $v_2$ will correspond to the additional constraints $x_3 \leq 11.138$ and $x_3 \geq 11.138$, respectively. After defining the functions $r_j(x_j)$ and solving the new problems, we obtain $L(v_1) = -2289.611$, $U(v_1) = 448.384$, $L(v_2) = -1995.718$ and $U(v_2) = 489.812$. $U(v_1) > U_h$ $U(v_2) > U_h$, so $U_h$ does not change (and $U_x$ remains the optimal solution of $P(v_0)$). Now the two active nodes are $v_1$ and $v_2$. The function $L$ is minimized at $v_1$ and $|U_h - L(v_1)| > \varepsilon$, so we perform branching and continue in this way. At some moment we obtain $U(v_2) = 224.145$, which means that starting from this moment $U_h = 224.145$ and $U_x$ becomes the optimal solution of $P(v_0)$. After a few more iterations, after branching at $v_{13}$, we obtain, in particular, that at $v_{22}$ we have $L(v_{22}) = 259.418$, which means that $L(v_{22}) > U_h - \varepsilon$ and we close node $v_{22}$. The details for the first 51 nodes have been collected in Table 3 below. In each row, the label of node $v_j$ is followed by the label of the parent node; two child nodes, order of branching, branching variable and its value (if the branching was performed at $v_j$); the values of both objectives: $U(v_j)$ and $L(v_j)$; and the actual value of $U_h$. At the stage presented in the table, 25 nodes are still active (A), four have been closed (C), and the branching has been performed at the other nodes.
Table 3: Beginning of the algorithm

<table>
<thead>
<tr>
<th>Node (v)</th>
<th>Parent node</th>
<th>Child nodes</th>
<th>Checking order</th>
<th>Branching variable</th>
<th>Branching value</th>
<th>U(v)</th>
<th>L(v)</th>
<th>Uh</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>v0</td>
<td>none</td>
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6 Computational experiments

Test problems were randomly generated and solved with the proposed method. Two types of demand distributions were considered: uniform $U(0, u)$ and exponential $Exp(\lambda)$, where $u$ and $\lambda$ were chosen uniformly at random from the intervals $[15, 20)$ and $[0.5, 0.6)$, respectively. In both cases unit transportation costs were chosen from the interval $[5, 10)$, surplus costs from the interval $[1, 2)$, shortage costs from the interval $[5, 10)$, reduction ratios from the interval $[0.8, 0.9)$ and the supply from each source point from the interval $[10, 20)$. The algorithm was implemented in Java SE and run on a personal computer with Intel(R) Core(TM) i7-2670 QM CPU @2.20 GHz. For both types of distributions, 100 randomly generated problems of four sizes were solved: $(m, n) = (10, 10), (10, 20), (10, 50) \text{ and } (20, 50)$, that is, 800 test problems in total. The running times in seconds (average, standard deviation, minimum and maximum) are presented in Table 4:

Table 4: Running times in seconds

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<th>$U(0, u)$ 10x50</th>
<th>$U(0, u)$ 20x50</th>
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As we can see, the algorithm can be regarded as fast: the running times are less than a second or a few seconds in the case of the smaller problems and about one hour in the case of the bigger problems (up to 1000 variables). However, one needs to remember that the branch and bound methods are super-polynomial, which means that the solution times may grow very rapidly with the increasing size of the problem.

7 Final remarks

The algorithm presented above allows to find the Pareto-optimal solutions of the Bi-criteria Stochastic Generalized Transportation Problem. In this type of problem we assume that one of the criteria is the sum of the transportation cost and the expected total extra cost of all the deliveries. The second criterion is the risk measured by the variance of the expected extra cost. The resulting problem, which allows to find the efficient solutions, is a non-convex optimization problem that can be solved with a branch-and-bound method described in the paper. The subproblems solved in the nodes of the solution tree are of SGTP form and therefore can be solved using the Equalization Method. The numerical evidence shows that the presented algorithm allows to solve problems of average size in a reasonable time.

References

Anholcer M. (2013), Algorithm for Bi-criteria Stochastic Generalized Transportation Problem, Multiple Criteria Decision Making, Vol. 8, 5-17.


Li J. (2000), *A Dynamic Transportation Model with Multiple Criteria and Multiple Constraint Levels*, Mathematical and Computer Modelling, 32, 1193-1208.


