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BI-CRITERIA STOCHASTIC GENERALIZED TRANSPORTATION PROBLEM: EXPECTED COST AND RISK MINIMIZATION[1]

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Abstract

In the paper we consider a bi-criteria version of the Stochastic Generalized Transportation Problem, where one goal is the minimization of the expected total cost, and the second one is the minimization of the risk. We present a model and a solution method for this problem.

Keywords: Stochastic Generalized Transportation Problem, Bi-criteria Stochastic Generalized Transportation Problem, expected cost, variance of the cost, Equalization Method, branch and bound.

1 Introduction

The Generalized Transportation Problem (GTP) and its generalizations can be used in many real-life applications, in particular in modeling of transportation of perishable products, see e.g. Nagurney et al. (2013). One can look at the GTP as a special kind of the Generalized Minimum Cost Flow Problem or as a generalization of the ordinary Transportation Problem. The generalized flows, as well as some solution methods, can be found e.g. in Ahuja et al. (1993). The generalized flows were also studied by Glover et al. (1972), Goldberg et al. (1988), and Wayne (2002), among others. The particular case of the GTP was studied in particular by Balas (1966), Balas and Ivanescu (1964), and Lourie (1964). Anholcer

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and Kawa (2012) considered the two-stage GTP and its application in the supply chain in which complaints are involved.

The transportation of perishable goods is not the only application of generalized flows. In Ahuja et al. (1993) several others have been discussed. In particular, they may be used in the modeling of conversions of physical entities in financial, mineral and energy networks or machine loading. Nagurney et al. (2013) discuss, in turn, the application of generalized flows in the modeling of selected kinds of logistic chains, in particular in the distribution process of medical materials, food, pharmaceuticals and clothes.

Very often (also in the above mentioned papers) it is assumed that the demand is fixed. In fact, it is usually impossible to predict *a priori* the exact values of demand. However, in many cases it is possible to estimate its probability distribution.

The Stochastic Generalized Transportation Problem (SGTP) is the generalized version of the GTP, where one assumes that the values of demand are given as random variables. At least two approaches can be applied to transform this kind of problem into an equivalent, deterministic form. One could assume that the probability of satisfying the demand constraints has to be not less than some fixed value. This, together with the demand distribution, allows to transform the constraints (and hence the problem) into a deterministic form. However, in the case of transportation problems, another approach is more common. In this approach we remove the demand constraints and use them to introduce a new cost function, including the expected extra cost, increasing with the discrepancy between the actual value of the demand and the size of delivery. This approach has been used in such classic papers as Williams (1963), Cooper and LeBlanc (1977), but also in more recent ones, such as Holmberg and Jörnsten (1984), Holmberg (1995), Qi (1985, 1987) and Anholcer (2012, 2015). It is also worth mentioning that this approach is related to the classical Newsvendor Problem which has been known at least from the moment of the publication of Edgeworth (1888), and then analyzed and generalized by numerous authors, see e.g. Khouja et al. (1996), Şen and Zhang (1999), Chen and Chuang (2000), Yang et al. (2007), Goto (2013) (in fact, the Newsvendor Problem can be considered as an instance of the Stochastic Transportation Problem with one source and one destination).

A more general version of the Nonlinear Transportation Problem (where any convex costs at the destination points are applicable) was discussed by Anholcer (2005, 2008a, 2008b), Sikora (1993) and Sikora et al. (1991), among others. In those papers the Equalization Method was considered and it was proved to be convergent in Anholcer (2005, 2008a). The convergence of the general versions for the Nonlinear and Stochastic GTP was also proved by Anholcer (2012, 2015).

In all the above papers only the expected costs were taken under consideration. It can be useful, however, to involve also the risk, measured by variance.

This makes the problem bi-criterial. The problem of stochastic programming involving both expected cost and variance has been recently studied by Li et al. (2014) who transformed this problem into a quasi-linear form and applied it to the Transportation Problem. A version of the bi-criteria SGTP, this time with expected cost and time criteria, has been studied by Anholcer (2013). Also Nagurney et al. (2013) studied the generalized flows where two criteria (expected cost and risk) were involved (the authors assumed that the risk can be represented by a function convex with respect to the flow, which is, however, not always true; see below). Bi- and Multi-criteria Transportation Problems were discussed also e.g. by Aneja and Nair (1979), Gupta and Gupta (1983), Shi (1995), Li (2000), Basu and Acharya (2002), Khurana and Arora (2011), Kesavarz and Khorram (2011) and Kumar et al. (2012). The (linear) Generalized Transportation Problem in the multi-criteria version was studied by Gen et al. (1999), among others.

In this paper we present a method for finding efficient solutions of the Bi-criteria Stochastic Generalized Transportation Problem with two criteria: expected cost and variance. In Section 2 the problem is formulated. In Sections 3 and 4 the algorithm, together with its theoretical justification, is presented. Section 5 contains an illustrative example. The results of computational experiments are presented in Section 6. Section 7 contains final remarks.

2 Problem formulation

In the Generalized Transportation Problem, the goal is to minimize the transportation costs of a uniform good delivered from *m* supply points to *n* destination points. The amount of the transported good changes during the transportation process. More precisely, the amount delivered to demand point *j* from supply point *i* is equal to $r_{ij}x_{ij}$, where x_{ij} is the amount of the good that leaves supply point *i* and r_{ij} is the *reduction ratio*. The unit transportation costs c_{ij} are constant, the demand b_j of every demand point *j* has to be satisfied and the supply a_i of any supply point *i* cannot be exceeded. The model looks as follows:

$$\min\left\{f(x)=\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}\right\},$$

s. t.

$$\sum_{i=1}^{m}r_{ij}x_{ij}=b_j, j=1,\dots,n,$$

$$\sum_{j=1}^{n}x_{ij}\le a_i, i=1,\dots,m,$$

$$x_{ij}\ge 0, i=1,\dots,m, j=1,\dots,n.$$

In the Stochastic GTP (SGTP), the demands b_j are independent continuous random variables X_j with density functions φ_j. We will assume that for every $j = 1, \dots, n$ and for every $x > 0$,

$$\varphi_j(x) > 0.$$

The unit surplus cost $s_j^{(1)}$ and the unit shortage cost $s_j^{(2)}$ are defined for every destination point j. This implies that the expected extra cost at the destination j is equal to:

$$f_j(x_j) = s_j^{(1)} \int_0^{x_j} (x_j - t)\varphi_j(t)dt + s_j^{(2)} \int_{x_j}^{\infty} (t - x_j)\varphi_j(t)dt.$$

Using elementary transformations and integrating by parts, we obtain that:

$$\begin{aligned} f_j(x_j) &= s_j^{(2)} \int_0^{\infty} (t - x_j)\varphi_j(t)dt + \left(s_j^{(1)} + s_j^{(2)}\right) \int_0^{x_j} (x_j - t)\varphi_j(t)dt = \\ &= s_j^{(2)} \left(\int_0^{\infty} t\varphi_j(t)dt - x_j \int_0^{\infty} \varphi_j(t)dt\right) + \\ &+ \left(s_j^{(1)} + s_j^{(2)}\right) \left(\left[(x_j - t)\Phi_j(t)\right]_0^{x_j} + \int_0^{x_j} \Phi_j(t)dt\right) = \\ &= s_j^{(2)}\left(E(X_j) - x_j\right) + \left(s_j^{(1)} + s_j^{(2)}\right) \int_0^{x_j} \Phi_j(t)dt, \end{aligned}$$

where Φ_j is the cumulative distribution function of the demand at destination j (the last equality uses the fact that $\Phi_j(0) = 0$).

Finally, the SGTP takes the form:

$$\min\left\{f(x) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j(x_j)\right\},$$

s. t.

$$\sum_{i=1}^{m} r_{ij}x_{ij} = x_j, j = 1, \dots, n,$$

$$\sum_{j=1}^{n} x_{ij} \le a_i, i = 1, \dots, m,$$

$$x_{ij} \ge 0, i = 1, \dots, m, j = 1, \dots, n.$$

The first derivative of the expected cost function has the form:

$$f_j^{'}(x_j) = -s_j^{(2)} + \left(s_j^{(1)} + s_j^{(2)}\right) \Phi_j(t),$$

while the second derivative is equal to:

$$f_j^{''}(x_j) = \left(s_j^{(1)} + s_j^{(2)}\right) \varphi_j(t).$$

This means that each function f_j is twice differentiable and strictly convex on the interval where $\varphi_j(t) > 0$. This allows to use the corresponding version of the Equalization Method (Anholcer, 2012 and 2015) to solve this problem.

Of course it may happen that a Decision Maker considers the transportation costs, shortage costs and surplus costs as not equally important. In such a situation one could use three criteria instead of one, or even when using one objective, one could still introduce weights, reflecting the Decision Maker's preferences. However, this would not change the structure or the general form of the resulting weighting problem, discussed in Section 3 (Observation 1).

The second criterion of interest is variance. The formula for variance for destination j is:

$$g_j(x_j) = p_j(x_j) - q_j(x_j),$$

where:

$$p_j(x_j) = \left(s_j^{(1)}\right)^2 \int_0^{x_j} (x_j - t)^2 \varphi_j(t)dt + \left(s_j^{(2)}\right)^2 \int_{x_j}^{\infty} (t - x_j)^2 \varphi_j(t)dt$$

and:

$$q_j(x_j) = \left(f_j(x_j)\right)^2.$$

One can see that:

$$p_j'(x_j) = 2\left(s_j^{(1)}\right)^2 \int_0^{x_j} (x_j - t)\varphi_j(t)dt + 2\left(s_j^{(2)}\right)^2 \int_{x_j}^{\infty} (x_j - t)\varphi_j(t)dt$$

and:

$$p_j''(x_j) = 2\left(s_j^{(1)}\right)^2 \int_0^{x_j} \varphi_j(t)dt + 2\left(s_j^{(2)}\right)^2 \int_{x_j}^{\infty} \varphi_j(t)dt.$$

Moreover:

$$q_j'(x_j) = 2f_j(x_j)f_j'(x_j)$$

and:

$$q_j''(x_j) = 2[f_j'(x_j)]^2 + 2f_j(x_j)f_j''(x_j).$$

This means that each of the functions $g_j(x_j)$ is a twice differentiable DC-function. Namely, it is the difference of two convex functions, which are strictly convex if $\varphi_j(x_j) > 0$. However, in general, the functions g_j do not need to be convex.

As the demands are independent random variables, the variance of total extra cost is equal to the sum of the variances at the destination points. Thus the bi-criteria problem (BSGTP) takes the form:

$$\min\left\{f(x) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j(x_j)\right\},$$

$$\min\left\{g(x) = \sum_{j=1}^{n} g_j(x_j)\right\},$$

s. t.

$$\sum_{i=1}^{m} r_{ij} x_{ij} = x_j, j = 1, \dots, n,$$

$$\sum_{j=1}^{n} x_{ij} \le a_i, i = 1, \dots, m,$$

$$x_{ij} \ge 0, i = 1, \dots, m, j = 1, \dots, n.$$

Usually the two objective functions have different minima. Our goal is to find a solution method that finds the efficient (Pareto-optimal) solutions.

3 Algorithm – the main idea

Let S denote the set of all feasible solutions of the BSGTP. The problem may be rewritten as:

$\min f(x),$

$\min g(x),$

s. t.

$x \in S.$

The following observation is a corollary from the well-known result about the efficiency of the solution to the weighting problem (see e.g. Miettinen, 1998, p. 78, Theorem 3.1.2).

Observation 1

If $x^\star$ is, for some $\lambda > 0$, an optimal solution to the problem:

$\min h(x) = f(x) + \lambda g(x)$

s. t.

$x \in S,$

then it is a Pareto-optimal solution of the BSGTP.

Minimizing $h(x)$ on S always leads to an efficient solution. The problem obtains then the form of a GTP with a nonlinear objective function. The function $h(x)$ is not necessarily convex, but it is a separable function in which each summand is a DC-function. Thus one can use a branch-and-bound method to determine an exact solution. We will discuss such a method in the next section.

4 Algorithm – the details

The method that we are going to present uses the ideas discussed by Falk and Soland (1969), as well as by Holmberg and Tuy (1999). Assume that the variable x_j is bounded from below and from above: $l_j \le x_j \le u_j$. Since the function $q_j(x_j)$ is convex, we have:

$$q_j(x_j) \le r_j(x_j; l_j, u_j)$$

for $l_j \le x_j \le u_j$, where:

$$r_j(x_j; l_j, u_j) = q_j(l_j) + \frac{x_j - l_j}{u_j - l_j}\left(q_j(u_j) - q_j(l_j)\right)$$

is a linear function such that:

$$r_j(l_j; l_j, u_j) = q_j(l_j)$$

and:

$$r_j(u_j; l_j, u_j) = q_j(u_j).$$

This means that for each index j we have:

$$g_j(x_j) = p_j(x_j) - q_j(x_j) \ge p_j(x_j) - r_j(x_j; l_j, u_j) = g_j^\star(x_j; l_j; u_j).$$

One can see that $g_j^\star(x_j; l_j; u_j)$ is a lower estimate of $g_j(x_j)$ on the interval $[l_j, u_j]$. Let l be the vector of the lower bounds and u the vector of the upper bounds. Let:

$$h^\star(x; l; u) = \sum_{j=1}^{n} \left(f_j(x_j) + \lambda g_j^\star(x_j; l_j; u_j)\right).$$

Of course, $h^\star(x; l; u)$ is a lower estimate of $h(x)$ on the generalized rectangle defined by the inequalities $l \le x \le u$. This means that the new problem:

$\min h^\star(x; l; u)$

s. t.

$x \in S,$

has the form of an SGTP and can be solved using the Equalization Method (see Anholcer, 2012 and 2015). Note that no additional constraints are introduced, so the set of feasible solutions does not change.

The rule of branching is as follows. After solving the problem with function $h^\star(x; l; u)$, we check whether the solution is satisfactory for some predefined accuracy level ε. If it is not, we choose j for which the difference $r_j(x_j; l_j; u_j) - - q_j(x_j)$ is the largest and define two child problems by setting $l_j \coloneqq x_j$ and $u_j \coloneqq x_j$, respectively, for the new problems (recall that we do not change the set of feasible solutions; those values are used only to find the formula of the lower estimate function).

Finally, we can write the algorithm as follows (U_h and U_x denote the upper bound on the optimal value of the objective and the point at which this value is reached, respectively; for a given node v of the solution tree, $L(v)$ and $P(v)$ denote the lower bound on the optimal value of the objective and the corresponding convex problem).

Algorithm 1: The Branch and Bound Method for BSGTP
Input: initial problem, the value of $\lambda > 0$, accuracy level ε.
Output: Pareto-optimal solution $x^\star$.

1. *Initial solution*. Let the initial bounds for each x_j be:
$$l_j = 0, u_j = \sum_{i=1}^{m} r_{ij} a_i.$$
Solve (using the Equalization Method) the corresponding problem $P(v_0)$:
$\min h^\star(x; l; u)$
s. t.
$$x \in S.$$
Assume that the obtained optimum is $x^\star$. Set $U_x = x^\star$, $U_h = h(x^\star)$, and $L(v_0) = h^\star(x^\star; l; u)$, where v_0 is the root of the solution tree.
Go to step 2.
2. *Checking the optimality*. Find an active node $v^\star$, for which $L(v)$ has the minimum value. If:
$$|U_h - L(v^\star)| < \varepsilon,$$
then STOP. The solution U_x is satisfactory. Otherwise go to step 3.
3. *Branching and bounding*. Consider the problem $P(v^\star)$. Let $j^\star$ be an index j for which the difference $r_j(x_j; l_j; u_j) - q_j(x_j)$ is the largest. Remove the node $v^\star$ from the set of active nodes. Add two new active nodes v' and v'' and define the corresponding convex problems. To obtain $P(v')$, set $u_{j^\star} = x^\star_{j^\star}$ in $P(v^\star)$. To obtain $P(v'')$, set $l_{j^\star} = x^\star_{j^\star}$ in $P(v^\star)$. Let us denote the new bounding vectors by l', u', l'', u'', respectively.
Solve $P(v')$ and $P(v'')$ using the Equalization Method. Assume that the obtained optima are x' and x'', respectively. If $U_h > h(x')$, then set $U_x = x'$ and $U_h = h(x')$. Set $L(v') = h^\star(x'; l'; u')$. If $U_h > h(x'')$, then set $U_x = x''$ and $U_h = h(x'')$. Set $L(v'') = h^\star(x''; l''; u'')$.
Close all the active nodes v for which $L(v) > U_h - \varepsilon$.
Go back to step 2.

5 Illustrative example

Let us analyze a simple example that illustrates the algorithm. Assume that there are two supply points with the supply equal to $a_1 = a_2 = 15$ and three destinations, with uniform demand distribution given by the density functions:
$$\varphi_1(x_1) = \begin{cases} \frac{1}{10}, x \in [0,10], \\ 0, x \notin [0,10], \end{cases}$$

$$\varphi_2(x_2)=\begin{cases}\frac{1}{12}, x\in[0,12],\\ 0, x\notin[0,12],\end{cases}$$

$$\varphi_3(x_3)=\begin{cases}\frac{1}{14}, x\in[0,14],\\ 0, x\notin[0,14].\end{cases}$$

The unit transportation costs c_{ij}, the reduction ratios r_{ij}, the surplus costs $s_j^{(1)}$ and the shortage costs $s_j^{(2)}$ are given in the Table 1.

Table 1: Problem parameters

Parameter	$j = 1$	$j = 2$	$j = 3$	Parameter	$j = 1$	$j = 2$	$j = 3$
c_{1j}	5	3	2	r_{1j}	0.92	0.95	0.93
c_{2j}	2	1	4	r_{2j}	0.91	0.87	0.92
$s_j^{(1)}$	1	4	5	$s_j^{(2)}$	4	6	10

Assume that we are interested in finding the solution for $\lambda = 0.5$ and $\varepsilon = 0.01$. The functions of expected costs are given by:

$$f_1(x_1)=\begin{cases}\frac{1}{4}x^2-4x+20, x\in[0,10],\\ x-5, x>10,\end{cases}$$

$$f_2(x_2)=\begin{cases}\frac{5}{12}x^2-6x+36, x\in[0,12],\\ 4x-24, x>12,\end{cases}$$

$$f_3(x_3)=\begin{cases}\frac{15}{28}x^2-10x+70, x\in[0,14],\\ 5x-35, x>14.\end{cases}$$

The functions p_j have the form:

$$p_1(x_1)=\begin{cases}-\frac{1}{2}x^3+16x^2-160x+\frac{1600}{3}, x\in[0,10],\\ x^2-10x+\frac{100}{3}, x>10,\end{cases}$$

$$p_2(x_2)=\begin{cases}-\frac{5}{9}x^3+36x^2-432x+1728, x\in[0,12],\\ 16x^2-192x+768, x>12,\end{cases}$$

$$p_3(x_3)=\begin{cases}-\frac{25}{14}x^3+100x^2-1400x+\frac{19600}{3}, x\in[0,14],\\ 25x^2-350x+\frac{4900}{3}, x>14.\end{cases}$$

The first bounds on the variables (corresponding to the node v_0) are defined by $0 \leq x_1 \leq 27.45$, $0 \leq x_2 \leq 27.3$ and $0 \leq x_3 \leq 27.75$. The respective linear estimates of q_j are equal to:

$$r_1(x_1) = 400 + \frac{x_1 - 0}{27.45 - 0}(504.0025 - 400)$$
$$r_2(x_2) = 1296 + \frac{x_2 - 0}{27.3 - 0}(7259.04 - 1296)$$
$$r_3(x_3) = 4900 + \frac{x_3 - 0}{27.75 - 0}(10764.0625 - 4900)$$

The solution of the problem $P(v_0)$ is as follows:

Table 2: Solution

Value	$j = 1$	$j = 2$	$j = 3$	**Value**	$j = 1$	$j = 2$	$j = 3$
x_{1j}	0.00	3.02	11.98	$p_j(x_j)$	74.03	628.64	878.24
x_{2j}	5.40	9.60	0.00	$q_j(x_j)$	40.67	446.96	628.89
x_j	4.92	11.22	11.14	$r_j(x_j)$	418.63	3747.27	7253.62
$f_j(x_j)$	6.38	21.14	25.08	$r_j(x_j) - q_j(x_j)$	377.96	3300.30	6624.73

The objectives of the initial problem and of the convex problem are $h(x^\star) = 338.215$ and $h^\star(x^\star) = -4813.279$. This means that $L(v_0) = -4813.279$ and $U_h = U(v_0) = 338.215$. Since v_0 is the only (active) node and $|U_h - L(v_0)| > \varepsilon$, we perform branching with respect to the variable x_3 (the maximum difference $r_j(x_j) - q_j(x_j)$ is $r_3(x_3) - q_3(x_3)$). Since $x_3 = 11.138$, the new nodes v_1 and v_2 will correspond to the additional constraints $x_3 \leq 11.138$ and $x_3 \geq 11.138$, respectively. After defining the functions $r_j(x_j)$ and solving the new problems, we obtain $L(v_1) = -2289.611$, $U(v_1) = 448.384$, $L(v_2) = -1995.718$ and $U(v_2) = 489.812$. $U(v_1) > U_h$ $U(v_2) > U_h$, so U_h does not change (and U_x remains the optimal solution of $P(v_0)$). Now the two active nodes are v_1 and v_2. The function L is minimized at v_1 and $|U_h - L(v_1)| > \varepsilon$, so we perform branching and continue in this way. At some moment we obtain $U(v_8) = 224.145$, which means that starting from this moment $U_h = 224.145$ and U_x becomes the optimal solution of $P(v_8)$. After a few more iterations, after branching at v_{13}, we obtain, in particular, that at v_{22} we have $L(v_{22}) = 259.418$, which means that $L(v_{22}) > U_h - \varepsilon$ and we close node v_{22}. The details for the first 51 nodes have been collected in Table 3 below. In each row, the label of node v_j is followed by the label of the parent node; two child nodes, order of branching, branching variable and its value (if the branching was performed at v_j); the values of both objectives: $U(v_j)$ and $L(v_j)$; and the actual value of U_h. At the stage presented in the table, 25 nodes are still active (A), four have been closed (C), and the branching has been performed at the other nodes.

Table 3: Beginning of the algorithm

Node (v)	**Parent node**	**Child nodes**	**Checking order**	**Branching variable**	**Branching value**	**U(v)**	**L(v)**	**Uh**
1	2	3	4	5	6	7	8	9
v0	none (root)	v1, v2	1	x3	11.138	338.215	– 4813.279	338.215
v1	v0	v3, v4	2	x2	12.393	448.384	– 2289.611	338.215
v2	v0	v5, v6	3	x2	9.393	489.812	– 1995.718	338.215
v3	v1	v7, v8	4	x3	6.089	367.325	– 1080.794	338.215
v4	v1	v9, v10	5	x3	5.672	541.545	– 936.929	338.215
v5	v2	v11, v12	6	x3	18.044	445.815	– 892.492	338.215
v6	v2	v13, v14	7	x3	15.495	558.560	– 587.665	338.215
v7	v3	v17, v18	9	x2	6.072	580.604	– 357.557	224.145
v8	v3	v15, v16	8	x2	6.072	224.145	– 447.747	224.145
v9	v4	v25, v26	13	x2	18.395	762.404	– 194.699	224.145
v10	v4	v23, v24	12	x2	16.601	387.967	– 222.306	224.145
v11	v5	v19, v20	10	x2	5.100	375.752	– 271.061	224.145
v12	v5	v31, v32	16	x2	3.641	523.392	– 128.550	224.145
v13	v6	v21, v22	11	x2	14.436	411.793	– 249.963	224.145
v14	v6	A	A	A	A	551.582	110.603	224.145
v15	v8	v29, v30	15	x1	7.933	281.030	– 134.819	224.145
v16	v8	v27, v28	14	x1	7.933	240.024	– 134.905	224.145
v17	v7	v39, v40	20	x3	4.163	637.550	– 44.627	224.145
v18	v7	v37, v38	19	x3	4.161	590.147	– 49.242	224.145
v19	v11	v35, v36	18	x1	8.462	429.345	– 67.279	224.145
v20	v11	v33, v34	17	x1	6.805	345.258	– 85.043	224.145
v21	v13	v49, v50	25	x1	5.181	344.501	56.284	224.145
v22	v13	C	C	C	C	445.887	259.418	224.145
v23	v10	v41, v42	21	x1	5.625	321.107	– 3.570	224.145
v24	v10	A	A	A	A	445.700	115.497	224.145
v25	v9	A	A	A	A	708.643	120.357	224.145
v26	v9	A	A	A	A	819.072	186.977	224.145
v27	v16	v43, v44	22	x3	8.893	245.454	9.312	224.145
v28	v16	A	A	A	A	246.101	57.107	224.145
v29	v15	v45, v46	23	x2	3.950	296.393	19.732	224.145
v30	v15	v47, v48	24	x2	3.884	303.556	55.917	224.145
v31	v12	A	A	A	A	583.376	65.850	224.145
v32	v12	A	A	A	A	499.660	102.411	224.145
v33	v20	A	A	A	A	368.272	80.426	224.145
v34	v20	A	A	A	A	340.735	115.675	224.145
v35	v19	A	A	A	A	441.503	76.627	224.145
v36	v19	A	A	A	A	440.998	126.633	224.145
v37	v18	C	C	C	C	721.613	259.574	224.145

Table 3 cont.

1	2	3	4	5	6	7	8	9
v38	v18	A	A	A	A	325.146	77.547	224.145
v39	v17	C	C	C	C	768.302	263.785	224.145
v40	v17	A	A	A	A	372.663	82.115	224.145
v41	v23	A	A	A	A	342.190	160.389	224.145
v42	v23	A	A	A	A	319.241	190.339	224.145
v43	v27	A	A	A	A	255.496	87.144	224.145
v44	v27	A	A	A	A	245.696	81.619	224.145
v45	v29	A	A	A	A	336.275	126.426	224.145
v46	v29	A	A	A	A	241.175	85.048	224.145
v47	v30	A	A	A	A	343.076	163.929	224.145
v48	v30	A	A	A	A	251.624	125.184	224.145
v49	v21	A	A	A	A	404.270	214.555	224.145
v50	v21	C	C	C	C	329.764	246.344	224.145

6 Computational experiments

Test problems were randomly generated and solved with the proposed method. Two types of demand distributions were considered: uniform $U(0,\ u)$ and exponential $Exp(\lambda)$, where u and λ were chosen uniformly at random from the intervals $[15,\ 20)$ and $[0.5,\ 0.6)$, respectively. In both cases unit transportation costs were chosen from the interval $[5,\ 10)$, surplus costs from the interval $[1,\ 2)$, shortage costs from the interval $[5,\ 10)$, reduction ratios from the interval $[0.8,\ 0.9)$ and the supply from each source point from the interval $[10,\ 20)$. The algorithm was implemented in Java SE and run on a personal computer with Intel(R) Core(TM) i7-2670 QM CPU @2.20 GHz. For both types of distributions, 100 randomly generated problems of four sizes were solved: (m, n) = (10, 10), (10, 20), (10, 50) and (20, 50), that is, 800 test problems in total. The running times in seconds (average, standard deviation, minimum and maximum) are presented in Table 4:

Table 4: Running times in seconds

Problem type	**$U(0, u)$ 10×10**	**$U(0, u)$ 10×20**	**$U(0, u)$ 10×50**	**$U(0, u)$ 20×50**	**$Exp(\lambda)$ 10×10**	**$Exp(\lambda)$ 10×20**	**$Exp(\lambda)$ 10×50**	**$Exp(\lambda)$ 20×50**
AVG	0.16	0.95	159.79	2316.95	2.35	8.10	1039.79	5445.41
ST DEV	0.45	2.72	181.66	1255.08	10.82	45.34	1183.32	3449.44
MIN	0.02	0.12	16.40	289.35	0.12	0.52	96.99	718.73
MAX	6.64	37.41	1538.15	10128.26	207.66	1204.13	9029.58	29031.47

As we can see, the algorithm can be regarded as fast: the running times are less than a second or a few seconds in the case of the smaller problems and about one hour in the case of the bigger problems (up to 1000 variables). However, one needs to remember that the branch and bound methods are super-polynomial, which means that the solution times may grow very rapidly with the increasing size of the problem.

7 Final remarks

The algorithm presented above allows to find the Pareto-optimal solutions of the Bi-criteria Stochastic Generalized Transportation Problem. In this type of problem we assume that one of the criteria is the sum of the transportation cost and the expected total extra cost of all the deliveries. The second criterion is the risk measured by the variance of the expected extra cost. The resulting problem, which allows to find the efficient solutions, is a non-convex optimization problem that can be solved with a branch-and-bound method described in the paper. The subproblems solved in the nodes of the solution tree are of SGTP form and therefore can be solved using the Equalization Method. The numerical evidence shows that the presented algorithm allows to solve problems of average size in a reasonable time.

References

Ahuja R.K., Magnanti T.L., Orlin J.B. (1993), *Network Flows. Theory, Algorithms and Applications*, Prentice Hall.

Aneja Y.P., Nair K.P.K. (1979), *Bicriteria Transportation Problem*, Management Science 25, 73-78.

Anholcer M. (2005), *Zbieżność metody wyrównań dla nieliniowych zadań alokacji* [in:] K. Piasecki, W. Sikora (eds.), *Z prac Katedry Badań Operacyjnych*, Zeszyty Naukowe Akademii Ekonomicznej w Poznaniu 64, 183-198.

Anholcer M. (2008a), *Analiza porównawcza wybranych algorytmów rozwiązywania nieliniowych zadań alokacji dóbr jednorodnych*, Wydawnictwo Akademii Ekonomicznej, Poznań.

Anholcer M. (2008b), *Porównanie działania wybranych algorytmów rozwiązywania nieliniowych zadań alokacji* [in:] D. Kopańska-Bródka (ed.), *Metody i zastosowania badań operacyjnych '2007*, Prace Naukowe Akademii Ekonomicznej, Katowice, 9-25.

Anholcer M. (2012), *Algorithm for Stochastic Generalized Transportation Problem*, Operations Research and Decisions, Vol. 22, No. 4, 9-20.

Anholcer M. (2013), *Algorithm for Bi-criteria Stochastic Generalized Transportation Problem*, Multiple Criteria Decision Making, Vol. 8, 5-17.

Anholcer M. (2015), *The Nonlinear Generalized Transportation Problem with Convex Costs*, Croatian Operational Research Review, Vol. 6, No. 1, 225-239.

Anholcer M., Kawa A. (2012), *Optimization of Supply Chain via Reduction of Complaints Ratio*, Lecture Notes in Computer Science, Vol. 7327, 622-628.

Balas E. (1966), *The Dual Method for the Generalized Transportation Problem*, Management Science, Vol. 12, No. 7, Series A, Sciences, 555-568.

Balas E., Ivanescu P.L. (1964), *On the Generalized Transportation Problem*, Management Science, Vol. 11, No. 1, Series A, Sciences, 188-202.

Basu M., Acharya D.P. (2002), *On Quadratic Fractional Generalized Solid Bi-criterion Transportation Problem*, The Korean Journal of Computational & Applied Mathematics, 10 (1-2), 131-143.

Chen M.S., Chuang C.C. (2000), *An Extended Newsboy Problem with Shortage-level Constraints*, International Journal of Production Economics, 67, 269-337.

Cooper L., LeBlanc L. (1977), *Stochastic Transportation Problems and Other Network Related Convex Problems*, Naval Research Logistics Quarterly, 24 (2), 327-337.

Edgeworth F.Y. (1888), *The Mathematical Theory of Banking*, Journal of the Royal Statistical Society, 53, 113-127.

Falk J.E., Soland R.M. (1969), *An Algorithm for Separable Nonconvex Programming Problems*, Management Science, 15 (9), 550-569.

Gen M., Ida K., Choi J. (1999), *Improved Genetic Algorithm for Bicriteria Generalized Transportation Problem*, Research Reports Ashikaga Institute of Technology, 28, 297-303.

Glover F., Klingman D., Napier A. (1972), *Basic Dual Feasible Solutions for a Class of Generalized Networks*, Operations Research, Vol. 20, No. 1, 126-136.

Goldberg A.V., Plotkin S.A., Tardos E. (1988), *Combinatorial Algorithms for the Generalized Circulation Problem*, SFCS'88 Proceedings of the 29th Annual Symposium on Foundations of Computer Science, 432-443.

Goto H. (2013), *Multi-Item Newsvendor Problem with an Equality Resource Constraint*, Asia-Pacific Journal of Operational Research, 30, 1250041-1−1250041-17.

Gupta B., Gupta R. (1983), *Multi-criteria Simplex Method for a Linear Multiple-objective Transportation Problem*, Indian Journal of Pure and Applied Mathematics, 14 (2), 222-232.

Holmberg K. (1995), *Efficient Decomposition and Linearization Methods for the Stochastic Transportation Problem*, Computational Optimization and Applications, 4, 293-316.

Holmberg K., Jörnsten K.O. (1984), *Cross Decomposition Applied to the Stochastic Transportation Problem*, European Journal of Operational Research, 17, 361-368.

Holmberg K., Tuy H. (1999), *A Production-Transportation Problem with Stochastic Demand and Concave Production Costs*, Mathematical Programming, 85, 157-179.

Khouja M., Mehrez A., Rabinowitz G. (1996), *A Two-item Newsboy Problem with Substitutability*, International Journal of Production Economics, 44, 267- 275.

Khurana A., Arora S.R. (2011), *Fixed Charge Bi-criterion Indefinite Quadratic Transportation Problem with Enhanced Flow*, Revista Investigación Operacional, 32, 133-145.

Kumar S.K., Lal I.B., Lal S.B. (2012), *Fixed-charge Bi-criterion Transportation Problem*, International Journal of Computer Application, 2 (1), 39-42.

Li J. (2000), *A Dynamic Transportation Model with Multiple Criteria and Multiple Constraint Levels*, Mathematical and Computer Modelling, 32, 1193-1208.

Li F., Jin C., Wang L. (2014), *Quasi-linear Stochastic Programming Model Based on Expectation and Variance and Its Application in Transportation Problem*, Applied Mathematical Modelling, 38, 1919-1928.

Lourie J.R. (1964), *Topology and Computation of the Generalized Transportation Problem*, Management Science, Vol. 11, No. 1, Series A, Sciences, 177-187.

Miettinen K. (1998), *Nonlinear Multiobjective Optimization*, International Series in Operations Research & Management Science, 12, Springer.

Nagurney A., Yu M., Masoumi A.H., Nagurney L.S. (2013), *Networks Against Time. Supply Chain Analytics for Perishable Products*, Springer Briefs in Optimization, Springer.

Qi L. (1985), *Forest Iteration Method for Stochastic Transportation Problem*, Mathematical Programming Study, 25, 142-163.

Qi L. (1987), *The A-Forest Iteration Method for the Stochastic Generalized Transportation Problem*, Mathematics of Operations Research, Vol. 12, No. 1, 1-21.

Shi Y. (1995), *A Transportation Model with Multiple Criteria and Multiple Constraint Levels*, Mathematical and Computer Modelling, 21 (4), 13-28.

Sikora W. (1993), *Modele i metody optymalnej dystrybucji dóbr*, Zeszyty naukowe − seria II, Prace doktorskie i habilitacyjne, Akademia Ekonomiczna, Poznań.

Sikora W., Runka H., Pyrzyński D. (1991), *Optymalizacja przepływów w sferze dystrybucji dóbr jednorodnych*, Grant No. H\12\209\90-2, Akademia Ekonomiczna, Poznań.

Şen A., Zhang A.X. (1999), *The Newsboy Problem with Multiple Demand Classes*, IEE Transactions, 31, 431-444.

Yang P.C., Wee H.M., Chung S.L., Kang S.H. (2007), *Constrained Optimization of a Newsboy Problem with Return Policy Using KKT Conditions and GA*, New Trends in Applied Artificial Intelligence, Lecture Notes in Computer Science, Vol. 4570, 227-237.

Wayne K.D. (2002), *A Polynomial Combinatorial Algorithm for Generalized Minimum Cost Flow*, Mathematics of Operations Research, Vol. 27, No. 3, 445-459.

Williams A.C. (1963), *A Stochastic Transportation Problem*, Operations Research, 11, 759-770.