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THREE WELFARE ORDERINGS THAT ARE FULLY COMPARABLE REVISITED

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Abstract

We are concerned with welfare orderings on the set of evaluation vectors. In our framework the number of agents, criteria or states of nature is fixed and an evaluation vector assigns a real valued evaluation to each criteria, agent or state of nature. Hence the space of evaluation vectors is a finite dimensional Euclidean space. In such a context we provide axiomatic characterizations of the utilitarian, maximin and leximin welfare orderings. The axiomatic characterization of the utilitarian welfare ordering is based on a quasi-linearity property. The axiomatic characterizations of the maximin and leximin welfare orderings are obtained by suitably modifying the axioms used by Barbera and Jackson (1988).

Keywords: social Welfare Orderings, Maximin, Leximin.

1 Introduction

In this paper we are concerned with axiomatic characterizations of orderings (reflexive, complete and transitive binary relations) defined on the set of finite dimensional evaluation vectors. We refer to these orderings as welfare orderings. The economic interpretation of an evaluation vector depends on the context. In the case that the context is the traditional one discussed in Amartya Sen's extension of Arrowian social welfare function (which Sen refers to as social welfare functional), then an evaluation vector is the vector of utilities obtained (or evaluations assigned) by each individual in a society to a particular social state. In the case that the context is the one about rational decision making by a single individual, there are two possible sub-cases each with its own interpretation and terminology. One is the scenario of multi-criteria or multi-attribute decision

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making. In this case each coordinate of an evaluation vector is the evaluation along a particular criterion and the evaluation vector itself is the ensemble of criterion-wise evaluations for the entire list of criteria one is concerned with. The second scenario is the one concerning decision making under complete uncertainty (more popularly known as ambiguity these days) where an agent faces the possibility of confronting in a future period exactly one of a finite set of uncertain states of nature. In this case each coordinate of an evaluation vector is the utility that one attains if a particular state of nature is realised and the evaluation vector itself is the ensemble of state dependent utilities for the entire list of states of nature exactly one of which is going to be realized at a future date. In this final scenario that we consider, the decision maker has no prior probabilities over the set of future states of nature. In each case that we have discussed so far, the problem is to obtain a welfare ordering over the set of evaluation vectors.

We consider three different welfare orderings which are very well known in welfare economics and share a common characteristic in that all three of them satisfy full comparability. Full comparability says if the evaluation vector $\mathbf{x}$ is at least as good or favourable as evaluation vector $\mathbf{y}$, then the evaluation vector $\mathbf{x}'$ should also be at least as good or favourable as the evaluation vector $\mathbf{y}'$, where the evaluation vectors $\mathbf{x}'$ and $\mathbf{y}'$ are obtained from $\mathbf{x}$ and $\mathbf{y}$ by multiplying each and every coordinate of $\mathbf{x}$ and $\mathbf{y}$ by the same positive real number (i.e. making the same change of scale along all directions) and then shifting the origin by the same amount in all directions.

The first welfare ordering that we consider is the utilitarian welfare ordering. Using well known results for numerical representation of quasi-linear preferences of a consumer as discussed in microeconomic consumer choice theory and with minimal investment in new technology on our part we are able to arrive at a completely new axiomatic characterization of the utilitarian welfare ordering. The result we establish says that a welfare ordering satisfies strict domination, continuity and quasi-linearity in every component if and only if it is a utilitarian welfare ordering. Quasi-linearity in a component means that the relation between two evaluation vectors is preserved if the evaluation at that component (or coordinate) is increased by the same real number for the two evaluation vectors. Quasi-linearity in every component means that that this property holds for every component.

The next two welfare orderings we consider are the maximin welfare ordering and the leximin welfare ordering. Our analysis of these two welfare orderings parallels the discussion of these two welfare orderings that is reported in Barbera and Jackson (1988). Unlike Barbera and Jackson (1988), in our framework the number of components (agents/criteria/states of nature) is fixed. In such a context we provide axiomatic characterizations of the maximin and leximin welfare orderings by suitably modifying the axioms used by Barbera and Jackson (1988). In the context of social welfare functionals and hence social welfare orderings, the maximin welfare ordering is best described as the "dictatorship of the least well off" and is therefore incompatible with any concept of negative liberty. On the other hand the leximin welfare ordering ranks one evaluation vector over the other, if and only if at the least rank where the two evaluation vectors disagree, the first vector has a higher evaluation than the second. Four of the seven axioms that we use are exactly those used by Barbera and Jackson (1988). Our proof of theorem 1 coincides almost word for word with the proof of theorem 1 in their paper. However, since our framework is different our results are different from their results and proofs of results merit mentioning, however close they may be to the corresponding proofs in the earlier work.

Of all the axioms we use in the characterization of maximin and leximin welfare orderings, only two are really unfamiliar to those who are acquainted with the literature on welfare orderings and therefore require some motivation. These two axioms are convexity with respect to duplicated evaluations and improvement impatience. Convexity is best explained in a two agent social welfare framework. In a two agent social welfare framework, convexity says that if an evaluation vector is preferred to a given perfectly egalitarian evaluation vector, then a third evaluation vector that is obtained from the first by replacing the evaluation of the "better off" agent by the average evaluation of the first vector is also preferred to the perfectly egalitarian evaluation vector. Hence, reasons for preferring a non-egalitarian evaluation vector to a perfectly egalitarian one are required to be quite compelling. The motivation for improvement impatience is much simpler. If there are two evaluation vectors sharing a common minimum evaluation, and there are just two different evaluation values in each evaluation vector, then the one which has fewer components getting the minimum evaluation is the preferred evaluation vector. In the context of social welfare functionals, this clearly points towards a social welfare ordering with a favourable bias towards utility distributions with fewer "least well-off" individuals.

For the broad framework and general definitions of utilitarian, maximin and leximin welfare orderings as defined in our paper one may refer to d'Aspremont (1985). We however try to adhere to the equivalent definitions of maximin and leximin that is available in Barbera and Jackson (1988). Since this paper relates to work done thirty years ago, a more recent survey of the literature such as the one by Bossert and Weymark (2004), should convince the reader that our results are original and no duplication of past effort occurs in our work.

2 The Model

Let N = {1,2,...,L} for some positive integer L, denote the set of individuals/ criteria/states of nature. An **evaluation vector** is an element of $\mathbb{R}^N$. A binary relation R on $\mathbb{R}^N$ is a subset of $\mathbb{R}^N \times \mathbb{R}^N$. If $(x,y) \in R$, then we write it as xRy. An ordering on $\mathbb{R}^N$ is a complete, reflexive and transitive binary relation on $\mathbb{R}^N$. If R is a binary relation on $\mathbb{R}^N$ then let P denote the asymmetric part and I denote the symmetric part of R. Henceforth we shall refer to binary relations on $\mathbb{R}^N$ as binary relations. Given a binary relation R and $X \subset \mathbb{R}^N \times \mathbb{R}^N$, let R|X denote $R \cap X$. R|X is called the **restriction of** R to X. We will refer to orderings on $\mathbb{R}^N$ as **welfare orderings.**

It may appear that the concept of a welfare ordering is very restrictive since we require a welfare ordering to be an ordering on $\mathbb{R}^N$. In welfare economics, we are often confronted with orderings on $\mathbb{R}^N_+$ as for instance the Nash (1950) welfare ordering. However such orderings can be harmlessly extended to all of $\mathbb{R}^N$ as the following definition reveals.

The **Nash welfare ordering** R_{Na} is defined as follows: let u be the real valued function defined on $\mathbb{R}^N$ such that for all $x \in \mathbb{R}^N_+$, $u_{Na}(x) = \prod_{i=1}^{L} x_i$ and for all $x \in \mathbb{R}^N \backslash \mathbb{R}^N_+$, $u_{Na}(x) = 0$. Then for all $x,y \in \mathbb{R}^N$, $xR_{Na}y$ if and only if $u_{Na}(x) \geq u_{Na}(y)$.

We will not dwell further on the Nash welfare ordering.

Given $x \in \mathbb{R}^N$ and $i \in N$, let x_{-i} denote the vector in $\mathbb{R}^{N\backslash\{i\}}$ such that for all $j \in N\backslash\{i\}$, the j^th^ coordinate of x_{-i} is equal to the j^th^ coordinate of x, i.e. x_j. The vector x can also be written as (x_i, x_{-i}).

Given, $x,y \in \mathbb{R}^N$, (a) $x \geq y$ denotes $x_i \geq y_i$ for all $i \in N$; (b) $x \leq y$ denotes $x_i \leq y_i$ for all $i \in N$; (c) $x > y$ denotes $x \geq y$ and $x \neq y$; (c) $x < y$ denotes $x \leq y$ and $x \neq y$; (d) $x >> y$ denotes $x_i > y_i$ for all $i \in N$.

Notation: Let e denote the vector in $\mathbb{R}^N_+$ all whose coordinates are equal to 1 and for $k \in \{1,\ldots,N\}$, let $e^{(k)}$ denote the k^{th} unit coordinate vector, i.e. the vector whose k^{th} coordinate is equal to 1 and all other coordinates are equal to zero. Then given any $x \in \mathbb{R}^N$, $x = \sum_{k=1}^{L} x_k e^{(k)}$. Further, if $k \in \{1,\ldots,N\}$, then x_{-k} is the vector in $\mathbb{R}^{\{1,\ldots,N\}\backslash\{k\}}$ whose j^{th} coordinate is x_j for $j \neq k$. We may represent x as (x_k, x_{-k}).

For $a \in \mathbb{R}$ and $x \in \mathbb{R}^N$ let $J(a,x) = \{i \in N | x_i \leq a\}$ and let #J(a,x) denote the cardinality of J(a,x).

The **utilitarian welfare ordering** R_U is defined as follows: there exists positive real numbers $\alpha_1, \alpha_2, \ldots, \alpha_L$ such that for all $x,y \in \mathbb{R}^N$, xRy if and only if $\sum_{k=1}^{L} \alpha_k x_k \geq \sum_{k=1}^{L} \alpha_k y_k$.

The **maximin welfare ordering** R_{Mm} is defined as follows: $\forall x,y \in \mathbb{R}^N$, $xP_{Mm}y$ if and only if $\exists a \in \mathbb{R}$ such that $J(a,x) = \phi$ and $J(a,y) \neq \phi$.

The **leximin welfare ordering** R_{Lm} is defined as follows: $\forall x,y \in \mathbb{R}^N$, $xP_{Lm}y$ if and only if $\exists a \in \mathbb{R}$ such that $\#J(a,x) < \#J(a,y)$ and $\#J(b,x) = \#J(b,y)$ for all $b < a$.

We shall be concerned with the following axioms on welfare orderings.

A welfare ordering R is said to satisfy:

(1) **full-comparability** if for all $x^1, x^2, y^1, y^2 \in \mathbb{R}^N$ satisfying $x_k^2 = ax_k^1 + b$, $y_k^2 = ay_k^1 + b$ for all $k \in N$, where a is a strictly positive real number and b is any real number, it is the case that x^1Ry^1 implies x^2Ry^2.

All the orderings discussed in this paper satisfy full-comparability.

A welfare ordering R is said to satisfy:

(2) **domination** if for all $x,y \in \mathbb{R}^N$, $x \geq y$ implies xRy and $x >> y$ implies xPy.

(3) **strict domination in the k^{th} component (or component k)** if for all $x,y \in \mathbb{R}^N$, [$x_j = y_j$ for all $j \neq k$ and $x_k > y_k$] implies [x P y];

(4) **strict domination** if for all $x,y \in \mathbb{R}^N$, $x > y$ implies xPy.

(5) **continuity** if for all sequences $\langle x^n | n \in \mathbb{N} \rangle$ and $\langle y^n | n \in \mathbb{N} \rangle$ in $\mathbb{R}^N$ with $\lim_{n\to\infty} x^n = x \in \mathbb{R}^N$ and $\lim_{n\to\infty} y^n = y \in \mathbb{R}^N$, x^nRy^n for all $n \in \mathbb{N}$ implies xRy.

(6) **quasi-linearity in component k** if for all $x,y \in \mathbb{R}^N$, x R y implies $(x + \alpha e^{(k)})R(y + \alpha e^{(k)})$ for all $\alpha > 0$.

(7) **quasi-linearity in all components** if it satisfies quasi-linearity in coordinate k for all $k \in N$.

(8) **symmetry** if for all permutations σ on N such that $\forall x,y, x', y' \in \mathbb{R}^N$ satisfying $x_i' = x_{\sigma(i)}$ and $y_i' = y_{\sigma(i)}$ $\forall i \in N$ it is the case that xRy if and only if $x'Ry'$.

(9) **convexity with respect to duplicated evaluations** if for all $a,b,c \in \mathbb{R}$ with $a \leq b$ and $x,y,z \in \mathbb{R}^N$, $x_1 = y_1 = a$, $x_i = b$, $y_i = \frac{a+b}{2}$ for $i > 1$, $z_i = c$ for all $i \in N$, it is the case that xPz implies yPz.

(10) **strong convexity with respect to duplicated evaluations** if for all $a,b,c \in \mathbb{R}$ and $x,y,z \in \mathbb{R}^N$, $x_1 = y_1 = a$, $x_i = b$, $y_i = \frac{a+b}{2}$ for $i > 1$, $z_i = c$ for all $i \in N$, it is the case that xPz implies yPz.

(11) **improvement impatience** if for all $a,b,c \in \mathbb{R}$, with $b > a$, $c > a$, $x,y \in \mathbb{R}^N$ and $K \in \{1,...,L-1\}$: $x_i = y_i = a$ $\forall i = 1,...,K$, $y_{K+1} = a$, $x_i = b$ $\forall i \in \{K+1,...,L\}$ and $y_i = c$ $\forall i \in \{K+2,...,L\}$ only if $K + 2 \leq L$, implies xPy.

(12) **shuffling** if for all permutations σ, ρ on N such that $\forall x,y, x', y' \in \mathbb{R}^N$ satisfying $x_i' = x_{\sigma(i)}$ and $y_i' = y_{\rho(i)}$ $\forall i \in N$ it is the case that xRy if and only if $x'Ry'$.

(13) **ascending order separability** if for all $x,y, x', y' \in \mathbb{R}^N$ with $x_j \leq x_{j+1}$, $y_j \leq y_{j+1}$, $x_j' \leq x_{j+1}'$, $y_j' \leq y_{j+1}'$ for all $j = 1,...,L-1$ and any $i \in N$ satisfying $x_i = y_i$, $x_i' = y_i'$, $x_{-i} = x_{-i}'$, $y_{-i} = y_{-i}'$ it is the case that xRy if and only if $x'Ry'$.

(14) **separability** if for all $x,y, x', y' \in \mathbb{R}^N$ and any $i \in N$ satisfying $x_i = y_i$, $x_i' = y_i'$, $x_{-i} = x_{-i}'$, $y_{-i} = y_{-i}'$ it is the case that xRy if and only if $x'Ry'$.

Barbera and Jackson (1988) refer to something very similar to separability as the "sure thing principle". Clearly, separability implies ascending order separability, although the converse is not true. For instance R_{Mm} satisfies ascending order separability but not separability. That R_{Mm} does not satisfy separability is established in the following example.

Example 1

Let $x = (4,3)$. $y = (4,2)$, $x' = (1,3)$, $y' = (1,2)$. Thus, $xP_{Mm}y$ but $x'I_{Mm}y'$. Thus, $y'R_{Mm}x'$ but not $xR_{Mm}y$. This hold in spite of $x_1 = y_1$, $x_1' = y_1'$, $x_2 = x_2'$, $y_2 = y_2'$.

Note that shuffling implies symmetry. However, the converse is not true and shuffling is a much stronger property than symmetry. This will be shown in example 11.

Note further that both symmetry and shuffling are implied by the property known as anonymity.

A welfare ordering R is said to satisfy **anonymity** if for all x, $y \in \mathbb{R}^N$ and permutation σ on N, $y_i = x_{\sigma(i)}$ for all $i \in N$ implies xIy.

It is also the case that our main results remain intact if we replace symmetry and shuffling by anonymity. In fact, since we are concerned with <u>orderings</u> on $\mathbb{R}^N$ anonymity and shuffling are equivalent properties.

Let $\mathfrak{J} = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N | \min_{i \in N} x_i \neq \min_{i \in N} y_i\}$.

The restrictions of R_{Mm} and R_{Lm} to $\mathfrak{J}$ agree with each other.

3 Some well known preliminary results

In this section we present some well known preliminary results which immediately lead to an axiomatic characterization of the utilitarian welfare ordering.

The following two propositions along with their proofs can be found in Rubinstein (2012). Proposition 2 requires proposition 1 for its proof.

Proposition 1

Let R be a welfare ordering that satisfies continuity, domination and strict domination in component k. If R is quasi-linear in component k then there exists a function $v: \mathbb{R}_+^{N \setminus \{k\}} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}_+^N$, x R y if and only if $x_k + v(x_{-k}) \geq y_k + v(y_{-k})$.

Proposition 2

Let R be a welfare ordering that satisfies continuity, domination and strict domination. If R is quasi-linear in all components then there exists positive real numbers $\alpha_1, \alpha_2, \ldots, \alpha_L$ such that the function $u: \mathbb{R}_+^N \to \mathbb{R}$ defined by $u(x) = \sum_{k=1}^{L} \alpha_k x_k$ for $x \in \mathbb{R}_+^N$ satisfies the following property: for all $x, y \in \mathbb{R}_+^N$, xRy if and only if $u(x) \geq u(y)$.

4 Quasi-linearity and utilitarian welfare orderings

In this section we present and prove an axiomatic characterization of the utilitarian welfare ordering using quasi-linearity. Before doing so we introduce the following lemma.

Lemma 1
Let R be a welfare ordering that satisfies dominance, continuity and quasi--linearity in all components. Then for all $x,y\in\mathbb{R}^N$ and $z\in\mathbb{R}^N_+$: xRy implies (x+z)R(y+z) and xPy implies (x+z)P(y+z).

Proof
Suppose R is a welfare ordering that satisfies dominance, continuity and quasi--linearity in all components. Let $x,y\in\mathbb{R}^N$ and $z\in\mathbb{R}^N_+$.

Suppose xRy.

Let $M = \{k\in N|z_k > 0\}$. If $M = \phi$, then $z = 0$ so that (x+z)R(y+z). Hence suppose, $M \neq \phi$. Without loss of generality suppose, $M = \{1,...,K\}$ for some positive integer $K \leq L$. Thus, $x + z = x + \sum_{k=1}^{K} z_k e^{(k)}$ and $y + z = y + \sum_{k=1}^{K} z_k e^{(k)}$. By quasi-linearity, xRy implies $(x+z_1e^{(1)})R(y+z_1e^{(1)})$ and if $K > 1$, then for all $J < K$, $(x + \sum_{k=1}^{J} z_k e^{(k)})$ R $(y + \sum_{k=1}^{J} z_k e^{(k)})$ implies $(x + \sum_{k=1}^{J+1} z_k e^{(k)})$ R $(y + \sum_{k=1}^{J+1} z_k e^{(k)})$. Thus by a standard finite induction argument we get (x+z)R(y+z).

Now suppose xPy and towards a contradiction suppose (y+z)R(x+z). By quasi- linearity we have (x+z)R(y+z), so that we have (y+z)I(x+z). By continuity of R, xPy implies that there exists $\varepsilon > 0$ sufficiently small, so that we have $(x-\varepsilon e)$Py. By quasi-linearity we get $(x + z-\varepsilon e)$R(y+z). Transitivity of R along with $(x + z-\varepsilon e)$R(y+z) and (y+z)I(x+z) implies $(x + z-\varepsilon e)$R(x+z). This contradicts dominance since $x + z >> x + z-\varepsilon e$. Hence we must have, (x+z)P(y+z). Q.E.D.

Proposition 3
Let R be a welfare ordering. Then R satisfies continuity, domination, strict domination and quasi-linearity in all components if and only if it is utilitarian.

Proof
Let $R = R_U$. Then it is easily verified that it is a welfare ordering that satisfies continuity, domination, strict domination and quasi-linearity in all components. Hence suppose R is a welfare ordering that satisfies continuity, domination, strict domination and quasi-linearity in all components. Then by proposition 2, there exists positive real numbers $\alpha_1, \alpha_2,\ldots, \alpha_L$ such that for all $x,y\in\mathbb{R}^N_+$, xRy if and only if $\sum_{k=1}^{L} \alpha_k x_k \geq \sum_{k=1}^{L} \alpha_k y_k$.

Let $x,y \in \mathbb{R}^N$. Then there exists a non-negative real number b such that x + be and y + be both belong to $\mathbb{R}^N_+$. Suppose xRy. Then by quasi-linearity and lemma 1 it must be the case that (x + be)R(y+be). From the previous paragraph it follows that (x + be)R(y+be) if and only if $\sum_{k=1}^{L} \alpha_k(x_k + b) \geq \sum_{k=1}^{L} \alpha_k(y_k+b)$. However, $\sum_{k=1}^{L} \alpha_k(x_k + b) \geq \sum_{k=1}^{L} \alpha_k(y_k+b)$ if and only if $\sum_{k=1}^{L} \alpha_k x_k + b\sum_{k=1}^{L} \alpha_k \geq \sum_{k=1}^{L} \alpha_k y_k + b\sum_{k=1}^{L} \alpha_k$, while the latter holds if and only if $\sum_{k=1}^{L} \alpha_k x_k \geq \sum_{k=1}^{L} \alpha_k y_k$. Thus, xRy implies $\sum_{k=1}^{L} \alpha_k x_k \geq \sum_{k=1}^{L} \alpha_k y_k$.

Conversely suppose $x,y \in \mathbb{R}^N$ and $\sum_{k=1}^{L} \alpha_k x_k \geq \sum_{k=1}^{L} \alpha_k y_k$. Towards a contradiction suppose yPx. Now, there exists a non-negative real number b such that x + be and y + be both belong to $\mathbb{R}^N_+$. By lemma 1 we have (y+be)P(x+be). Thus, we have $\sum_{k=1}^{L} \alpha_k(y_k + b) > \sum_{k=1}^{L} \alpha_k(x_k+b)$. This leads to $\sum_{k=1}^{L} \alpha_k y_k > \sum_{k=1}^{L} \alpha_k x_k$, contradicting $\sum_{k=1}^{L} \alpha_k x_k \geq \sum_{k=1}^{L} \alpha_k y_k$. Thus we must have xRy. Thus, $R = R_U$, i.e. R is utilitarian.

5 Some preliminary results concerning maximin and leximin welfare orderings

In this section we present some preliminary results concerning maximin and leximin welfare orderings.

Lemma 2

Both R_{Mm} and R_{Lm} satisfy strong convexity with respect to duplicated evaluations and hence convexity with respect to duplicated evaluations.

Proof

Let $a,b,c \in \mathbb{R}$ and $x,y,z \in \mathbb{R}^N$ with $x_1 = y_1 = a$, $x_i = b$, $y_i = \frac{a+b}{2}$ for $i > 1$, $z_i = c$ for all $i \in N$. (a) Suppose we have $xP_{Mm}z$. Thus, $\min\{a,b\} > c$. But $\min\{a, \frac{a+b}{2}\} \geq \min\{a,b\} > c$. Thus we have $yP_{Mm}z$.

(b) Suppose we have $xP_{Lm}z$. If $xP_{Mm}z$ then by (a) we have $yP_{Mm}z$ which implies $yP_{Lm}z$.

If it is not the case that $xP_{Mm}z$, then $\min\{a,b\} = c$ and $\max\{a,b\} > c$.

Case 1: $\min\{a,b\} = a$.

Then, $b > a = c$ and so $\frac{a+b}{2} > a = c$.

Thus, $\min\{a, \frac{a+b}{2}\} = c$ and $\max\{a, \frac{a+b}{2}\} > c$ and so $yP_{Lm}z$.

Case 2: $\min\{a,b\} = b$.

Thus, $a > b = c$ and so $\frac{a+b}{2} > b = c$. Thus, $\min\{a, \frac{a+b}{2}\} > c$ and so we have $yP_{Lm}z$.

Q.E.D.

Lemma 3
The restrictions of R_{Mm} and R_{Lm} to $\mathfrak{J}$ agree with each other. Let R be a welfare ordering that satisfies symmetry, domination and convexity with respect to duplicated evaluations. Then $R|\mathfrak{J} = R_{Mm}|\mathfrak{J} = R_{Lm}|\mathfrak{J}$.

Proof
That $R_{Mm}|\mathfrak{J} = R_{Lm}|\mathfrak{J}$ follows from the respective definitions. Hence let us suppose, R is a welfare ordering that satisfies symmetry, domination and convexity with respect to duplicated evaluations. We need to show that $R|\mathfrak{J} = R_{Mm}|\mathfrak{J} = R_{Lm}|\mathfrak{J}$.
We first prove that for $a,b,c,d \in \mathbb{R}$, it is the case that $[b \geq a, c > a, d > a]$ implies xPy where $x_1 = c$, $y_1 = a$, $x_i = d$ and $y_i = b$ $\forall i > 1$. Call this statement (i).
Suppose not. Then yRx. Let $\varepsilon > 0$ be such that $c > a + \varepsilon$, $d > a + \varepsilon$.
Then by dominance $xP(a+\varepsilon)e$, where e is the unit vector in $\mathbb{R}^N$.
By transitivity of R, we have $yP(a+\varepsilon)e$.
By convexity with respect to duplicated evaluations we have $z^1P(a+\varepsilon)e$, where $z_1^1 = a$ and $z_i^1 = \frac{a+b}{2}$ $\forall i > 1$.
By convexity with respect to duplicated evaluations again we have $z^2P(a+\varepsilon)e$, $z_1^2 = a$ and $z_i^2 = \frac{1}{2}a + \frac{1}{2}(\frac{a+b}{2}) = \frac{3a+b}{4} = \frac{(2^2-1)a+b}{2^2}$ $\forall i > 1$.
On the n^{th} repetition of convexity with respect to duplicated evaluations again we have $z^nP(a+\varepsilon)e$, $z_1^n = a$ and $z_i^n = \frac{(2^n-1)a+b}{2^n}$ $\forall i > 1$.
Now, $\lim_{n\to\infty} \frac{(2^n-1)a+b}{2^n} = a$. Hence there exists $K \in \mathbb{N}$, such that $\forall n \geq K$, $a+\varepsilon > \frac{(2^n-1)a+b}{2^n}$.
Thus, $(a + \varepsilon)e \gg z^n$ $\forall$ $n \geq K$ and hence by domination $(a + \varepsilon)ePz^n$ $\forall$ $n \geq K$, leading to a contradiction.
Hence we have xPy.
Now let $(x,y) \in \mathfrak{J}$ and suppose $a = \min_{i\in N} x_i$ and $b = \min_{i \in N} y_i$. Suppose without loss of generality that $b > a$. Let $d = \frac{2b+a}{3}$ and $c = \max_{i\in N} x_i$. Thus, $b > d > a$ and $c \geq a$.
By symmetry we can suppose $x_1 = a$. Then by domination we have $(a,c,c,\dots,c)Rx$.
By (i) we have $deP(a,c,c,\dots,c)$.
By domination we have bePde as well as yRbe.
Thus, yRbe, bePde, $deP(a,c,c,\dots,c)$, $(a,c,c,\dots,c)Px$ and transitivity of R implies yPx.

Since it is easy to verify that the restriction to $\mathfrak{I}$ of R_{Mm} agrees with the restriction to $\mathfrak{I}$ of R_{Lm} and both R_{Mm} and R_{Lm} satisfy symmetry, domination and convexity with respect to duplicated evaluations, our lemma is proved. Q.E.D.

6 An axiomatic characterization of the maximin welfare ordering and logical independence of the axioms

In this section we obtain an axiomatic characterization of the maximin welfare ordering and provide examples to show that the axioms we use are logically independent.

Proposition 4
The welfare ordering R_{Mm} is uniquely characterized by symmetry, domination, convexity with respect to duplicated evaluations and continuity.

Proof
It is easy to see that R_{Mm} satisfies symmetry and continuity. Convexity with respect to duplicated evaluations of R_{Mm} follows from lemma 2. Let us verify that R_{Mm} satisfies domination. Let $x,y\in\mathbb{R}^N$ and suppose $x \geq y$. Without loss of generality suppose $x_i \leq x_{i+1}$ for all $i = 1,..., n-1$. Thus, $\min_{i\in N} x_i = x_1 \geq y_1 \geq \min_{i\in N} y_i$. Thus, $xR_{Mm}y$. Further if $x >>y$, then $\min_{i\in N} x_i = x_1 > y_1 \geq \min_{i\in N} y_i$ and so $xP_{Mm}y$. Thus R_{Mm} satisfies domination.

Hence suppose R is a welfare ordering that satisfies symmetry, domination, convexity with respect to duplicated evaluations and continuity. We know from lemma 3, that $R|\mathfrak{I} = R_{Mm}|\mathfrak{I}$. Hence suppose $(x,y)\notin\mathfrak{I}$. Thus, $\min_{i\in N} x_i = \min_{i\in N} y_i$. Let $<\varepsilon^n|n\in\mathbb{N}>$ be sequence of strictly positive real numbers converging to 0. Let $<x^{n-}|n\in\mathbb{N}>$ and $<x^{n+}|n\in\mathbb{N}>$ be two sequences in $\mathbb{R}^N$ such that $\forall n\in\mathbb{N}$ and $i\in N$, $x_i^{n-} = x_i - \varepsilon^n$ and $x_i^{n-} = x_i - \varepsilon^n$. Then for all $n\in\mathbb{N}$, $\min_{i\in N} x_i^{n-} = \min_{i\in N} x_i - \varepsilon^n < \min_{i\in N} y_i < \min_{i\in N} x_i + \varepsilon^n = \min_{i\in N} x_i^{n+}$.

By lemma 3, x^{n+} Py and yPx^{n-} $\forall n\in\mathbb{N}$. Further, $\lim_{n\to\infty} x^{n+} = x = \lim_{n\to\infty} x^{n-}$. Thus by continuity, we have xRy and yRx, i.e. xIy.
Thus, $R = R_{Mm}$. Q.E.D.

Let us show that the properties we use in proposition 4 are logically independent.

Example 2
(A welfare ordering that satisfies symmetry, domination, convexity with respect to duplicated evaluations but not continuity): Let $R = R_{Lm}$. Then R satisfies symmetry, domination, convexity with respect to duplicated evaluations. But it does not satisfy continuity. Let $L = 2$ $x^n = (\frac{1}{n}, 1)$ and $y^n = (0,2)$ for all $n\in\mathbb{N}$. Thus,

x^nRy^n for all $n \in \mathbb{N}$. However, $\lim_{n\to\infty} x^n = (0,1)$, $\lim_{n\to\infty} y^n = (0,2)$ and (0,2)P(0,1). Thus R does not satisfy continuity.

Example 3

(A welfare ordering that satisfies symmetry, domination and continuity but not convexity with respected to duplicated evaluations): Let R be such that for all $x,y \in \mathbb{R}^N$, xRy if and only if $\sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n} y_i$. Clearly R satisfies symmetry, domination and continuity. Let L = 3, x = (1, 2, 2), y = $(1, \frac{3}{2}, \frac{3}{2})$ and z = (1.65, 1.65, 1.65). Then $\sum_{i=1}^{3} x_i = 5$, $\sum_{i=1}^{3} z_i = 4.95$ and so we have xPz. However, $\sum_{i=1}^{3} y_i = 4 < 4.95 = \sum_{i=1}^{3} z_i$ and so zPy. Thus R violates convexity with respect to duplicated evaluations.

Example 4

(A welfare ordering that satisfies symmetry, convexity with respect to duplicated evaluations and continuity but not domination): Let R be such that for all $x,y \in \mathbb{R}^N$, xRy if and only if $\max_{i\in N} x_i \leq \max_{i\in N} y_i$. It is easy to verify that R satisfies symmetry, convexity with respect to duplicated evaluations and continuity. Let $a, b \in \mathbb{R}$ with a < b. Let e be the vector in $\mathbb{R}^N$ with all its coordinates equal to 1. Thus be >> ae, but aePbe. Thus, R violates domination.

Example 5

(A welfare ordering that satisfies domination, convexity with respect to duplicated evaluations, continuity but not symmetry): Let R be such that for all $x,y \in \mathbb{R}^N$, xRy if and only if $x_1 \geq y_1$. Clearly, R satisfies domination, convexity with respect to duplicated evaluations and symmetry. However, R does not satisfy symmetry. Let $L \geq 2$, $x,y \in \mathbb{R}^N$ with $x_1 > y_1$ and $x_2 < y_2$. Let σ be the one--to-one function from N to N, such that σ(1) = 2, σ(2) = 1 and σ(i) = i for all $i \in N\backslash\{1,2\}$. Let $x', y' \in \mathbb{R}^N$ with $x'_i = x_{\sigma(i)}$ and $y'_i = y_{\sigma(i)}$ for all $i \in N$. Then, xPy but $y'Px'$ contradicting symmetry.

7 An axiomatic characterization of the leximin welfare ordering and logical independence of the axioms

Now let us consider the leximin welfare ordering.

Lemma 4

R_{Lm} satisfies separability (and hence ascending order separability).

Proof

To show that R_{Lm} satisfies separability let us consider $x, y, x', y' \in \mathbb{R}^N$ and $i \in N$ satisfying $x_i = y_i$, $x'_i = y'_i$, $x_{-i} = x'_{-i}$, $y_{-i} = y'_{-i}$ and $xR_{Lm}y$. Let ξ, η, ξ', η' be the arrangement of x,y, x', y' in ascending order. If $xI_{Lm}y$, then ξ = η. Thus when we replace x_i by x_i' and y_i by y_i' we get $\xi' = \eta'$, since $x_i = y_i$ and $x_i' = y_i'$. This is because in the ascending order arrangements the position of x_i is the same as the position of y_i and position of x_i' is same as the position of y_i'. Hence suppose, $xP_{Lm}y$. Thus there exists $a\in\mathbb{R}$ such that $\#J(a,x) < \#J(a,y)$ and $\#J(b,x) = \#J(b,y)$ for all $b < a$. Since $x_i = y_i$, $\#(J(a,x)\setminus\{x_i\}) < \#(J(a,y)\setminus\{y_i\})$ and $\#(J(b,x)\setminus\{x_i\}) = \#(J(b,y)\setminus\{y_i\})$ for all $b < a$.

Now $\forall b\in\mathbb{R}$, $x_i' \leq b$ if and only if $y_i' \leq b$. This is because $x_i' = y_i'$. Thus, $\forall b\in\mathbb{R}$, $x_i' \in J(b,x)\setminus\{x_i\}$ if and only if $y_i' \in J(b,y)\setminus\{y_i\}$. Thus, $\#J(a,x') < \#J(a,y',)$ and $\#J(b,x') = \#J(b,y')$ for all $b < a$. Hence, $x'P_{Lm}y'$. Q.E.D.

We are now in a position to state and prove the following proposition.

Proposition 5

R_{Lm} is uniquely characterized by shuffling, domination, convexity with respect to duplicated evaluations, improvement impatience and ascending order separability.

Proof

It is easy to see that R_{Lm} satisfies shuffling and improvement impatience. Convexity with respect to duplicated evaluations of R_{Lm} follows from lemma 2. That it satisfies ascending order separability follows from lemma 4. Let us verify that R_{Lm} satisfies domination. Let $x,y\in\mathbb{R}^N$ and suppose $x \geq y$. Without loss of generality suppose $x_i \leq x_{i+1}$ for all $i = 1,..., n-1$. Let $\sigma:N\rightarrow N$ be a one-to-one function such that $y_{\sigma(i)} \leq y_{\sigma(i+1)}$ for all $i = 1,..., n-1$. Now $x_1 \geq y_1 \geq y_{\sigma(1)}$ so that if $x_1 > y_1$ or $y_1 > y_{\sigma(1)}$, then $xP_{Lm}y$. Thus, $x >> y$ implies $xP_{Lm}y$. Hence suppose, $x_1 = y_1 = y_{\sigma(1)}$. If $x_i = y_{\sigma(i)}$ for all $i\in N$, then $xI_{Lm}y$ and so $xR_{Lm}y$. Hence suppose, $K = \min\{i\in N|\ x_i \neq y_{\sigma(i)}\}$. Clearly $K > 1$ and $K \leq L$. Towards a contradiction suppose, $x_K < y_{\sigma(K)}$ so that $y_K = y_{\sigma(i)}$ for some $i < K$. Now, for $i \leq K$, $y_i \leq x_i \leq x_K < y_{\sigma(K)}$. Thus, $\sigma(i)\in\{1,...,K\}$ for $i\in\{1,...,K\}$. Thus, $x_K \geq y_{\sigma(i)}$ for $i\in\{1,...,K\}$ and so $x_K \geq y_{\sigma(K)}$, leading to a contradiction. Along with $x_K \neq y_{\sigma(K)}$, $x_K \geq y_{\sigma(K)}$ implies $x_K > y_{\sigma(K)}$. Since $x_i = y_{\sigma(i)}$ for all $i\in\{1,...,K-1\}$, we get $xP_{Lm}y$.

Hence suppose R is a welfare ordering that satisfies shuffling, domination, convexity with respect to duplicated evaluations, improvement impatience and separability. Since shuffling implies symmetry, by lemma 3 we get that $R|\mathfrak{J} = R_{Lm}|\mathfrak{J}$. Hence suppose $(x,y)\notin\mathfrak{J}$. Thus, $\min_{i\in N} x_i = \min_{i\in N} y_i = a$ (say). By shuffling we may assume $x_i \leq x_{i+1}$ and $y_i \leq y_{i+1}$ for $i = 1,..., L-1$. Thus, $x_1 = y_1 = a$. Suppose $x_i = y_i$ $\forall i = 1,...,K$. If $K = L$, then by reflexivity of R we have xIy and so $xI_{Lm}y$. Hence suppose $K < L$.

Case 1: $K = L-1$.

Thus, $y_L \neq x_L$. Without loss of generality suppose, $x_L > y_L$.

By improvement impatience the evaluation vector $(y_L,...,y_L, x_L)$ is preferred to the evaluation vector $y_L e$.
By separabilty the evaluation vector $x = (x_1,...,x_{L-1}, x_L)$ is preferred to the evaluation vector $y = (y_i,...,y_{L-1},y_L)$ since $x_i = y_i$ for $i = 1,..., L-1$ and so in this case R agrees with R_{Lm}.
Case 2: $K < L-1$.
Thus, $K + 1 \leq L-1 < L$ and $y_{K+1} \neq x_{K+1}$. Without loss of generality suppose, $x_{K+1} > y_{K+1}$.
By improvement impatience the evaluation vector $(y_{K+1},...,y_{K+1}, x_{K+1},....,x_{K+1})$ is preferred to the evaluation vector $(y_{K+1},...., y_{K+1}, y_{K+1}, y_L,...., y_L)$.
In the first vector the first K co-ordinates are y_{K+1} and the remaining L-K coordinates are x_{K+1}. In the second vector the first K+ 1 coordinates are y_{K+1} and the remaining L- (K+1) coordinates are y_L.
By separability the evaluation vector $(x_1,...,x_K, x_{K+1},....,x_{K+1})$ is preferred to the evaluation vector $(y_1,...., y_K, y_{K+1}, y_L,...., y_L)$, since $x_i = y_i$ for $i = 1,...,K$.
Thus we can write $(x_1,...,x_K, x_{K+1},....,x_{K+1})P(y_1,...., y_K, y_{K+1}, y_L,...., y_L)$.
By dominance we have $xR(x_1,...,x_K, x_{K+1},....,x_{K+1})$ and $(y_1,...., y_K, y_{K+1}, y_L,...., y_L)Ry$.
By transitivity of R, we get xPy.
Once again P agrees with P_{Lm}.
This proves the proposition. Q.E.D.

It is worth observing that R_{Mm} does not satisfy improvement impatience. This observation is immediate from the fact that if $x,y\in\mathbb{R}^N$ satisfies the conditions in the definition of improvement impatience, then it must be the case that $xI_{Mm}y$, contrary to the requirement xPy.

Let us now show that the properties we use in proposition 5 are logically independent.

Example 6

(A welfare ordering that satisfies shuffling, domination, convexity with respect to duplicated evaluations, improvement impatience but not ascending order separability): Let L =3 and $\mathcal{Z} = \{x\in\mathbb{R}^3|$ there exists $i,j\in\{1,2,3\}$ with $i\neq j$ and $x_i = x_j\}$. Let R be a binary relation on $\mathbb{R}^N$ such that $R|\mathcal{Z}\times\mathcal{Z} = R_{Lm}|\mathcal{Z}\times\mathcal{Z}$ and for all $(x,y)\in(\mathbb{R}^3\times\mathbb{R}^3)\backslash(\mathcal{Z}\times\mathcal{Z})$, xRy if and only if $xR_{Mm}y$. It is easy to verify that R is an ordering which satisfies shuffling, domination, convexity with respect to duplicated evaluations, improvement impatience. However, R does not satisfy separability. Let $x = (2,2,3)$, $y = (2,3,3)$. Thus, $x,y\in\mathcal{Z}$ and we have yPx since it is the case that $yP_{Lm}x$. Let $x' = (1,2,3)$ and $y' = (1,3,3)$. Thus, $(x', y')\in(\mathbb{R}^3\times\mathbb{R}^3)\backslash(\mathcal{Z}\times\mathcal{Z})$ and so $x' I_{Mm} y'$ implies $x'Iy'$. This happens in spite of $x_1 = y_1 = 2$, $x_1' = y_1' = 1$, $x_2 = x_2' = 2$, $y_2 = y_2' = 3$, $x_3 = x_3' = 3$, $y_3 = y_3' = 3$. Thus, R does not satisfy separability.

Example 7

(A welfare ordering that satisfies shuffling, domination, convexity with respect to duplicated evaluations, ascending order separability but not improvement impatience). Let R = R_{Mm}. Then R satisfies all the properties required in proposition 5 except for improvement impatience.

Example 8

(A welfare ordering that satisfies shuffling, domination, improvement impatience, ascending order separability but not convexity with respect to duplicated evaluations). Let L = 2 and let R be a binary relation on $\mathbb{R}^N$ such that for all $x,y\in\mathbb{R}^N$, xRy if and only if $x_1 + x_2 \geq y_1 + y_2$. Then R satisfies all the properties required in the statement of proposition 5, other than convexity with respect to duplicated evaluations. Let x = (1,7), y = (1,4), z = (3,3). Then we have a = 1, b = 7, c = 3, xPz and zPy violating convexity with respect to duplicated evaluations.

Example 9

(A welfare ordering that satisfies shuffling, convexity with respect to duplicated evaluations, improvement impatience, ascending order separability but not domination). Let L= 2 and let R be a binary relation on $\mathbb{R}^N$ such that for all $x,y\in\mathbb{R}^N$, xPy if and only if either (i) $\min\{x_1, x_2\} < \min\{y_1, y_2\}$; or (ii) $\min\{x_1, x_2\} = \min\{y_1, y_2\}$ but $\max\{x_1, x_2\} > \max\{y_1, y_2\}$. It is easy to see that R is an ordering that satisfies shuffling, improvement impatience and ascending order separability. Let us show that R satisfies convexity with respect to duplicated evaluations. Let $a, b\in\mathbb{R}$ with $a \leq b$, x = (a,b), y = (a, $\frac{a+b}{2}$) and z = (c,c). Suppose xPz.

Case 1: $\min\{x_1, x_2\} = a$ and $a < c$.

Thus, $b \geq a$, so that $\frac{a+b}{2} \geq a$. Thus, $\min\{y_1, y_2\} = a < c$ and so yPz.

Case 2: $\min\{x_1, x_2\} = a$ and $a = c$.

Thus, $b \geq a$ and $b > c$. Hence $b > a$. Thus, $\frac{a+b}{2} > a = c$. Thus, yPz.

Thus R satisfies convexity with respect to duplicated evaluations.

However R does not satisfy domination. Let x = (1,2), y = (3,4).Since, $\min\{x_1, x_2\} < \min\{y_1, y_2\}$ we have xPy, in spite of $y >> x$. Thus R violates domination.

Example 10

(A welfare ordering that satisfies domination, convexity with respect to duplicated evaluations, improvement impatience, ascending order separability but not shuffling). Let L = 2 and let R be the lexicographic ordering on $\mathbb{R}^N$, i.e. for all $x,y\in\mathbb{R}^N$, xPy if and only if either (i) $x_1 > y_1$; or (ii) $x_1 = y_1$ and $x_2 > y_2$. It is easily verified that R satisfies domination, convexity with respect to duplicated evaluations, improvement impatience, ascending order separability. However if

$x = (1,2)$, $y = (2,1)$, $x' = (2,1)$ and $y' = (1,2)$, then we have yPz and $x'Py'$ although x' and y' are obtained from x and y respectively, by interchanging the coordinates.

The above examples show that the axioms used in proposition 5 are logically independent. The next example shows that in proposition 5, we cannot replace shuffling with symmetry in order to obtain an axiomatic characterization of R_{Lm}.

Example 11

(A welfare ordering different from R_{Lm} that satisfies symmetry, domination, convexity with respect to duplicated evaluations, improvement impatience, ascending order separability but not shuffling). Let L = 2 and R be a binary relation on $\mathbb{R}^N$ such that for all $x,y\in\mathbb{R}^N$, xRy if and only if $\#\{i|x_i \geq y_i\} \geq \#\{i|y_i \geq x_i\}$. Then clearly R is an ordering and satisfies symmetry, domination, convexity with respect to duplicated evaluations, improvement impatience, ascending order separability. To show that R does not satisfy shuffling, let x = (1,3) and y = (2,1). Then clearly, xIy. However, if we let $\sigma:\{1,2\}\rightarrow\{1,2\}$ be the identity function and $\rho:\{1,2\}\rightarrow\{1,2\}$ to be such that $\rho(1) = 2$, $\rho(2) = 1$,then we get $x'P\,y'$, where $x'_i = x_{\sigma(i)}$ and $y'_i = y_{\rho(i)}$ $\forall i\in\{1,2\}$.

We already know that R_{Mm} satisfies symmetry, domination, convexity with respect to duplicated evaluations, ascending order separability, but not improvement impatience which the ordering defined in example 11 (i.e. majority rule on $\mathbb{R}^N$) satisfies. Similarly majority rule on $\mathbb{R}^N$ satisfies symmetry, domination, convexity with respect to duplicated evaluations, improvement impatience, ascending order separability but not continuity that R_{Mm} satisfies. That majority rule on $\mathbb{R}^N$ does not satisfy continuity is shown in the following example.

Example 12

Let L = 2 and R be the ordering defined in example 11. Let x = (1,0) and for $n\in\mathbb{N}$, let $y^n = (0, \frac{1}{n})$. Then, y^nIx for all $n\in\mathbb{N}$ which implies y^nRx for all $n\in\mathbb{N}$. However, $y = (0,0) = \lim_{n\rightarrow\infty}(0,\frac{1}{n}) = \lim_{n\rightarrow\infty} y^n$ and we have xPy. Thus, R is not continuous.

8 Conclusion

In this paper we obtain new axiomatic characterizations for three different welfare orderings. The interesting fact about these three welfare orderings is that they satisfy full-comparability- a desirable property that is easily established as in the surveys that we cite in this paper and a fact that we do not need to use in our axiomatic characterizations. The three welfare orderings we consider are of considerable importance in group decision theory as well as in the theory of choice in the presence of ambiguity. These orderings play a very significant role in applied multi-criteria decision making too. Hence researchers have periodically come up with new characterizations of these welfare orderings in order to understand them better and convey their importance to others whose work have an interface with group and multicriteria decision making. We hope that this paper will also serve the same purpose, and prove itself to be incerementally useful.

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