MULTIPLE CRITERIA DECISION MAKING '05

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MULTIPLE CRITERIA DECISION MAKING ‘05

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ul. 1 Maja 50, 40-287 Katowice, tel. +48 032 25 77 635, fax +48 032 25 77 643
www.ae.katowice.pl e-mail:wydawucz@ae.katowice.pl

CONTENTS

PREFACE

The book includes theoretical and applicational papers from the field of multicriteria decision making. The following approaches are used as theoretical tools: goal programming, data envelopment analysis, interactive multiple goal programming, multiobjective linear and nonlinear programming, multiobjective dynamic programming, fuzzy approach, Electre methodology, group decision making and bicriteria robust approach. These methods are applied
in such practical problems as: project management, manpower planning, social choice problems, risk analysis and investment in construction, restoration of historical organs and search for negotiations strategies.

Short description of the articles presented in the volume are given below.

1. Trade-off between several conflict goals is a typical multi-criteria decision making (DM) problem. It appears in the majority of real-life DM situations, including bid preparation. In such situations the process of preparing the best bids and obtaining contracts belongs mainly to the project performers who are oriented towards the external customers. In the paper ***The Target Costing Approach in Multi-Criteria Project Bidding*** (**T. Błaszczyk**) a synthesis of two well known methodologies: target costing (up to now used in new product development planning) and goal programming is proposed. The proposed method is intended in particular as an aid for individual decision makers participating in the competitive bidding process.

2. In a goal programming problem with linear fractional criteria, the resulting problem is not easy to solve due to the non-linear constraints inherent in its formulation. The paper ***Goal Programming with Linear Fractional Criteria: An Application to a Forest Problem*** (**R. Caballero, T. Gómez, M. Hernández, M.A. León**) introduces a simple and reliable test to establish whether a linear fractional goal problem has solutions satisfying all the goals; if this is the case, the solutions are found by solving a linear programming problem. This scheme has been applied to a forest planning problem which made economic and ecological objectives compatible with data provided by the Integral Forest Enterprise of Pinar del Río (Cuba).

3. A shift is defined as a fixed-length duty consisting of a fixed number of contiguous work hours in a day with a rest break preferably around the middle of the day. In the paper ***Effective Heuristics vs GP Solutions for Shift Duties Generation*** (**S. CK Chu, C. SY Yuen**) this fixed-length property is exploited to formulate a straightforward yet flexible goal programming model with integer variables. To address the computational issues the authors propose a very competitive heuristics and give such a comparative analysis.

4. Decision making under uncertainty implies that in certain situations a person does not have the information which adequately describes, prescribes or predicts a system, its behavior or other characteristics, deterministically and numerically. Thus uncertainty is related to a state of the human mind, i.e. lack of complete knowledge about something. In the paper ***Multicriteria Decision Aid under Uncertainty*** (**C. Dominiak**) the author considers the "traditionally understood" problem of decision making under uncertainty and therefore he assumes that the probabilities of the states of nature are not known. A discrete set of alternatives and a discrete set of scenarios have been selected for the purpose of evaluating alternatives. A dominance relation for the type of problem considered in the paper is proposed. This relation enables us to define the optimal solution and an efficient one. Next, three decision aiding procedures are introduced: the hierarchy and quasi-hierarchy procedure, decision aiding on the basis of distance function procedure, and interactive multicriteria decision aiding procedure. Each method is illustrated by a simple numerical example.

5. Supplier selection processes have received considerable attention in business. Most production systems are organized as networks of units. Determining suitable suppliers in supply chain networks has become a key strategic issue. Supplier-customer relationships are changing into a cooperative form. The impact of information sharing plays a crucial role. The supplier selection problem is a multiple criteria group decision making problem. In the paper ***Multiple Criteria Supplier Selection Network Model*** (**P. Fiala**) a multiple criteria model of cooperative decision making and a solution method based on the combination of ANP and multi-objective programming is presented. The proposed approach considers the network and dynamic environment for supplier selection process and manages relationships among units.

6. Data Envelopment Analysis (DEA) models compare relative efficiency of decision making units (DMU) described by multiple inputs and outputs. In most DEA models the best DMUs reach the efficiency score of 100%. De-

pending on the number of units and/or inputs and outputs used in the model, the number of efficient units can be relatively high. The necessity of classification of efficient units leads to several definitions of the so-called super-efficiency. In the paper ***A Slack Based Model for Measuring Super-Efficiency in Data Envelopment Analysis*** (**J. Jablonsky**) an original definition of super-efficiency in DEA models is proposed. The definition works directly with positive slacks of inputs and negative slacks of outputs and uses standard methodology known from goal programming models. The results of a test example of the model are compared with other super-efficiency definitions (Andersen and Petersen model and SBM model). All the computational experiments are done by the original MS Excel DEA support system that uses as the optimisation engine the internal MS Excel solver.

7. The notion of "robust approach" covers many different approaches to decision making. Generally, in robust decision making a decision has to be made when not all the parameters of the problem are known. The question consists in making a decision which will be satisfactory when all the parameters become known and fixed. In the paper ***Bicriterial Robust Approach in Project Management*** (**D. Kuchta**) a robust schedule construction in project management is discussed and a new approach is proposed.

8. Eco-efficiency can be characterised as follows: goods and services can be produced with less energy and resources and less waste emission. Let the production possibility frontier of an economy be determined by the input-output model extended by primary inputs, pollution generation and abatement. The degree by which a net-output vector could be extended for given primary inputs and environmental standards can be considered as a measure of eco-inefficiency. This could equivalently be achieved by a reduction of primary inputs for given environmental standards and given final demand. In the first part of the paper ***Measuring Eco-Efficiency in a Leontief Input-Output Model*** (**M. Luptáčik, B. Böhm**) radial and slack based measures of eco-efficiency are derived. In the second part the authors propose another approach based on the construction of a production possibility frontier. For this purpose a multi-objective optimization problem with maximisation of final demand for each commodity subject to constraints on the primary inputs is formulated. Then, using data envelopment analysis with the production possibility frontier as the standard envelope, the eco-efficiency of an economy is estimated. The eco-efficiency values for the radial efficiency measure and the one based on the DEA model coincide. A demonstration of the viability of these methods and their relationships is given using input-output from Austria and NAMEA data.

9. In the paper ***IND-NIMBUS for Demanding Interactive Multiobjective Optimization*** (**K. Miettinen**) a new software package for solving demanding multi-objective optimization problems, named IND-NIMBUS, is introduced. Its main features and principles are described. The software package in question can be connected to different modeling and simulation tools and therefore it can be used to solve nonlinear, even nonconvex problems, where the function evaluations require the solution of some underlying system, for example, a system of partial differential equations. In addition, the underlying interactive multi-objective optimization method NIMBUS and, in particular, its synchronous version are described.

10. One of the main purposes of construction contracts is to allocate risk and liability clearly, comprehensibly and unambiguously. Otherwise, disputes (also legal) between contract parties are inevitable, which always leads to extra expenses incurred by both parties. The aim of the paper ***Graphical Risk Allocation Model in Construction Contracts for Changes in Market Prices*** (**S. Mitkus**) is to analyse the allocation of risk between the participants of a construction process (the contractor and the client) when implementing construction projects.

11. The evaluation of each project usually involves multiple objectives, including economic desirability, technical novelty, social impact, ecological consequences and others. While financial criteria are usually of quantitative nature, others are often qualitative. In the paper ***An Interactive Procedure for Project Selection*** (**M. Nowak**) a new procedure for this problem is proposed. The method uses simulation technique for economic desirability evaluation and takes into account experts' assessments in relation to qualitative criteria. Thus, project selection can be analyzed as a multiple criteria decision making problem, in which outcomes of projects are evaluated by vectors of probability distributions. In the paper an interactive technique is employed for solving this problem. The method combines two basic approaches usually used for comparing uncertain projects: mean-risk analysis and stochastic dominance. The decision maker expresses his/her preferences by defining restrictions specifying minimal or maximal values of scalar criteria measuring either expected outcome or variability of outcomes. As such restrictions are, in general, not consistent with stochastic dominance rules, a procedure for identifying and eliminating such inconsistencies is described.

12. Recently one can notice an increasing interest in equity or fairness issues in the area of Operations Research. Several research publications dealing with this issue in various application areas have appeared. Some of them directly relate the fairness and equity concepts to the multiple criteria optimization methodology. Multiple criteria optimization traditionally starts with an assumption that the criteria are incomparable. However, many applications arise from situations which present equitable criteria. Moreover, some aggregations of criteria are often applied to select efficient solutions in multiple criteria analysis. The latter enforces comparability of criteria (possibly rescaled). Finally, the novel and distinct mathematical approach denoted by equitable efficiency has been developed to provide solutions to these examples of multiple criteria optimization. The concept of equitable multiple criteria techniques is a specific refinement of the Pareto-optimality. Hence, equitable multiple criteria techniques focus on a selection of Pareto-optimal solutions. It turns out, however, that the techniques are often applied to select efficient solutions in general multiple criteria optimization. The paper ***Equity, Fairness and Multicriteria Optimization*** (**W. Ogryczak**) deals with generation techniques for equitably efficient solutions of multiple criteria optimization problems.

13. In the paper ***Duality in Fuzzy Multiple Objective Linear Programming with Possibility and Necessity Relations*** (**J. Ramík**) a class of fuzzy multiple objective linear programming (FMOLP) problems with fuzzy coefficients based on fuzzy relations is introduced. The concepts of feasible and (α,β) – maximal and minimal solutions are defined. The class of crisp (classical) multiple objective linear programming (MOLP) problems can be embedded into the class of FMOLP. Moreover, for FMOLP problems a new concept of duality is introduced and the weak and strong duality theorems are derived. The concepts and results introduced in the paper are illustrated and discussed on a simple numerical example.

14. Social choice problems may be interpreted as multi-criterial decision making where individual preferences are the criteria and the social planner is the decision maker. A notion of coalitional strategy-proofness for transferable utility scenarios is introduced in such problems. In the paper ***Impossibility of Strategy-Proofness with Coalition Formation under Transferable Utility*** (**J. Roy, H. Sosnowska**) deterministic and probabilistic cardinal social schemes are studied. It is shown that there is no society which is coalitionally strategy-proof.

15. The aim of the paper ***A Model for Evaluating the Investment in the Construction of Dwelling Houses Based on Multiple Criteria Decision Synthesis Methods*** (**E. Šarkienė, V. Šarka, L. Ustinovichius**) is to answer the question whether it is more advisable to invest in individual dwelling-houses construction in Vilnius or in the construction of apartment houses. A two-stage model for evaluating the profitability of investment in the construction of dwelling-houses is selected. At the first stage, the land plot for individual dwelling-houses is evaluated, while at the second stage, a comparative analysis of the possible construction variants of individual dwelling-houses and apartment houses from economical perspective is made.

16. In the paper ***Developments in Multi-Attribute Portfolio Selection*** (**R.E. Steuer, Y. Qi, M. Hirschberger**) the authors explain why it is possible that finance professional view conventional portfolio selection as a single criterion problem, while multiple criteria optimization professionals view it as a bi-criterion problem. Next, they show how, for more complex investors, the theory of mean-variance portfolio selection can be extended to include additional objectives such as dividends, liquidity, turnover, number of securities in a portfolio, and so forth. This is followed by a discussion of the nature of the nondominated sets of multiple objective portfolio selection problems and current developments for the solution of such problems.

17. Decision problems with conflicting objectives and multiple stages can be considered as multi-objective dynamic programming problems. Another way of generalization of single-criterion dynamic programming models is to consider outcomes in partially ordered criteria set, which can be defined for instance by means of triangular norms. The paper ***Triangular Norms in Discrete Dynamic Programming*** (**T. Trzaskalik, S. Sitarz**) is devoted to investigate this possibility.

18. In the 17th- and 18th-century Poland the portable organ, called the positive organ, was a very popular instrument. Unfortunately, only 18 copies of this once so common instrument are nowadays extant in Poland. One of the extant instruments from this group, found only recently, comes from Sokoły near Łapy in the Podlasie region. The purpose of the paper ***Application of Multicriteria Analysis to Restoration of Historical Portable Organ*** (**M. Trzaskalik-Wyrwa, M. Nowak, T. Trzaskalik**) is the joint application of the analysis evaluating an historic organ and the Electre I method in the selection of the guidelines for conservation efforts in the case of the recently discovered organ.

19. Risk is an integral element of any economic project. It is impossible to avoid, and therefore it is necessary to be able to estimate and minimize it. Any investment in construction can be risky. The basic purpose of risk analysis may be formulated as follows: to give to potential partners of the project the facts on the issue related to making a decision whether to participate in the project and which method to choose so that financial losses are avoided. The paper ***Classification of Real Alternatives and Its Application to the Investment Risk in Construction*** (**L. Ustinovichius, G. Ševčenko, D. Kochin**) presents a verbal method of determining investment risk in construction. A new way to solve the problem, based on Verbal Decision Analysis approach, is offered.

20. It can be derived from the empirical works and some behavioural models that the negotiation strategy the parties use is one of the most important factors influencing the negotiation process and its outcome. Therefore the determination of the negotiation strategies that allow the negotiating subjects to best satisfy their goals is the major task for the mediator or the negotiation support system in the externally supported negotiations. In the paper ***Application of Multiple Attribute Stochastic Dominance to Selection of Negotiation Strategies in E-negotiations*** (**T. Wachowicz**) the author proposes describing the negotiation situation as a two-person game, the strategies of which correspond to all the possible negotiation strategies that the parties can apply during the negotiation process. This way we find, by determining the solution of the game, the efficient mix of negotiation strategies which maximises the negotiation outcomes for both parties. To compare the payoffs given as the vectors of value distribution the author applies the model of multi-attribute stochastic dominance proposed by Zaraś and Martel. To find the game solution the procedure for determining the negotiation set of the game with the combination of the Zaraś and Martel model is applied. The general procedure for determining the negotiation game solution and a numerical example of its application based on the Inspire empirical dataset are also presented.

Tadeusz Trzaskalik

Tomasz Błaszczyk

THE TARGET COSTING APPROACH IN MULTI-CRITERIA PROJECT BIDDING

INTRODUCTION

The main purpose of this paper is an experiment of modeling the project bid preparation by a single bidder using a method recently employed in new product development (NPD). The Target Costing method (TC) had been compiled in the 1980s in Japan as a planning tool to improve the NPD process in Toyota Corp. Its main goal was to aid in the preparation of the offer which will be evaluated by an assumed customer with respect to more than one criterion with simultaneous pursuance to achieving his or her own profits. Both the results achieved thanks to its application in early initiations and a fast growing number of companies exploiting this method argue that it is a good tool for multi-criteria planning of NPD in a competitive environment. The author of the paper proposes the introduction of TC approach to planning process of a project meant as a service rather than a product. The project bidding process is investigated as the main field of its application in project planning. In Chapter 1 the project bidding procedures are described in the perspective of multi-criteria decision analysis, and in Chapter 2 two recent approaches are invoked. Chapter 3 contains a proposal of TC combined with goal programming optimization model application in the given decision making problem.

1. PROJECT BIDDING

Nowadays, most of enterprises are delivered as projects, defined [7] as an intentional transformation of a system Ω from an initial state s to a specific state s'; therefore, s' should represent the goals to be achieved. As in every system in a project at the level of planning each evaluation should consider multiple objects and relations between them. Because of that several project characteristics depends on others. A common example is the time-cost trade-off where project completion time and realization cost are examined. Usually faster completion of several activities in the course of the entire project completion involves higher costs and vice versa. In every project it is usually necessary to evaluate more than two main criteria. In different applications their number rises depending on the specific criteria important in particular branches. Their number and character depend only on the evaluator. The most important feature of a project is its uniqueness which means that there was no identical project done the same way before. Even if several projects are similar in their scope, the conditions of their realization are usually different, so that the past data can't be sufficient for the planning of a new project. With uncertainty in estimations of future conditions this is the main source of risk in every planned project.

This paper emphasizes one specific case in which the project will be realized for an external (outside the company) owner. The project owner can choose a contractor from among several competing providers. In such situation the goals for each provider (bidder) are:

- to win the bidding,
- to achieve profits from the realization in case of winning.

The project owner can evaluate bids and chose the best one with respect to his criteria mostly by means of one of the following procedures or by their combination:

- price enquiry (e.g. different providers offer the same product, but with different prices) or other single-criterion enquiry,
- limited tender – multi-criteria analysis where only the authorized providers' bids are examined,
- unlimited tender, where every bid is examined, no matter who the provider is,
- negotiations (the best bid is negotiated with providers),
- full liberty order (the best bid is chosen without restrictions: any criterion matters),

In the present paper the discussion will be constrained to unlimited tender procedure, whose scheme is shown in Figure 1.

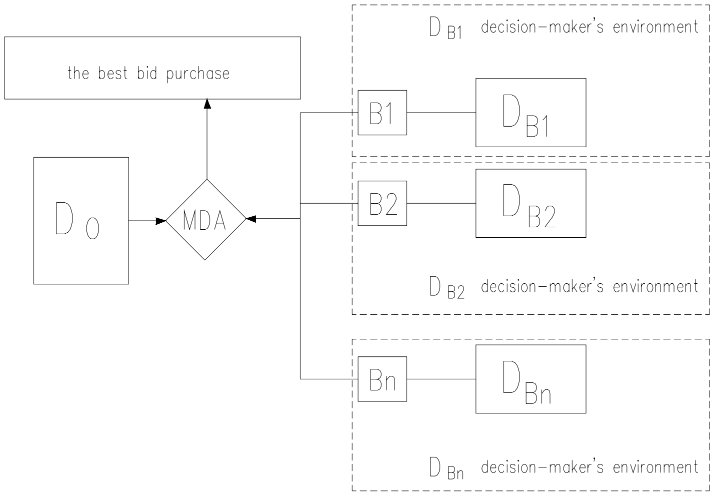

Fig. 1. Unlimited tender procedure

The main features of unlimited tender are:

– every bidder knows the criteria and procedure of examination while preparing bids,
– there is no information flow between decision-makers and bidders,
– every bid must meet the deadline,
– subjects of all bids are undisclosed until their simultaneous disclosure.

In such situation every individual bidder is treated equally: nobody is favored. Competitors don't know the number of competitive bids, the identity of the bidder, or the contents of the bids.

In such situation there is a recognized additional source of risk – the uncertainty of the outcome of the bidding competition. If a bidder has a full knowledge about other bidders, he can compare his own bid with all others, estimate the probability of winning, recognize his own strengths and weaknesses and reformulate the bid if necessary. Unfortunately in most cases such knowledge is limited. In the tender procedure, however, only the best bid will be awarded. For every single bidder this means that his offer must be better than the best of the other offers. Therefore there should be enough information to compare one's own bid with the expected best bid content. In the following sections two recent methods aiding bid preparation are presented. The first could be used in project bidding strategy planning; the second is useful for estimating the completion progress and due date after the bidding competition has been won. The third method presented here is a new proposal allowing the bidder to plan the whole bid, including the schedule before laying down the bids.

2. MULTI-CRITERIA PROJECT BIDDING – RECENT RESEARCHES

AHP based models

A possibility of using the Analytic Hierarchy Process (AHP) in project bidding was described by, Cagno, Caron and Perego [2]. Their paper shows the combined AHP and simulation model helping the individual bidder to define the bidding strategy. In their approach a basic knowledge of competitors is necessary, because every bid is compared with the others. The authors suggest planning the bid with respect to technical, financial, service, and contractual considerations and calculating its competitive value by assigning certain evaluation criteria and expected competitive bids to the project owner. Using this value the probability (P_{WIN}) of winning the bidding could be estimated. Knowing the price P and the estimated profit calculated on the basis of estimated costs C the authors obtained the expected bid profit contribution ($EPC = P_{WIN} (P - C)$). If this value is acceptable for the bidder, the bid can be offered. Otherwise, creating a new bid and repeating the analysis is suggested. The general algorithm of this approach is shown in Figure 2:

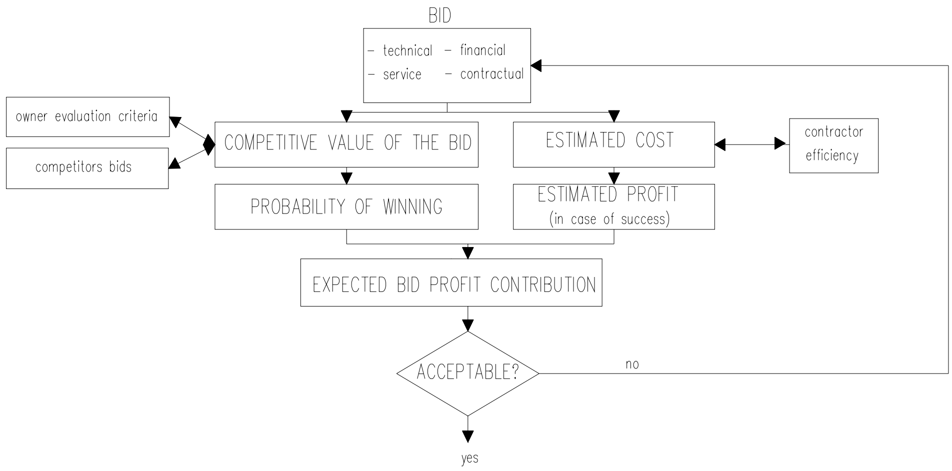

Fig. 2. Decision framework for AHP based model [2]

Further considerations show the effort of modeling the comparisons of several criteria between one's own bid and that of the competitors. There the AHP is used to calculate the priorities of each alternative for each criterion. Using a Monte Carlo simulation the authors obtained a distribution of priorities of final bids (which represents the competitive value of the bids); this way the comparison of one's own bid with competitive bids is achieved.

The Elmaghraby's approach

Elmaghraby [5] emphasizes the bidders' expectations in case of successful completion of bidding. His discussion refers to internal conditions of the company. It is assumed that the cost of project must be equal to total planned cost of its realization multiplied by *FaRM* (*Fair and Reasonable Markup*).

The idea of FaRM (which is exactly a factor that ensures the expected rate of return) came from bi-criteria time-cost trade-off in a project. It is also an attempt to allocate value of money changing in a time interval. To calculate the *FaRM* value it is necessary to identify the *key events* (*KE*) in the plan of the project. Key events are events whose appearance means that a part of work has been completed when a partial payment is requested. A special kind of *KE*s in every project are the last events. Favoring of *KE*s is necessary to identify cash flows (inflows and outflows) in a project and to assign them to the appropriate graph nodes. *KE* nodes are the last ones in subgraphs shares activities required to complete a part of a project. Further discussion could be conducted for the deterministic (based on a Critical Path Method) or the probabilistic (PERT) case. In both situations dues in every *KE*s are calculated as a sum of costs of completing shared activities increased by *FaRM*. In the deterministic case *FaRM* has the value $(1 + p)\alpha$, where p is the required profit of the contractor and α is a discounting factor complying money losing in value during realization of the part of project under consideration. The probabilistic case is somewhat more complex and requires the compliance of probabilities of completing several events at specified dates using mean time values of its predecessors.

The approach described above is not a complex solution for the project bidding problem. It does not comply with the requirement of choice of alternative by the customer. The financial flows are as a matter of fact the main factors in achieving profits but the choice of the objective bid is the necessary condition for further analysis.

3. TARGET COSTING WITH GOAL PROGRAMMING MODEL

In TC method a new way of price estimation was developed. The difference between this one and the classical one was that the price P (defined by marketing research) was an initial information from which the acceptable cost K^p of the production was calculated. The expected profit Z can be used to calculate K^p.

$$K^p = P - Z \tag{1}$$

In this approach the product can be treated as a set of n components, which are carriers of m consumer-evaluated attributes J_j ($j = 1,...,m$). Usually the present costs of all components (also named *drifting costs* – k_i^D) summarized to form the global drifting cost K^D:

$$K^D = \sum_{i=1}^{n} k_i^D \tag{2}$$

is higher than the acceptable cost K_p.

A model described by Kuchta [6] and modified by Błaszczyk [1] is based on the matrix M:

$$M = \begin{pmatrix} v_{11} & \cdots & v_{1m} \\ v_{n1} & \cdots & v_{n} \end{pmatrix} = \begin{pmatrix} f_{11} g_1 & \cdots & f_{1m} g_m \\ f_{n1} g_1 & \cdots & f_{nm} g_m \end{pmatrix} \tag{3}$$

where f_{ij} ($\sum_i f_{ij} = 1$) are estimated contributions of the component i ($i = 1,...,n$) in the whole project attribute j and g_j ($\sum_j g_j = 1,\ j = 1,...,m$) are weights representing the contribution of the attribute j in clients' expectations. The values v_{ij} are products of contributions f_{ij} multiplied by g_j for each component for each attribute weight. For each component i we can calculate its value of the utility evaluation function U_i, based on a function Y_i used by the customer to evaluate bids. In general, U_i can be written as follows:

$$U_i = Y_i(v_{ij}) \tag{4}$$

Referring k_i^D for each component to global value K^D, a relative drifting cost k_i^{D*} can be obtained:

$$k_i^{D*} = \frac{k_i^D}{K^D} \tag{5}$$

and the relation Z_i:

$$Z_i = \frac{U_i}{k_i^{D*}} \tag{6}$$

gives the information about the actual cost of each component compared to its utility for the client. Three cases are possible:

a) $Z_i = 1$ – ideal case, cost of component is adequate to its evaluation value, no modifications are expected,
b) $Z_i > 1$ – advantageous case, its evaluation value is disproportionately high,
c) $Z_i < 1$ – disadvantageous case, drifting cost is too high relative to its evaluation value, component costs need to be lowered.

In case of overdraft:

$$K_p < K^D \tag{7}$$

the improvement of the utility can be reached by modification of attributes in components, where $Z_i < 1$ to increase its value in clients' eyes. In other cases such operation is not feasible. The drifting cost reduction of those components is required to reach the level of acceptable cost. In the case where cost reduction is not feasible it is necessary to revise the value of the expected profit. If the expected profit has to be lowered below its minimal value, it may be suggested that further bid preparation be abandoned. The general scheme of target costing algorithm is shown in Figure 3:

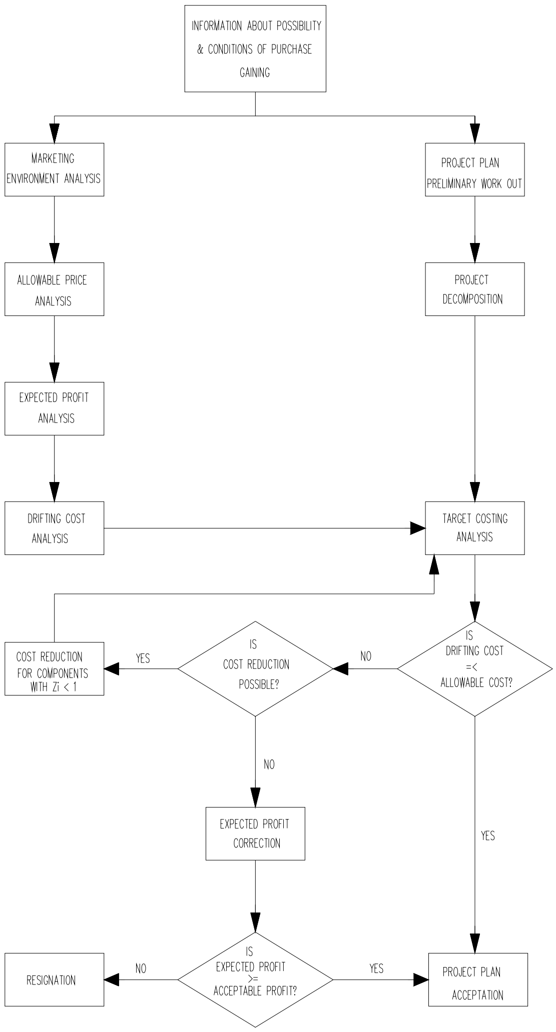

Fig. 3. Decision algorithm scheme for project bid preparation with target costing [1]

In the paper [1] an improvement of the target costing method in project planning is suggested by introducing the *goal programming* (*GP*) algorithm [3]. The idea of GP is to replace multiple criteria by a single metacriterion. The goal of each criterion can be represented by a single point or an interval; the objective function is a minimized sum of weighted lower and upper deviations. In the target costing approach we have multiple goals according to the number of project components. In an ideal situation $Z_i = 1$ for every component. The use of the GP procedure makes it possible to measure every deviation Z_i , both "z_i^+ – *in plus*" and "z_i^- –*in minus*":

$$z_i^+ = \begin{cases} Z_i - 1, & \text{for} \quad Z_i > 1 \\ 0, & \text{for} \quad Z_i \le 1 \end{cases} \tag{8}$$

$$z_i^- = \begin{cases} 0, & \text{for} \quad Z_i > 1 \\ 1 - Z_i, & \text{for} \quad Z_i \le 1 \end{cases} \tag{9}$$

The objective function could be defined as follows:

$$\sum_{i=1}^{n} (w_i^+ z_i^+ + w_i^- z_i^-) \to \min \tag{10}$$

where:

w_i^+ – relative weight of z_i^+ ,
w_i^- – relative weight of z_i^- .

while:

$$Z_i + z_i^+ - z_i^- = 1$$

and:

$$Z_i = \frac{\sum_{j=1}^{m} v_{ij}(x_1^{(i)},...,x_{l_i}^{(i)})}{k_i^{D*}(x_1^{(i)},...,x_{l_i}^{(i)})}$$

with constraints:

$$\forall_i \, \forall_k \quad 0 \le x_l^{(i)} \le \overline{x}_l^{(i)}$$

where:

$i = (1, 2, ..., n)$, $j = (1, 2, ..., m)$, $l = (1, 2, ..., l_i)$,

n – number of activities i project,
m – number of evaluated criteria,
li – number of decision variables related to activity i,
$x_l^{(i)}$ – value of 1 variable related to activity I,
$\overline{x}_l^{(i)}$ – maximal value of l variable related to activity *I*.

Example

The project company is preparing a project realization bid for an external customer. The bid specifications include: detailed project scope, earliest time of the beginning of completion, required quality and quantity standards (described in the attached technical documentation), and bidding procedure description. The project scope contains seven activities $A_1, A_2, \ldots, A_7$. The relationships between them are shown by the project network in Figure 4.

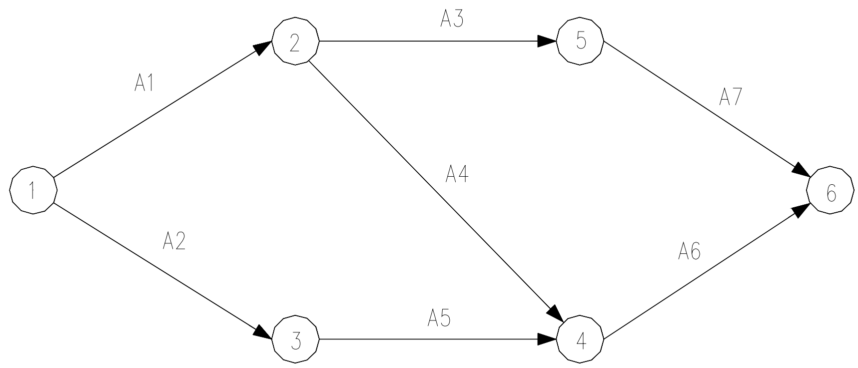

Fig. 4. Example of a project network

Bids will be evaluated with respect to two criteria: *time (date of project completion) T* and *price P.* Weights w_T i w_P are equal to 50%. The chosen bid will be characterized by the lowest value of the following evaluation function:

$$Y = \mathrm{w_T} \frac{T_{oi}}{T_B} + \mathrm{w_P} \frac{P_{oi}}{P_B} \rightarrow \min,$$

where, by analogy to the formula (1):

T_{oi}, P_{oi} – values of criteria T, P in the bid,
T_B, P_B – estimated best T, P values in all bids.

The decision maker assumes that the situation in which for any component i we have $Z_i > 1$ is much better than that in which $Z_i < 1$. For that reason, the weights are set to $w^+ = 0.9$ and $w^- = 0.1$.

As the components of the project, activities $A_1, \ldots, A_7$, characterized by the completion times t and the completion costs c, were adopted. The quantity of profit also has the status of a component but with time value equal to the constant 0.

Based on competitive environment analysis it was found that the shortest possible completion time in other bids will be at least 32 days and the lowest price, at least $1400. One's own possibilities are compared in Table 1:

Table 1

Drifting values of time and cost and possible compression of activities

Activity	„Drifting" time	Possible compression	„Drifting" cost	Compression cost per day
A_1	10	2	100	5
A_2	12	3	150	5
A_3	15	5	190	5
A_4	20	6	215	5
A_5	17	3	180	5
A_6	10	2	110	5
A_7	23	7	225	5

The expected profit was set on the level of $150. The columns „drifting time" and „drifting cost" refer to the values of the completion times and the corresponding cost of activities reachable at the present time. The column „possible compression" defines the maximal time compression of activities (in days) and „compression cost per day" denotes the quantity of the additional cost (in $) of the single day of compression.

To solve this model the Microsoft Excel with SOLVER was used. The recognized critical path consists of activities: A_1-A_3-A_7, with completion time 39 days. Drifting cost of project is calculated as follows:

$$K^D = \sum_{i=1}^{n} k_i^D = 100 + 150 + 190 + 215 + 180 + 110 + 225 = 1170$$

and the project price is equal to:

$$P = K^D + Z = 1170 + 200 = 1370$$

which makes the bid evaluation function value equal to:

$$Y = w_T \frac{T_{oi}}{T_B} + w_P \frac{P_{oi}}{P_B} = 0.5\frac{48}{32} + 0.5\frac{1370}{1400} = 1.099,$$

with the estimated best possible bid evaluation function value:

$$Y_B = w_T \frac{T_B}{T_B} + w_P \frac{P_B}{P_B} = 0.5\frac{32}{32} + 0.5\frac{1400}{1400} = 1.000,$$

Relative utilities of each activities are shown in Table 2:

Table 2

Relative evaluation values in first iteration

Component	Time		Cost		Z_i
	no weight	V_{iT}	No weight	k_i^{D*}	
A_1	0.21	0.1	0.07	0.04	2.85
A_2	0	0	0.11	0.05	0
A_3	0.31	0.16	0.14	0.07	2.25
A_4	0	0	0.16	0.08	0
A_5	0	0	0.13	0.07	0
A_6	0	0	0.08	0.04	0
A_7	0.48	0.24	0.16	0.08	2.91
zysk	0	0	0.15	0.07	0

The above results could be commented upon as follows: Z_i of critical activities are > 1, because its completion has a direct influence on the entire project realization time evaluated with respect to the *time* criterion. Non-critical activities do not change the project completion time – they are „useless" from this point of view, but their realization generate costs charging the *price* criterion. Therefore in those cases the values Z_i are equal to 0. A similar situation takes place as regards the *profit* component.

Thanks to the introduction of formulas (2)-(10) to the model, the following values were obtained:

Table 3

Activity completion times after analysis

Activity	Primary „drifting" time	Time after compression	Total cost of activity
A_1	10	8.00	110.00
A_2	12	9.10	164.50
A_3	15	11.91	205.47
A_4	20	15.10	239.50
A_5	17	14.10	194.50
A_6	10	8.10	119.50
A_7	23	16.17	259.17

This caused the change of the critical path: now it is A_1-A_4-A_6, with completion time 31.2 day and the drifting cost of the project equal to:

$$K^D = \sum_{i=1}^{n} k_i^D = 110 + 164.5 + 205.47 + 239.5 + 194.5 + 119.5 + 259.17 =$$

$$= 1292.64$$

and the project price can be calculated as follows:

$$P = K^D + Z = 1294.64 + 150 = 1442.64$$

which gives the following bid evaluation function value:

$$Y = w_T \frac{T_{oi}}{T_B} + w_P \frac{P_{oi}}{P_B} = 0.5 \frac{31{,}2}{32} + 0.5 \frac{1442{,}64}{1400} = 0.9604.$$

This indicates the growth of the total project utility and an increase of the possibility of winning the bidding. Moreover, the project evaluation fuction value lowered to the value $Y = 0.9604$, better than the estimated best bid value ($Y = 1.000$).

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Rafael Caballero

Trinidad Gómez

Mónica Hernández

María Amparo León

GOAL PROGRAMMING WITH LINEAR FRACTIONAL CRITERIA: AN APPLICATION TO A FOREST PROBLEM*

INTRODUCTION

In a goal programming (GP) problem, when the goals are linear fractional functions, the formulation of the goal programming problem to be solved is quite complex because of non-linear constraints. As Awerbuch et al. showed [1], solving this kind of problem is not as easy as it might appear. These authors offer an example showing that direct linearization of the problem is not suitable for solving it. In the literature, there are very few references to goal programming with fractional goals, except for the papers of Hannan [9; 10], Soyster and Lev [15], and an article by Kornbluth and Steuer [13].

In this work, we first review the difficulties encountered when solving a linear fractional goal programming problem, and suggest the use of an associated linear problem. Following this, a theoretical study is presented of the relationships that exist between the solutions of the linear problem associated with the true linear fractional goal programming problem. We show that the linear problem can be used as a search strategy for solutions that satisfy all goals, but that it is unsuitable when such solutions do not exist.

In the third section we present a linear fractional goal programming model for solving a timber harvest scheduling problem in order to obtain a balanced age class distribution of a forest plantation in Cuba. The forest area

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of Cuba has been severely reduced due to indiscriminate exploitation and natural disasters (fires, hurricanes etc.). Thus, in this particular case, the main goal is to organize and regulate the forest. This involves a significant change from its current distribution by ages to obtain a more even-aged structure over a planning horizon of 25 years. This has been formalized as fractional goals which take into account the dynamic aspect of the problem and ensure attaining a balanced age class distribution in a progressive and flexible way.

In Section 4 the model is applied to a specific case, the *San Juan and Martínez Management Unit*, which belongs to the *Integral Pinar del Río* forestry company. We then analyse the results obtained. Finally, in Section 5 we draw some conclusions followed by the references in Section 6.

1. GOAL PROGRAMMING IN MULTIOBJECTIVE LINEAR FRACTIONAL PROGRAMMING

The problem we deal with has p linear fractional objectives and a constraints set that is a convex polyhedron. Without loss of generality, we assume that the decision-maker imposes a minimum target value for each objective. Thus, the unwanted deviation variables are the negative ones, that is, n_i for $i = 1, \ldots, p$. Using the Lexicographic GP approach and assuming that the priority levels are imposed in such a way that in a given level N_s we find k goals numbered from 1 to k, (in the case of Weighted GP, $k = p$), in this level N_s, the problem to be solved is as follows:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{k} n_i \\
s.t \quad & Ax \leq b \\
& \frac{f_i(x)}{g_i(x)} + n_i - p_i = u_i \quad i = 1,\ldots,k \\
& x, n_i, p_i \geq 0 \qquad i = 1,\ldots,k
\end{aligned}
\tag{1}
$$

where $A \in M_{mxn}(\mathbb{R})$ and $b \in \mathbb{R}^m$. Let $X = \{x \in \mathbb{R}^n \,/\, Ax \leq b,\ x \geq 0\}$, which includes the goals in the previous levels[1]; and $f_i(x) = c_i^t x + \alpha_i$, $g_i(x) = d_i^t x + \beta_i$ where $c,d \in \mathbb{R}^n$, α_i, $\beta_i \in \mathbb{R}$. In addition, we assume that the denominators $g_i(x)$, $i = 1,..., p$ are strictly positive for every $x \in X$.

If the solution of problem (1), $(x^*, n_i^*, p_i^*)_{i=1,...,k}$ is such that $\sum_{i=1}^{k} n_i^* = 0$, then x^* is a solution that satisfies all the goals in this priority level. However, it is obvious that in principle this is not an easy problem to solve due to the non--linear constraints corresponding to the goals we aim to satisfy.

If we multiply all these constraints on both sides by the factor $g_i(x)$ – which is always positive in X for every i per hypothesis – we obtain the formulation of the following linear programming problem, since $f_i(x)$ and $g_i(x)$ are all linear functions:

$$\begin{aligned} \min \quad & \sum_{i=1}^{k} n_i' \\ s.t \quad & x \in X \\ & f_i(x) - g_i(x) \cdot u_i + n_i' - p_i' = 0 \\ & n_i', p_i' \geq 0 \qquad i = 1,...,k \end{aligned} \tag{2}$$

The relationship between these two problems has already been studied in the literature, which concludes that the use of (2) to solve (1) is not always a valid approach, since the solutions of the two problems might not be identical, even when there are unique solutions. A test problem can be used to verify whether the solutions for the two problems are identical or not (see [1; 15; 10]).

However, by setting aside the aim of solving problem (1) directly, we can focus on the search of points in X that verify all the goals imposed. Although it is true that the two problems are not equivalent, and therefore we cannot use (2) for solving (1) in every case, we can prove that (2) can be used for deducing the existence of solutions that satisfy all the problem's goals.

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[1] The constraints imposing the achievement of the goals in the previous levels are of the form $f_j(x)/g_j(x) \geq u_j$ for those j belonging to previous levels. However, these constraints become linear constraints, since they are equivalent to $f_j(x) - u_j g_j(x) \geq 0$, where $f_j(x)$ and $g_j(x)$ are linear functions, so they can be included in the formulation $Ax \leq b$.

Theorem 1

Given problems (1) and (2) as described earlier, the following statements are valid:

i. If, when solving (2), the solution is $(x^*, n_i'^*, p_i'^*)_{i=1,\dots,k}$ such that $\Sigma_i n_i'^* = 0$, then there is at least one point of X that verifies the fractional goals imposed by the decision-maker for the current priority level and this solution coincides with x^*.

ii. If, when solving (2), the solution is $(x^*, n_i'^*, p_i'^*)_{i=1,\dots,k}$ such that $\Sigma_i n_i'^* > 0$, then, there is **no** solution that satisfies all the goals of the linear fractional problem for the current priority level.

Proof. See [3].

Although this theorem does not attempt to show the equivalence between problems (1) and (2), it is obvious that there is a strong relationship between them. This is sufficient to allow us to use (2) (which is a linear problem) for finding a solution that will satisfy all the goals of the linear fractional goal programming problem.

In the examples provided by Awerbuch et al. [1], Hannan [9; 10], and Soyster and Lev [15] to show the lack of equivalence between (1) and (2), the statements of Theorem 1 are verified.

Finally, we establish a result similar to that of Theorem 1 using the Minimax GP approach. In this instance the problem to solve is as follows:

$$\begin{aligned} \min \quad & d \\ s.t \quad & Ax \le b \\ & \varphi_i(x) + n_i - p_i = u_i \quad i = 1,\dots,k \\ & \lambda_i n_i \le d \qquad i = 1,\dots,k \\ & x, n_i, p_i \ge 0 \qquad i = 1,\dots,k \end{aligned} \tag{3}$$

Once again, we are dealing with the non-linearity of the constraints associated with the goals because the initial functions are fractional functions. Again, we deal with this drawback by solving the linear problem associated with the original problem:

$$\begin{aligned}
&\min \quad d' \\
&s.t \quad Ax \le b \\
&\qquad f_i(x) - g_i(x) \cdot u_i + n'_i - p'_i = 0 \quad i = 1,\dots,k \\
&\qquad \lambda_i n'_i \le d' \quad i = 1,\dots,k \\
&\qquad x, n'_i, p'_i \ge 0 \quad i = 1,\dots,k
\end{aligned} \tag{4}$$

As shown for Theorem 1, it is possible to prove that solving this linear problem is sufficient to guarantee the existence − or non-existence − of a solution that will satisfy all the problem's goals using the minimax approach. However, the two problems − (3) and (4) − are not equivalent problems.

Theorem 2

Given problems (3) and (4) as described earlier, the following statements are valid:

i. If, when solving (4), the solution is $(x^*, d'^*, n_i'^*, p_i'^*)_{i=1,\dots,k}$ with $d'^* = 0$, then there is at least one point of X that verifies the fractional goals imposed by the decision-maker and this solution coincides with x^*.
ii. If, when solving (4), the solution $(x^*, d'^*, n_i'^*, p_i'^*)_{i=1,\dots,k}$ is such that $d'^* > 0$ then there is no solution that satisfies all the goals of the original problem.

Proof. See [3].

2. THE APPLICATION

2.1. The problem

Decision making in forest planning has currently become a multi-dimensional decision context, concerned with multiple and sustainable use of the forests. They are not envisaged simply as a source of goods and services; rather, the preservation of biodiversity and environmental protection are also factors to be taken into account. In fact, the term *sustainability* goes beyond the steady supply of timber products in order to include other goods and services provided by forestry systems [6]. Therefore, we need multiple criteria decision-making models to the management of any forest system.

Field [7] was a pioneer in this area who analysed a forest planning problem using a multicriteria framework. From this time onwards many other works applying multicriteria techniques to forestry problems were published. Kao and Brodie [12], Field et al. [8] and Hotvedt [11] are some of the authors who have used Goal Programming for timber production planning. On the other hand, Díaz-Balteiro and Romero [4; 5] designed a multigoal programming model and obtained the best-compromise solutions validated in terms of optimal utility.

The characteristics common to all these models is that they have been applied to European and North-American even-aged forests.

In this application we deal with a very different forest management problem, placed in the Cuban context. The forestry area of Cuba has suffered dramatically due to indiscriminate exploitation and natural disasters (fires, hurricanes etc.). This situation, together with the fear of greater ecological disasters, has given rise to conservationist policies which lead to old growth forests with subsequent financial losses and other problems. This also means that Cuban forests have a highly uneven-aged distribution and thus an important objective in the Cuban context will be to plan a redistribution of the forest into even-aged stands.

However, Cuba is making great efforts in reforesting and caring for its natural forests. Some Cuban forestry companies have focused on achieving an even-aged structure for their plantations. In this line, the Pinar del Río University has been authorized to carry out this kind of work with the forestry companies of the region. The present study is framed within this approach, and is preceded by the work of León et al. [14], where a Goal Programming model with linear goals was formulated into the management planning of a *Pinus Caribaea* plantation in this province. The authors obtained several solutions which satisfied the target values, but not all of them ensured a balanced even-aged distribution over the planning horizon. Thus, we propose fractional programming as a good alternative to take into account the dynamic aspect of the problem in order to achieve that the area covered by each age class must be the same by the planning horizon.

Therefore, we have designed a lexicographic GP model with fractional goals to regulate a pure plantation of *Pinus Caribaea* in Pinar del Río (Cuba).

2.2. The model

The model is initially formalized in a general way and then applied to the specific case of a Cuban plantation which belongs to the Integral Pinar del Río forestry company.

Let us assume that the plantation area to reorganize is managed for wood production and is classified according to productivity (site class) and by age of the stands (age class). Thus, the starting situation is given by the following matrix:

$$\mathbf{S}^0 = \begin{pmatrix} s_{11}^0 & s_{12}^0 & .... & s_{1I}^0 \\ s_{21}^0 & s_{22}^0 & .... & s_{2I}^0 \\ .... & .... & .... & .... \\ \\ s_{H1}^0 & s_{H2}^0 & .... & s_{HI}^0 \end{pmatrix}$$

where s_{hi}^0 is the total number of hectares of the site class h ($h = 1, 2, \ldots, H$) within the age class i ($i = 1, 2, \ldots, I$) at the starting point. The sum of the column elements of the matrix shows the available area at the starting point in each age class ($\mathbf{S}_i^0 = \sum_{h=1}^{H} s_{hi}^0$), whereas the sum by rows gives the available area in each site class ($\mathbf{S}_h^0 = \sum_{i=1}^{I} s_{hi}^0$).

The planning horizon *(T)* has been divided into periods, so that, when a period has elapsed, the trees in age class i become age class $i + 1$. Thus, if t is the number of years in each class (for reasons of simplicity we assume this number is constant), the number of periods under consideration, denoted by P, is equal to the number of years of the planning horizon divided by t.

The decision variables of our model represent the number of hectares of a specific site class h ($h = 1, 2, \ldots, H$) and age class ($i = 1, 2, \ldots, I$) with a forest intermediate treatment or a final cutting j ($j = 1, 2, \ldots, J$) at period p ($p = 1, 2, \ldots, P$), denoted by x_{hij}^p. The forest treatment to apply depends on age, and so the value of the subscript j depends on the value of i, $j \in N(i)$, where $N(i) = \{j / (i, j) \in N\}$ and $N = \{(i, j)/j$ is the forest treatment corresponding to age class $i\}$. Clearcutting is denoted by J, the last value of the subscript j.

Due to the evolution of the forest, s_{hi}^{p} depends on the area of the previous period in the following way:

$$s_{h1}^{p} = \sum_{i=1}^{I} x_{hiJ}^{p} \ , \qquad h = 1, 2,\ldots, H$$

$$s_{hi}^{p} = s_{h(i-1)}^{(p-1)} - x_{h(i-1)J}^{p} \ , \qquad i = 2,\ldots, I\text{-}1; \quad h = 1, 2,\ldots, H$$

$$s_{hI}^{p} = s_{h(I-1)}^{(p-1)} - x_{h(I-1)J}^{p} + s_{hI}^{(p-1)} - x_{hIJ}^{p} \, , \qquad h = 1, 2,\ldots, H$$

In our context, the following premises summarize the wishes of the decision-maker:

– Timber production should be such that non declining yield occurs, for each period into which the time horizon is divided.
– Whenever possible avoid clearcutting at early ages.
– The area covered by each age class should be roughly the same by the end of the planning horizon.
– The Net Present Value (NPV) must be higher than a certain threshold throughout the planning period.

We formalize the previous premises as goals, that is, as soft constraints and thus our model becomes a Goal Programming problem. The preferences regarding the satisfaction of the goals are modelled by using the lexicographic approach according to their priority and taking into account that they are the same in each period p ($p = 1, 2,\ldots, P$).

First priority level. The area to which clearcutting is applied *($j = J$)* should be kept to ecologically acceptable levels. Thus, the area which ensures the perpetuation of the forest harvest in site class *h*, for period *p,* Se_{h}^{p} should not be exceeded. This area Se_{h}^{p} is given by the total area in site class *h* divided by the rotation age and multiplied by the number of years in each class. Therefore, in each period, we have the following *H* goals:

$$\sum_{i=1}^{I} x_{hiJ}^{p} + n1_{h}^{p} - p1_{h}^{p} = Se_{h}^{p} \quad h = 1,\ldots, H$$

where $n1_{h}^{p}$ *and* $p1_{h}^{p}$ are the negative and positive deviation variables and for each site class the positive ones are unwanted.

In addition, given that all goals have the same relevance, the function to be minimized in this level is the sum of the positive deviation variables multiplied by a normalizing coefficient in order to prevent bias.

Second priority level. We aim at keeping harvest levels to the maximum sustained yield. Thus, if V^p represents the volume harvested at period p and $v^p{}_{hij}$ is the volume per hectare harvested from each site class, age, forest treatment and period, this goal can be expressed by the following equation:

$$\sum_{h=1}^{H} \sum_{(i,j)\in N} v_{hij}^{p} x_{hij}^{p} + n2^{p} - p2^{p} = V^{p}$$

As before, the positive deviation variable is the one to be minimized.

Third priority level. We try to regulate the forest without having to sacrifice young stands in the process, so no stand under age class I-1 should be cut. Consequently, this goal is formulated as follows:

$$\sum_{h=1}^{H} \sum_{i=1}^{I-2} x_{hiJ}^{p} + n3^{p} - p3^{p} = 0$$

where the positive deviation variable is the one to be minimized.

Fourth priority level. The area covered by each age class should be roughly the same by the end of the planning horizon. This is expressed by a goal establishing that the ratio between the number of hectares in the first age class and the last age class in each period must be above a target value. Thus, this is a fractional goal formulated as follows:

$$\frac{\mathbf{S}_{1}^{p}}{\mathbf{S}_{I}^{p}} + n4^{p} - p4^{p} = \frac{1}{P} p \quad p = 1,..., P$$

where $\mathbf{S}_{1}^{p} = \sum_{h=1}^{H} \sum_{i=1}^{I} x_{hiJ}^{p}$ and $\mathbf{S}_{I}^{p} = \sum_{h=1}^{H} s_{h(I-1)}^{(p-1)} - x_{h(I-1)J}^{p} + s_{hI}^{(p-1)} - x_{hIJ}^{p}$.

The target values increase within each period in such a way that in the last period the target value is 1. If this last value is reached, a balanced age class distribution by the end of the last planning period is ensured. In this case, the unwanted deviation variable is the negative one.

Fifth priority level. Finally, the following goal reflects the economic objective of the model. We want to exceed a value requested by the decision-makers in each period NPV^p,

$$\sum_{h=1}^{H} \sum_{(i,j)\in N} NPV_{hij}^{p} x_{hij}^{p} + n5^{p} - p5^{p} = NPV^{p}$$

where NPV_{hij}^{p} is the Net Present Value per each hectare harvested from site class h, age class i, and treatment j at period p. The negative deviation variable is the one to be minimized.

These priority levels are applied to each period of the planning horizon. Therefore, the objective function of the model is as follows:

$$LexMin\left(f^1,...,f^P\right) =$$

$$=\left(\left\{\sum_{h=1}^{H}\frac{p1_h^1}{Se_h^1}, p2^1, p3^1, n4^1, n5^1\right\},...,\left\{\sum_{h=1}^{H}\frac{p1_h^P}{Se_h^P}, p2^P, p3^P, n4^P, n5^P\right\}\right)$$

On the other hand, the feasible set of the model is defined by the following constraints.

We have area accounting constraints per site class and per age class during each period p ($p = 1, 2,\ldots, P$):

$$\sum_{j\in N(i)} x_{hij}^p \le s_{hi}^{(p-1)}; \quad h = 1,2,...,H; \quad i = 1,...,I; \; p = 1,2,...,P$$

We also impose constraints to control some of the model's key values. We establish constraints to control the lower bound of the total cutting area and thus guarantee the regeneration of the stands:

$$\sum_{i=1}^{I} x_{hiJ}^p \ge \beta\, Se_h^p \quad h = 1,2,...H; \, p = 1,2,..., P; \quad 0 \le \beta \le 1$$

Finally, we establish lower bounds for *Net Present Value*:

$$\sum_{h}^{H} \sum_{(i,j)\in N} NPV_{hij}^p x_{hij}^p \ge \gamma\, NPV^p \quad p = 1,2,..., P; \quad 0 \le \gamma < 1$$

The values of the parameters β and γ are calculated when the model is applied to a particular situation and depend on the decision-makers' requests.

2.3. Results and discussion

This model has been applied to the San Juan y Martínez Management Unit which has 3,984.3 hectares of *Pinus Caribaea*. The initial forest configuration is as follows:

$$\mathbf{S}^0 = \begin{pmatrix} 0.0 & 0.0 & 198.0 & 188.0 & 83.2 \\ 32.2 & 344.6 & 405.9 & 79.0 & 759.6 \\ 33.5 & 236.8 & 266.7 & 102.0 & 692.4 \\ 30.6 & 78.9 & 130.5 & 174.4 & 148.0 \end{pmatrix}$$

As indicated, the sum by columns corresponds to the number of hectares available in each age class at the starting situation, $\mathbf{S}_i^0 (i = 1,2,...,5)$

$$\begin{pmatrix} 96.3 & 660.3 & 1{,}001.1 & 543.4 & 1{,}683.2 \end{pmatrix}$$

and the sum by rows refers to the availability of each site class, $\mathbf{S}_h^0 (h = 1,2,...4)$

$$\begin{pmatrix} 469.2 & 1{,}621.3 & 1{,}331.4 & 562.4 \end{pmatrix}$$

There are four site classes in this plantation ($H = 4$) and five age classes ($I = 5$). The planning horizon, T, coincides with the rotation length and, in our context, is equal to 25 years [14]. The time unit for each planning period is 5 years and thus, we have a total of five periods ($P = 5$).

Besides applying clearcutting (treatment 4) in all age classes, the other intermediate treatments to be applied by age class are as follows: thinning 1 ($j = 1$) in age class 2, thinning 2 ($j = 2$) in age class 3 and thinning 3 ($j = 3$) in age class 4. Therefore, the problem has a total of 160 decision variables.

For the first priority level, the target values are given by $Se_h^p = Se_h^0 = \frac{1}{5}\mathbf{S}_h^0 \;\; h = 1,...,4$.

Regarding the second priority level, V^p is 138,328 m^3 for every period. For the third and fourth priority levels, the target values have already been specified in the model. Finally, for the fifth priority level, and in line with the decision-makers' requests, the minimum desired level of *NPV* is 790 000 pesos[2] for the first two periods and 760 000 pesos for the last three.

On the other hand, also in line with the decision-makers' requests, in order to guarantee the regeneration of stands, the value of parameter β takes the value 0.9.

Similarly, the value of γ is set to 0.9 to guarantee that the values of *NPV* in each period are always more than or equal to 90% of the set target values.

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[2] 25 Cuban pesos ≅ 1 $.

The resolution of the problem was done with the program *PFLMO* [2] using the resolution method described in Section 2. Given the high level of initial uneven age class distribution in the plantation we were forced to relax the target values of the fractional goal for period 3, from 0.6 to 0.5, which had no effect on the final equilibrium achieved. After this adjustment, *PFLMO* found solutions that satisfied all the goals, and therefore even-aged solutions by the end of the planning horizon.

Once the existence of solutions verifying all the target values was established, the efficiency of the solution obtained was restored. In this case the restoration technique used was the Interactive Restoration method that allows the decision-makers to work with several options at this new stage of problem resolution. The decision-makers chose *NPV*, the economic objective, to be the one to maximize within the set of solutions verifying all the problem goals. Thus, the solution obtained after restoration achieved a balance age class distribution by the end of the planning horizon and satisfied all the ecological goals of the problem while yielding the greatest *NPV* for the company.

The solution obtained is shown in Appendix 1 as Solution 1 (Table 2). As shown in Table 1, the *NPV* for the company is 4 151 784 pesos – which is quite high if we take into account that the target value was around 3 860 000 pesos. However, the decision-makers did not consider this solution to be ecologically acceptable because it meant applying clearcutting to a large number of hectares of age class 4 in all the periods.

The decision-makers wanted to impose a stricter constraint on clear-cutting in age class 4. Therefore, we choose a solution that satisfied the goals and such that only a maximum of 15% of the total age class 4 area available for each site class and in each period was available for clearcutting. The solution obtained in this case is given in Appendix 1 as Solution 2 (Table 3). Total *NPV* is 4 067 495 pesos and the total number of hectares cut in age class 4 is 109.26 (Table 1).

The decision-makers also wanted to obtain the solution which, while satisfying all the target values, involved the least amount of cutting of age class 4, in order to compare such a solution with the previous ones. This solution is shown in Appendix 1 as Solution 3 (Table 4). In this case, the clearcutting of age class 4 stands is only done during the first period and, as shown in Table 1, only a very small percentage of the total age class 4 area is involved, i.e. 0.6%. However, the *NPV* obtained with this solution is lower than in previous solutions, i.e. 4 000 371 pesos.

Table 1

Comparison between solutions

	Cutting in age 4 (ha.)	*NPV (pesos)*
SOLUTION 1	791.94	4 151 784
SOLUTION 2	398.77	4 067 495
SOLUTION 3	1.26	4 000 371

Bearing these solutions in mind, the decision-makers evaluated the different alternatives provided and chose Solution 2. This solution satisfies all the target values and only 15% of age class 4 underwent clearcutting. In addition, the *NPV* in this solution is 4 067 495 pesos. The decision-makers were fully satisfied with this solution and so the resolution process ended.

Figure 1 shows the evolution of each age class during the different planning periods for the solution chosen by the decision-makers. As we can see, the area covered by each age class has been balanced by the last period of the planning horizon.

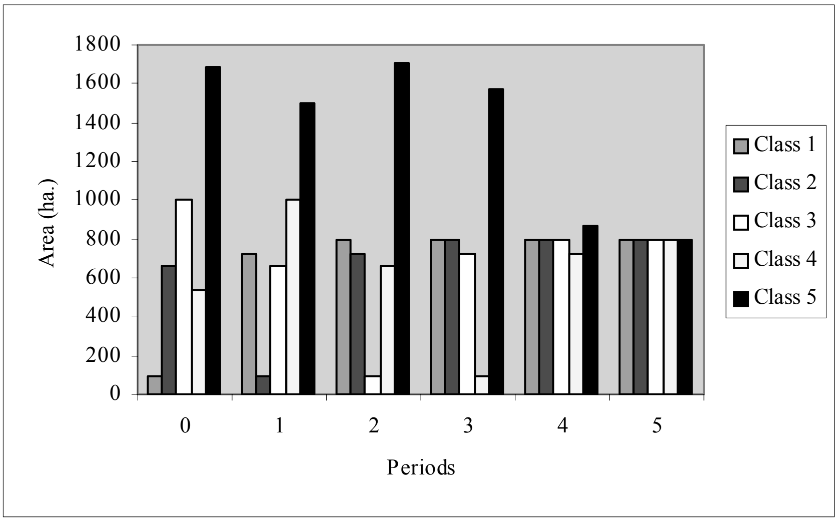

Fig. 1. Evolution of each age class during the planning periods

CONCLUSIONS

This work aims at providing a simple and useful technique for solving a goal programming problem with linear fractional criteria when there are points in the feasible set that satisfy all the goals. The theoretical results demonstrate that we can use an associated linear problem to check whether the problem has solutions that satisfy the goals and, in this case, to find them.

In addition, as an application, we have proposed a model to calculate the area to be harvested in a plantation in Cuba, in each site class during each period while maximizing profits, without having a harmful impact on the ecosystem. It ensures a balanced age class distribution in the plantation by the end of the planning horizon, which fully satisfies the wishes of the decision-makers, thereby solving the company's requirements.

The fractional goal models the decision-makers' desire for a balance age class distribution in a way that takes into account the dynamic aspect of the problem, also ensuring that those solutions which satisfy the goals fulfil this desire. All this is achieved without giving up the financial objective. Thus, the model we offer not only achieves an even-aged distribution of the forest, but also enables its efficient exploitation.

Furthermore, the model allows us to calculate the number of hectares undergoing different treatments (indicating the timber volume to be extracted in each planning period), to know the Net Present Value generated by such management planning, and also to reduce clearcutting during the planning horizon.

APPENDIX 1

The selected solutions are shown below. The first column (named FRACT) shows the value of each solution for the fractional goal at each period. Columns 2-5 show the hectares undergoing different management treatments. In column 6 we specifically show the number of hectares for clear cutting in age class 4 for each period. Finally, we show the *NPV* generated by the solutions in each period, expressed in Cuban pesos.

Table 2

Solution 1

	FRACT	T1 (ha.)	T2 (ha.)	T3 (ha.)	T4 (ha.)	T4 age 4 (ha.)	NPV (pesos)
P.1	0.51309	660.3	1001.1	335.5131	755.0499	207.8869	857,945
P.2	0.47551	96.3	660.3	738.5976	796.8596	93.84	848,004
P.3	0.5177	670.594	96.3	581.314	796.8601	78.9861	787,365
P.4	0.95014	703.02	755.05	2.9515	796.86	93.3485	792,759
P.5	1	703.02	796.86	437.1714	796.8604	317.879	865,711

Table 3

Solution 2

	FRACT	T1 (ha.)	T2 (ha.)	T3 (ha.)	T4 (ha.)	T4 age 4 (ha.)	NPV (pesos)
P.1	0.486205	660.3	1001.1	431.83	728.422	81.51	854,400
P.2	0.468075	30.6	660.3	532.183	796.86	110.16	823,388
P.3	0.508897	643.966	96.3	576.905	796.8598	83.3948	783,699
P.4	0.920908	423.7923	643.966	81.855	796.86	14.445	783,686
P.5	1	112.48	703.02	382.6762	796.8599	109.2633	822,322

Table 4

Solution 3

	FRACT	T1 (ha.)	T2 (ha.)	T3 (ha.)	T4 (ha.)	T4 age 4 (ha.)	NPV (pesos)
P.1	0.47513	660.3	1001.1	456.05	717.174	1.256	852,291
P.2	0.465003	30.6	486.889	397.2	796.86	0	795,441
P.3	0.505267	171.994	96.3	660.3	796.86	0	772,549
P.4	0.909091	357.815	632.718	96.3	796.86	0	780,418
P.5	1	112.48	501.481	340.884	796.86	0	799,672

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Sydney CK Chu

Christina SY Yuen

EFFECTIVE HEURISTICS VS GP SOLUTIONS FOR SHIFT DUTIES GENERATION*

INTRODUCTION

This paper reports comprehensive computation experiences on a general modeling framework for a flexible crew-shift duties generation problem (DGP) initially described in [5]. DGP arises naturally as a mathematical description for the crew (being bus drivers) deployment problem against the background of projects [13] conducted at the Busing and Baggage Departments of the Hongkong Airport Services (HAS), Ltd. HAS of the Hong Kong International Airport is the primary handler of all ground services and aircrafts support functions. Our resulting goal programming (GP) approach has, for the actual case study, exhibited its significant impact on the manpower planning issues albeit its apparent modeling simplicity [9]. The primary factor for its success is GP models' ease of handling frequent changes of flight schedules by modeling flexibility in the work patterns of workers' fixed-length duties [13].

Beyond case studies, there are two natural concerns on the GP models further examined in this paper: Concern on model robustness (or its modeling adaptiveness for different problem scenarios) and concern on computation robustness with *integer* variables. The first concern is addressed here by way of successful computations with different key control parameter values for 25 sets of randomly generated problem instances of input data. The second concern is then further addressed in terms of a "best-fitting" type heuristics and its comparative study (with GP) on the same 25 data sets. Finally the heuristics is tested on an extensive set of 1 000 additional randomized data instances.

Since the main focus of this paper is on computational experience, a detailed review on the vast literatures on manpower duties and crew planning/scheduling/rostering problems (DGP/CSP/CRP) is not given here. Instead, we mention two key review references: a classical review of Bodin, Golden, As-

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sad & Ball [3]; and a most updated collection of papers given in a whole issue of the "European Journal of Operational Research" (EJOR) 2004, Vol. 153(1). This February 2004 feature issue of EJOR provides a comprehensive review of the areas of "Timetabling and Rostering". Significant scientific interests are evidenced by the success of the EURO Working group on Automated Timetabling (WATT) and the international series of conferences on the Practice and Theory of Automated Timetabling (PATAT). A dozen or so papers in this special issue report on a wide range of rostering applications, with an editorial by Burke and Petrovic [4]. Examples include review paper of staff scheduling and rostering by Ernst et al. [10]; nurse rostering problem by Bellanti et al. [2]; local search for shift design by Musliu et al. [11]; and a case study of single shift planning and scheduling by Azmat and Widmer [1].

For an overview of DGP/CSP/CRP as such that is more closely related to our specific airport applications, the readers can refer to our forthcoming article to appear in a future issue of EJOR [7] or its conference proceedings versions of Chu [6] and Chu and Yuen [9].

1. GOAL PROGRAMMING FORMULATION

The modeling formulation of DGP that we describe here can be interpreted as the basic core – the planner – of a more sophisticated DGP/CSP/CRP integrated model in the following sense. DGP in its simplest form (computes and) allocates duties (of given fixed structure of work pattern, rather than crew or staff needing further varying requirements of scheduling) to cover known demands. Demands are given, for equally spaced (hourly) time intervals of (the working time of) a day.

As such, DGP is the *prerequisite* to CSP and CRP in that it provides the planning inputs needed in subsequent scheduling and rostering of staff. As its name implies, DGP allocates duties (performed by crew) in an optimal way to meet known demand over a contiguous number of time intervals. We study its *base* formulation in this paper as stated below. A more detailed account of DGP with its extensions is given in an earlier paper of Chu [5] mentioned above.

We use the following notations for our GP model. Let H be the working time horizon, and let $h = 1, \cdots, H$ index the individual hours. R_h denotes the demand for interval h and d_h represents the over allocation (or over-achievement deviation variable in a GP context) at interval h. The length of a duty is denoted by J. The primary decision variable x_{ij} is the number of allocated staff that starts duty from interval i and breaks at the j^{th} interval after the start of duty, $j = 1, \cdots, J$. Hence for a working horizon of intervals $1, \cdots, H$, we have for the index $i = S, \cdots, T$. The earliest start interval S is such that $S \geqslant 1$ whereas the latest start interval T is limited to $T \leqslant H - J + 1$ (to finish work at interval H). Note that normally $S = 1$ as long as $R_1 > 0$ (there is demand for the very first interval); and $T = H - J + 1$ whenever $R_H > 0$ (there is demand for the very last interval).

We are now ready to state the base model of DGP in terms of a (linear integer goal) programming formulation:

$$\text{Min} \qquad \sum_{i=S}^{T} \sum_{j=1}^{J} c_{ij} x_{ij} \quad + \quad WD \tag{0}$$

$$\text{Subject to} \quad \sum_{i=p}^{q} \sum_{j \neq h-i+1} x_{ij} \quad - \quad d_h \quad = \quad R_h \ , \qquad h = 1, \cdots, H \tag{1}$$

$$d_h \ \leqslant \ D \ , \qquad h = 1, \cdots, H \tag{2}$$

where $p \equiv \max\ \{h - J + 1, S\}$, $q \equiv \min\ \{h, T\}$, and the allocation plan $\{\ x_{ij}\ \}$ are non-negative integer variables.

We see that the LHS of constraint (1) is the total work contribution as a function of $\{\ x_{ij}\ \}$ since the summation over $j = 1, \cdots, J$ $(j \neq h - i + 1)$ spans the $J - 1$ working intervals while the summation over $p \leqslant i \leqslant q$ picks out the total number of staff covering interval h. The single variable D of constraint (2) records the maximum (i.e. over achievement) deviation over all time intervals. Hence (for a "smoothed" allocation) it is minimized, either non-preemptively with a weighting factor W as shown in (0) here, or preemptively as the second priority goal. The coefficients $\{\ c_{ij}\ \}$ can either represent the actual unit pays of staff or (V-shaped) weighting parameters for different time and/or meal break intervals.

Lastly, we give a brief explanation of the summation indices of i ranging from p to q in (1): At time interval h, the index i in x_{ij} would lead to a "covering" duty (i.e. x_{ij} contributing workforce supply at time h), if i satisfies $i + J - 1 \geqslant h$ (from the *earliest* possible start of time i). This implies $i \geqslant h - J + 1$. Similarly, i satisfies $i \leqslant h$ (to the *latest* possible start of time i). Hence for index i to cover time h, we must have $h - J + 1 \ \leqslant \ i \ \leqslant \ h$. Therefore, together with $S \leqslant i \leqslant T$, we have

$$p \equiv \max\ \{h - J + 1, S\} \ \leqslant \ i \ \leqslant \ \min\ \{h, T\} \ \equiv \ q,$$

as shown in (1) above.

As an illustration of its computation, we show, in Figure 1, a typical numerical (daily) output from the DGP computation using a simple Lingo code [12] for our bus drivers' fixed-length duties generation application. For this actual problem instance, the parameters used are: $H = 19, I = 11, J = 9, S = 1, T = 11, c_{ij} = 1$ (implying uniform pay rate) and $W = 1000$. Results for a total of 19 hourly time intervals are shown in Figure 1. In the figure, every 3 occurences of the symbol "#" denote one unit of manpower demand; whereas the counterparts of the symbol "0" refer to one unit of manpower over-allocation, for a specific time interval (each row).

```
=======================================================
Time-interval vs Demand(#) and Over-allocation(0)
-------------------------------------------------------
01 #########                                   ( 3/ 0)
02 ##############################              (10/ 0)
03 ##################000000000000              ( 6/ 4)
04 #################################           (11/ 0)
05 #########000000000000000                    ( 3/ 5)
06 #########000000000000000                    ( 3/ 5)
07 ##################000000000000              ( 6/ 5)
08 ############000000000000000                 ( 4/ 5)
09 #########000000000000000                    ( 3/ 5)
10 ##################000000                    ( 6/ 2)
11 ###############000000000000000              ( 5/ 5)
12 ############################000000000000    ( 8/ 4)
13 ########################################    (12/ 0)
14 ############################000000000000    ( 8/ 4)
15 ###############################000000000    ( 9/ 3)
16 ###############################000000000    ( 9/ 3)
17 ###############000000000000000              ( 5/ 5)
18 #########000000000000000                    ( 3/ 5)
19 ###############000000000000000              ( 5/ 5)
-------------------------------------------------------
Note: Time 01 = 0500-0600hr, .. , Time 19 = 0000-0100hr
```

Fig. 1. Calculated allocation vs demand (output of DGP)

A duty amounts to a continuous stretch of work contribution (from a single staff) for a number of time intervals, with a gap of one meal break. At certain time interval(s), the calculated allocation(s) will be over and above the corresponding demand(s). An optimal DGP solution nevertheless minimizes such total over allocation. This (daily) result in Figure 1 leads to a total over allocation ($\sum d_h$ in (1) above) of 65 man-hours. This yields, with a total demand ($\sum R_h$ in (1) above) of 119, an effective overall 'utilization' (ratio) of

$$\text{Ratio} \;\equiv\; 100 \times \frac{\sum R_h}{\sum R_h + \sum d_h} \;=\; 64.67\%.$$

We remark that this performance measure of Ratio is directly (or inversely) proportional to the base model's first criterion objective function value $\sum c_{ij}x_{ij}$, where the coefficients c_{ij} are taken to be 1 throughout this paper. This fact can be readily seen as follows. From (0), we have

$$\text{Min} \sum \, x_{ij} \;=\; \text{Max} \; \frac{1}{\sum \, x_{ij}} \;;$$

and from (1), summing over all time intervals gives $\sum x_{ij} - \sum d_h = \sum R_h$. (This is also clear from Figure 1 with its areas interpretation: Total demand (#) plus total over-allocation (O) equals total allocation.) Hence

$$\text{Ratio} \;=\; 100 \times \frac{\sum R_h}{\sum R_h + \sum d_h} \;=\; 100 \times \frac{\sum R_h}{\sum x_{ij}} \;=\; \frac{\text{Constant}}{\sum x_{ij}}.$$

Therefore Ratio is the normalized performance measure for the GP's first criteria of staff over-allocation. In the extreme (ideal) case of zero over-allocation ($\sum d_h = 0$), it attains its maximum 100% level. It is especially appropriate for comparing different problem instances of different demand patterns generated.

The 25 randomized data sets

Besides this illustration with our project's sample results in Figure 1 above, randomly generated numerical problem instances (with the same model parameters) are reported in Table 1 below to give evidence of the model's robustness. (We allowed the break index $j \in \{1, \cdots, J\}$ – implying a totally flexible break decision interval – for this particular randomization experiment. Further numerical results described later will study the effect on restriction of this break index parameter.) These 25 sets of randomly generated demand requirements are taken from Chu and So [8] where the details of the randomization are provided. For these sets of data (values given in the Appendix A.1), the mean 'utilization' ratio is in fact comparable at 74.83%, with a mean level 4.6 of maximum over allocation. Regarding the columns in Table 1, MinR, MaxR, AvgR and TtlR are, respectively, minimum, maximum, average and total requirements generated; while TtlD, Ratio and MaxD are the total (over-achievement) deviation, the 'utilization' ratio and the maximum deviation from the DGP model outputs for each data set.

Table 1
Results of 25 random problem instances

(i)	Set#	MinR	MaxR	AvgR	TtlR	TtlD	Ratio	MaxD
(1)	9	4	12	7.53	143	17	89.38%	2
(2)	18	4	13	9.84	187	29	86.57%	3
(3)	24	3	13	8.63	164	28	85.42%	3
(4)	7	3	13	9.26	176	32	84.62%	3
(5)	2	3	11	7.47	142	26	84.52%	3
(6)	10	3	13	7.63	145	31	82.39%	3
(7)	17	4	13	8.05	153	39	79.69%	3
(8)	6	3	12	7.42	141	43	76.63%	4
(9)	3	6	13	8.63	164	52	75.93%	4
(10)	25	3	13	7.74	147	53	73.50%	4
(11)	22	4	13	8.53	162	38	81.00%	5
(12)	23	3	13	8.68	165	51	76.39%	5
(13)	8	3	12	7.58	144	48	75.00%	5
(14)	21	3	13	7.84	149	51	74.50%	5
(15)	19	3	13	8.11	154	54	74.04%	5
(16)	11	3	13	7.95	151	57	72.60%	5
(17)	14	3	13	7.26	138	54	71.88%	5
(18)	13	3	13	8.16	155	61	71.76%	5
(19)	1	3	13	8.32	158	66	70.54%	5
(20)	4	3	13	7.42	141	59	70.50%	5
(21)	5	3	13	7.42	141	59	70.50%	5
(22)	15	4	13	9.63	183	81	69.32%	6
(23)	16	3	13	6.68	127	81	61.06%	6
(24)	20	3	13	7.63	145	111	56.64%	8
(25)	12	4	13	7.84	149	115	56.44%	8
	Avg	3.36	12.80	8.05	153.0	53.44	74.83%	4.60

We present the 25 sets of outcomes *re*-ordered in increasing value of D (=MaxD); and for the same MaxD, in decreasing utilization (=Ratio) order. Arranging the 25 sets in this way, it is very noticeable from the last two columns (Ratio, MaxD) that the performance of the computed results (i.e. the Ratio) is highly correlated with the resulting maximum (time-period specific) over allocation (i.e. the MaxD). This ranges from close to 90% utilization with a corresponding maximum over allocation of only 2, to the eighty some percents of MaxD = 3, to the seventy some percents when MaxD = 4 and 5, to the sixty some percents of MaxD = 6, and finally down to only 56% with our largest computed MaxD, being 8 for this 25 sets of randomized sample data. The additional insight gained from this experiment is therefore that the maximum over allocation is actually a rather important performance indicator, even though it is often treated simply as a smoothing measure of secondary priority goal.

Further numerical results (GP)

We further look at the numerical results of the effect on restriction of the important break index parameter j in the decision variables x_{ij}. With the length of a duty $J = 9$, three progressively move restrictive scenarios (as motivated by actual applications) are considered: $j \in [1, 9]$, $j \in [3, 7]$, $j \in [4, 6]$. These same 25 data sets (with their detailed values given in the Appendix A.1) lead to the GP outputs shown in Table 2.

Table 2
GP outputs for the 3 different scenarios

Time/Set	TtlR	GP Output [1,9]			GP Output [3,7]			GP Output [4,6]		
		TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD
1	158	66	70.54%	5	66	70.54%	10	66	70.54%	10
2	142	26	84.52%	3	26	84.52%	8	42	77.17%	8
3	164	52	75.93%	4	52	75.93%	7	52	75.93%	7
4	141	59	70.50%	5	59	70.50%	6	59	70.50%	8
5	141	59	70.50%	5	59	70.50%	8	59	70.50%	8
6	141	43	76.63%	4	43	76.63%	4	43	76.63%	5
7	176	32	84.62%	3	32	84.62%	8	40	81.48%	8
8	144	48	75.00%	5	48	75.00%	8	56	72.00%	8
9	143	17	89.38%	2	17	89.38%	3	17	89.38%	6
10	145	31	82.39%	3	31	82.39%	4	31	82.39%	5
11	151	57	72.60%	5	57	72.60%	7	57	72.60%	8
12	149	115	56.44%	8	115	56.44%	11	115	56.44%	11
13	155	61	71.76%	5	61	71.76%	8	61	71.76%	9
14	138	54	71.88%	5	54	71.88%	7	54	71.88%	8
15	183	81	69.32%	6	81	69.32%	9	81	69.32%	9
16	127	81	61.06%	6	81	61.06%	7	81	61.06%	7
17	153	39	79.69%	3	39	79.69%	7	39	79.69%	7
18	187	29	86.57%	3	29	86.57%	6	45	80.60%	7
19	154	54	74.04%	5	54	74.04%	6	62	71.30%	8
20	145	111	56.64%	8	111	56.64%	10	111	56.64%	11
21	149	51	74.50%	5	51	74.50%	8	59	71.63%	8
22	162	38	81.00%	5	38	81.00%	7	54	75.00%	8
23	165	51	76.39%	5	51	76.39%	9	51	76.39%	9
24	164	28	85.42%	3	28	85.42%	6	36	82.00%	6
25	147	53	73.50%	4	53	73.50%	6	53	73.50%	7
	Avg =		74.83%	4.60		74.83%	7.20		73.45%	7.84
	Min =		56.44%	2		56.44%	3		56.44%	5
	Max =		89.38%	8		89.38%	11		89.38%	11
	Range =		32.94%	6.00		32.94%	8.00		32.94%	6.00
	Std Dev =		8.53%	1.47		8.53%	1.89		7.60%	1.52

As expected, scenario [1,9] performs the best and is naturally taken as the benchmark for performance measures of *low* TtlD (first goal) and *low* MaxD (second goal). Scenario [3,7] comes next and, with a mild nice suprise, attains fully all 25 cases of the TtlD goal, but with now all cases except one (Set 6) of higher MaxD values. Scenario [4,6] reaches only 17 cases (except for Sets 2,7,8,18,19,21,22,24) of the TtlD goal, with all cases of higher MaxD values (and 12 cases compared to scenario [3,7]). Trade-offs between meal break restriction and performance measures, especially the maximum (interval specific) over-allocation level MaxD are clearly evident in this comparison.

2. HEURISTICS ALGORITHMIC APPROACH

Concern on computation robustness of DGP, a GP with *integer* variables, has led us to study the following "best-fitting" type heuristics. Indeed, the solution time for our case application (results in Figure 1) is very fast of less than 1 minute. However, the range of computational times for the 25 random instances in Table 1 is extremely large. For eight of the 25 cases, each takes more than 1.5 hours on a Pantium PC, while the remaining 17 cases average to less than 1 minute. Of the eight "hard" cases, four require manually fixing the single variable MaxD to be integer and solving its LP relaxation instead, due to the fact that each of their ILP times already exceeds our preset 10-hour limit.

(We remark here that the following heuristics has appeared in our electronic proceedings paper of a recent International MOPGP Conference [9]. It is explicitly included here considering the difficulty of readers' gaining access to a paper in electronic proceedings.)

Minimax time-reversible heuristics

Starting at time interval 1 and marching in a time-forward manner, we add each duty sequentially, selecting the break-hour which myopically minimizes its chosen interval's remaining demand (over its L covering intervals). Mathematically, denote

$$r_h \;\equiv\; \text{remaining demand at time } h,\; h = 1, \cdots, H.$$

As we consider a duty starting from time i (when $r_i \geqslant 1$), we add another duty $\triangle x_{ij} = 1$ such that j is chosen as the minimizing index in

$$r_{i+j-1} \;=\; \text{Min}_{k=1,\cdots L}\; r_{i+k-1}.$$

Time intervals are processed from $i = 1$ to $i = H+L-1$ (forward). Note that each chosen break-hour j is locally the time interval with the minimum r_j (as defined above) *and* the maximum d_j, which is the (current value of) over-allocation (when the minimum r_j is zero) at time interval j. (Note that $d_j \times r_j = 0$, or d_j can be positive only when $r_j = 0$.) Hence it is a minimax (and time-forward) greedy heuristics.

An obvious improvement is to process the time intervals starting at time $H + L - 1$ and working in a time-backward manner, i.e. from $i = H + L - 1$ to $i = 1$. This results in a minimax (and time-backward) greedy heuristics.

Combining these two by taking the better performance gives what we call a *Minimax Time-reversible Heuristics*. Its numerical performances for the same set of 25 random problem instances are given in Table 3. It can be seen from Table 3 that 20 out of 25 cases the (GP) optimal 'utilization' ratios are attained (with 12 from Forward alone, and 19 from Backward alone). A perhaps rather surprising further improvement is when we apply randomization to the choice of break-hour j (replacing the above minimax rule). Here this actually gives a higher number of 23 out of 25 cases of optimal ratios. (A side remark on computation: Randomization results naturally vary among different computer runs. In our case, the solution is taken from the best of iterations of 20,000 replicas, taking a total of about 45 minutes on a Pentium PC. Each heuristics trial, in either the minimax or the randomization case, takes negligible amount of computer time.)

Table 3
Heuristics Results of the 25 random problem instances

(0)	(1)	(2)	(3)	(4)
GP Solutions	25	74.83%	25	4.60
Heuristics (Minimax)				
- Forward	12	70.89%	2	7.20
- Backward	19	73.67%	7	6.52
Combined (better of F+B)	20	74.05%	8	5.92
Heuristics (Randomized)				
- Forward	20	74.12%	1	7.04
- Backward	22	74.38%	0	7.12
Combined (better of F+B)	22	74.38%	1	6.76
Complete Heuristics (Minimax+Randomized, with time reversibility)	25	74.83%	9	5.92

Note: Column (0) = Approach
(1) = Number of cases attaining optimal GP's (Maximum) Ratios
(2) = Computed average Ratios
(3) = Number of cases attaining optimal GP's (Minimum) MaxD's
(4) = Computed average MaxD's

For our 25 problem instances, it turns out that together with the final help from randomization, all 25 optimal (GP) ratios are attained by our heuristics. This of course can never be assumed for other different data sets and/or larger scale experiments. Indeed, in terms of the MaxD measure, our complete heuristics can only achieve 9 out of 25 cases of optimal (GP) MaxD's, as shown in Table 3. As expected, the minimax heuristics performs rather better than the randomization (alone) calculations on MaxD. Nevertheless, together all these point to the robust computing benchmark for the DGP optimization model, backed by efficient heuristics as such.

Table 4
Heuristics outputs for the 3 different scenarios

Time/ Set	TtlR	The Best Heuristic Output [1,9]			The Best Heuristic Output [3,7]			The Best Heuristic Output [4,6]		
		TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD
1	158	66	70.54%	8	66	70.54%	11	66	70.54%	12
2	142	26	84.52%	3	26	84.52%	8	42	77.17%	12
3	164	52	75.93%	6	52	75.93%	7	52	75.93%	7
4	141	59	70.50%	6	59	70.50%	7	59	70.50%	9
5	141	59	70.50%	7	59	70.50%	8	59	70.50%	8
6	141	43	76.63%	5	43	76.63%	5	43	76.63%	5
7	176	32	84.62%	3	32	84.62%	8	40	81.48%	8
8	144	48	75.00%	6	48	75.00%	8	56	72.00%	9
9	143	17	89.38%	2	17	89.38%	4	17	89.38%	6
10	145	31	82.39%	3	31	82.39%	5	31	82.39%	5
11	151	57	72.60%	7	57	72.60%	9	57	72.60%	12
12	149	115	56.44%	11	115	56.44%	13	115	56.44%	12
13	155	61	71.76%	5	61	71.76%	8	61	71.76%	9
14	138	54	71.88%	6	54	71.88%	7	54	71.88%	9
15	183	81	69.32%	9	81	69.32%	10	81	69.32%	10
16	127	81	61.06%	8	81	61.06%	8	81	61.06%	10
17	153	39	79.69%	5	39	79.69%	7	39	79.69%	7
18	187	29	86.57%	7	29	86.57%	6	53	77.92%	10
19	154	54	74.04%	5	54	74.04%	7	54	74.04%	14
20	145	111	56.64%	11	111	56.64%	12	111	56.64%	14
21	149	51	74.50%	5	51	74.50%	9	59	71.63%	10
22	162	38	81.00%	5	38	81.00%	12	46	77.88%	13
23	165	51	76.39%	7	51	76.39%	9	51	76.39%	10
24	164	28	85.42%	3	28	85.42%	11	52	75.93%	10
25	147	53	73.50%	5	53	73.50%	8	53	73.50%	10
	Avg =		74.83%	5.92		74.83%	8.28		73.33%	9.64
	Min =		56.44%	2		56.44%	4		56.44%	5
	Max =		89.38%	11		89.38%	13		89.38%	14
	Range =		32.94%	9.00		32.94%	9.00		32.94%	9.00
	Std Dev =		8.53%	2.33		8.53%	2.26		7.36%	2.50
	Count =	25		9	25		10	21		8

Further numerical results (heuristics)

Similar to the consideration given to GP computations, we further look at the numerical results of the effect on restriction of the important break index parameter j in the decision variables x_{ij}, for the heuristics. Again, with the length of a duty $J = 9$, three progressively move restrictive scenarios (as motivated by actual applications) are considered: $j \in [1, 9]$, $j \in [3, 7]$, $j \in [4, 6]$. These same 25 input data sets (with their detailed values given in the Appendix A.1) lead to the numerical outputs shown in Table 4. The nice "surprise" here is that Table 4 reads rather consistently similar to Table 2. (See the shaded entries, which indicate common values in both tables.) This consistency is also both in terms of the three different cases attaining their individual different levels of the two goals TtlD and MaxD, *and* the relative progressive degrees of performances (in the goals). For these data sets at least, the heuristics are observed to be also very robust with respect to the key parameter (break index) j of central importance, besides the obvious first goal of TtlD.

3. THE DETAILED COMPARISONS

We have seen from Table 3 the competing performance of the heuristics with respect to the GP solutions for the situation of $j \in [1, 9]$. Perfect performance is scored on the first goal of TtlD, while worse-off level is recorded on the second goal of MaxD (in Table 3). Here Table 5 gives a summary of the detailed comparisons of all three scenarios of $j \in [1, 9]$, $j \in [3, 7]$, $j \in [4, 6]$. It can be seen that out of 25 cases, the (best outputs of the) three heuritics achieve, respectively 25,25,21 cases of their GP counterpart (optimal) solutions on TtlD; and 9,10,8 cases on MaxD. (These are high-lighted as shaded entries in Table 5, which combines Tables 2 and 4.) While the heuristics is confirmed as very competetive on TtlD, it is (intuitively expected to be) much less so on the second goal MaxD. This is an vivid illustration of the superiority of a GP approach, whenever its computation can be completed within an acceptable amount of computer time and resources.

Three additional tables providing all details to the contributing (or relative) performance of forward vs backward as well as potential benefit in randomization are given in the Appendix A.3 for completeness. Table 6 below gives an overall summary concerning these two aspects.

Table 5
Comparison of heuristics vs GP performances for the 3 scenarios

Time/ Set	TtlR	GP Output [1,9]			The Best Heuristic Output [1,9]			GP Output [3,7]			The Best Heuristic Output [3,7]			GP Output [4,6]			The Best Heuristic Output [4,6]		
		TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD
1	158	66	70.54%	5	66	70.54%	8	66	70.54%	10	66	70.54%	11	66	70.54%	10	66	70.54%	12
2	142	26	84.52%	3	26	84.52%	3	26	84.52%	8	26	84.52%	8	42	77.17%	8	42	77.17%	12
3	164	52	75.93%	4	52	75.93%	6	52	75.93%	7	52	75.93%	7	52	75.93%	7	52	75.93%	7
4	141	59	70.50%	5	59	70.50%	6	59	70.50%	6	59	70.50%	7	59	70.50%	8	59	70.50%	9
5	141	59	70.50%	5	59	70.50%	7	59	70.50%	8	59	70.50%	8	59	70.50%	8	59	70.50%	8
6	141	43	76.63%	4	43	76.63%	5	43	76.63%	4	43	76.63%	5	43	76.63%	5	43	76.63%	5
7	176	32	84.62%	3	32	84.62%	3	32	84.62%	8	32	84.62%	8	40	81.48%	8	40	81.48%	8
8	144	48	75.00%	5	48	75.00%	6	48	75.00%	8	48	75.00%	8	56	72.00%	8	56	72.00%	9
9	143	17	89.38%	2	17	89.38%	2	17	89.38%	3	17	89.38%	4	17	89.38%	6	17	89.38%	6
10	145	31	82.39%	3	31	82.39%	3	31	82.39%	4	31	82.39%	5	31	82.39%	5	31	82.39%	5
11	151	57	72.60%	5	57	72.60%	7	57	72.60%	7	57	72.60%	9	57	72.60%	8	57	72.60%	12
12	149	115	56.44%	8	115	56.44%	11	115	56.44%	11	115	56.44%	13	115	56.44%	11	115	56.44%	12
13	155	61	71.76%	5	61	71.76%	5	61	71.76%	8	61	71.76%	8	61	71.76%	9	61	71.76%	9
14	138	54	71.88%	5	54	71.88%	6	54	71.88%	7	54	71.88%	7	54	71.88%	8	54	71.88%	9
15	183	81	69.32%	6	81	69.32%	9	81	69.32%	9	81	69.32%	10	81	69.32%	9	81	69.32%	10
16	127	81	61.06%	6	81	61.06%	8	81	61.06%	7	81	61.06%	8	81	61.06%	7	81	61.06%	10
17	153	39	79.69%	3	39	79.69%	5	39	79.69%	7	39	79.69%	7	39	79.69%	7	39	79.69%	7
18	187	29	86.57%	3	29	86.57%	7	29	86.57%	6	29	86.57%	6	45	80.60%	7	53	77.92%	10
19	154	54	74.04%	5	54	74.04%	5	54	74.04%	6	54	74.04%	7	62	71.30%	8	54	74.04%	14
20	145	111	56.64%	8	111	56.64%	11	111	56.64%	10	111	56.64%	12	111	56.64%	11	111	56.64%	14
21	149	51	74.50%	5	51	74.50%	5	51	74.50%	8	51	74.50%	9	59	71.63%	8	59	71.63%	10
22	162	38	81.00%	5	38	81.00%	5	38	81.00%	7	38	81.00%	12	54	75.00%	8	46	77.88%	13
23	165	51	76.39%	5	51	76.39%	7	51	76.39%	9	51	76.39%	9	51	76.39%	9	51	76.39%	10
24	164	28	85.42%	3	28	85.42%	3	28	85.42%	6	28	85.42%	11	36	82.00%	6	52	75.93%	10
25	147	53	73.50%	4	53	73.50%	5	53	73.50%	6	53	73.50%	8	53	73.50%	7	53	73.50%	10
	Avg =		74.83%	4.60		74.83%	5.92		74.83%	7.20		74.83%	8.28		73.45%	7.84		73.33%	9.64
	Min =		56.44%	2		56.44%	2		56.44%	3		56.44%	4		56.44%	5		56.44%	5
	Max =		89.38%	8		89.38%	11		89.38%	11		89.38%	13		89.38%	11		89.38%	14
	Range =		32.94%	6.00		32.94%	9.00		32.94%	8.00		32.94%	9.00		32.94%	6.00		32.94%	9.00
	Std Dev =		8.53%	1.47		8.53%	2.33		8.53%	1.89		8.53%	2.26		7.60%	1.52		7.36%	2.50
	Count =				25		9				25		10				21		8

Note: 1) S1[1,9] - meal break = randomly assigned with [ESB,LSB]=[1,9]
2) S2[1,9] - meal break = 1st available least demand hour with [ESB,LSB]=[1,9]
3) S1[3,7] - meal break = randomly assigned with [ESB,LSB]=[3,7]
4) S2[3,7] - meal break = 1st available least demand hour with [ESB,LSB]=[3,7]
5) S1[4,6] - meal break = randomly assigned with [ESB,LSB]=[4,6]
6) S2[4,6] - meal break = 1st available least demand hour with [ESB,LSB]=[4,6]
7) Count = no. of cases (out of 25) that the Best Heuristic Output yield the same TtlD and MaxD as that of GP Output.

Table 6
Forward vs backward & randomization aspect for the three scenarios

Scenario	S1:(Forward)/(Backward)	S2:(Forward)/(Backward)	Best of 4
$j \in [\,1,9\,]$	$(20,11)\ /\ (22,\ \ 9)$	$(12,\ \ 8)\ /\ (19,14)$	$(25,\ \ 9)$
$j \in [\,3,7\,]$	$(17,14)\ /\ (19,13)$	$(24,13)\ /\ (24,\ \ 7)$	$(25,10)$
$j \in [\,4,6\,]$	$(16,13)\ /\ (17,14)$	$(16,10)\ /\ (20,\ \ 9)$	$(21,\ \ 8)$

In Table 6, an ordered pair of entries (m, n) represent the numbers of cases for the measures (TtlD, MaxD) computed by the heuristics succeed in attaining the optimal GP results. $S1$ denotes the heuristics randomly assigning the meal break index over $j \in [1, 9]$, $j \in [3, 7]$, $j \in [4, 6]$ — the three tested scenarios indicated in Table 6 as row labels. $S2$ refers to the (heuristic) rule of assigning j as the first available least (residual) demand time interval. (Hence the higher values of (m, n) nearer and up to (25, 25) the better.) While more insight can be gained from the detailed values from the tables in the Appendix A.3, it is already evident from Table 6 that:

– It concurs with intuition that comparable MaxD values are regardless of the direction of forward or backward computation (due to the symmetric or time-reversibility problem nature).

– It is much less expected to see that the computed values of TtlD always benefit from *backward* computations in all six pairs of scenario by time-reversibility settings.

– Assigning meal break by the least demand heuristic rule ($S1$) works well only when the choice of break time interval is wide (i.e. in the case of $j \in [1, 9]$) and is actually worse in the more restrictive cases (i.e. $j \in [3, 7]$ and $j \in [4, 6]$), for the TtlD goal, in comparison to randomized choices ($S2$).

– However, ($S1$) does work better on the whole with respect to the MaxD goal than ($S2$), as might be expected heuristically.

Taking all of the above together, all four combinations of S1:(forward/backword) and S2:(forward/backward) are essential components of our minimax time-reversible heuristics. The complete heuristics fares very well indeed, especially for the case of $j \in [3, 7]$ which is particularly important from an application point of view rendering it the best choice of (meal break) implementation. This choice is further echoed in our final extensive test of 1 000 new cases of similarly randomly generated input demand data. The 1 000 cases were run by the heuristics for the three break time restriction scenarios for a total of 3 000 runs. Out of them, 10 cases for each scenario were solved to optimality by our DGP-GP code. Of these 30 selected cases, only three fail to attain the GP's optimal TtlD performance (Sets 301, 431, 751); and they all belong to the most restrictve scenario of $j \in [4, 6]$. The results are provided in the Appendix A.2, again for the sake of completeness (and additional information on the performance issue on the MaxD goal as well).

CONCLUDING REMARKS

The purpose of this paper is to report by way of DGP modeling and its further extensive computational experience, the advantage of DGP's readily producing improvement over existing manual staff assignment. In this context, we contrast this paper with an earlier work of Chu (2001), where the sole purpose there was to apply DGP and its extended version for a single instance of real data of the airport case study. Exact solutions (as shown in Figure 1 before) were easily computed then for its set of input data.

The integer programming nature of DGP has since then led us from the application to examining much more into DGP as an independent problem, with its more intriguing computational robustness issue. Thus the key contribution of this paper is the construction and the extensive computation experience of the (now proven) effective heuristics, whenever exact GP computations are facing difficulty with certain problem data instances. In short, the model's usefulness to the users is also strengthened by its computational robustness, in both exact solutions and heuristics calculations.

APPENDIX

A.1. The 25 randomly generated input data sets

Time/ Set	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	TtlR
1	7	8	10	6	13	8	12	9	4	13	5	13	6	3	9	11	5	8	8	158
2	8	11	11	9	6	11	7	9	3	5	5	7	7	6	8	10	8	3	8	142
3	13	6	8	11	8	6	12	12	6	6	11	8	9	10	7	8	8	7	8	164
4	9	3	7	3	4	9	3	13	5	5	10	5	5	11	13	9	12	11	4	141
5	13	8	13	8	6	4	6	10	6	3	4	9	3	3	7	9	11	9	9	141
6	10	6	5	4	8	6	11	11	6	7	9	9	3	6	11	4	12	8	5	141
7	3	12	10	12	13	4	5	11	13	11	4	11	11	8	8	13	8	7	12	176
8	8	12	5	9	11	12	6	8	10	5	8	4	10	6	7	6	3	3	11	144
9	5	7	8	7	6	4	9	10	6	5	9	12	8	10	4	11	4	10	8	143
10	8	4	8	4	3	10	7	11	13	8	13	9	11	7	3	12	3	8	3	145
11	9	8	3	11	13	9	4	5	8	8	5	9	9	8	6	12	6	9	9	151
12	11	4	11	8	8	6	5	11	6	9	6	10	8	8	6	11	4	4	13	149
13	13	9	11	10	7	10	4	6	5	11	12	9	7	6	12	7	5	8	3	155
14	12	5	5	5	13	11	11	5	6	6	12	3	4	8	4	6	9	10	3	138
15	13	13	10	12	4	8	7	13	10	9	5	8	13	7	11	10	9	10	11	183
16	9	5	5	5	7	4	11	4	8	9	13	8	3	5	8	4	8	3	8	127
17	4	12	8	6	5	13	11	11	5	8	6	6	10	6	7	8	11	8	8	153
18	7	13	8	13	6	10	10	8	7	9	9	12	4	13	12	11	13	11	11	187
19	5	5	9	8	8	13	11	9	12	8	5	8	7	12	6	4	13	8	3	154
20	8	10	3	3	5	8	5	6	6	13	6	8	4	11	13	6	9	10	11	145
21	11	3	10	12	11	3	5	9	7	5	5	9	8	11	6	13	5	12	4	149
22	9	10	13	8	7	7	12	4	5	5	9	7	10	9	10	12	10	8	7	162
23	12	3	9	11	5	11	3	13	8	7	6	4	10	11	9	13	12	10	8	165
24	5	3	10	10	9	9	9	12	8	3	8	9	7	13	12	4	10	12	11	164
25	4	3	8	7	8	13	9	9	5	7	9	7	11	5	3	12	8	10	9	147

A.2. The 30 selected samples from 1 000 randomized cases

Time/ Set	TtlR	The Best Heuristic Output [1,9]			The Best Heuristic Output [3,7]			The Best Heuristic Output [4,6]		
		TtlD	MaxD(H)	MaxD	TtlD	MaxD(H)	MaxD	TtlD	MaxD(H)	MaxD
1	157	51	5	5	51	9	9	59	10	10
101	154	54	6	4	54	8	8	54	12	8
201	159	41	5	4	41	10	9	57	12	10
301	168	32	4	3	32	9	7	(40<) 48	9	8
431	144	40	4	3	40	9	8	(48<) 56	10	8
511	148	60	7	6	60	6	6	76	10	6
671	145	47	5	5	47	8	8	47	9	9
751	161	31	8	4	31	7	7	(47<) 55	9	8
801	155	45	8	5	45	8	8	53	8	8
901	161	47	5	4	47	7	7	71	10	10
	Remark:	Shaded	cells are	worse-off	heuristics	results.				

A.3. The complete GP and heuristics outputs of the 25 data sets

Time/ Set	TtlR	GP Output [1,9]			S1[1,9] Forward			S1[1,9] Backward			S2[1,9] Forward			S2[1,9] Backward			The Best Heuristic Output [1,9]		
		TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD
1	158	**66**	**70.54%**	**5**	66	70.54%	9	66	70.54%	8	82	65.83%	10	66	70.54%	8	**66**	**70.54%**	**8**
2	142	**26**	**84.52%**	**3**	34	80.68%	5	34	80.68%	5	50	73.96%	6	26	84.52%	3	**26**	**84.52%**	**3**
3	164	**52**	**75.93%**	**4**	52	75.93%	6	52	75.93%	7	76	68.33%	7	60	73.21%	8	**52**	**75.93%**	**6**
4	141	**59**	**70.50%**	**5**	59	70.50%	7	59	70.50%	7	59	70.50%	6	59	70.50%	7	**59**	**70.50%**	**6**
5	141	**59**	**70.50%**	**5**	59	70.50%	7	59	70.50%	7	75	65.28%	7	83	62.95%	9	**59**	**70.50%**	**7**
6	141	**43**	**76.63%**	**4**	43	76.63%	6	43	76.63%	6	43	76.63%	5	43	76.63%	5	**43**	**76.63%**	**5**
7	176	**32**	**84.62%**	**3**	40	81.48%	7	40	81.48%	6	80	68.75%	9	32	84.62%	3	**32**	**84.62%**	**3**
8	144	**48**	**75.00%**	**5**	48	75.00%	6	48	75.00%	6	72	66.67%	8	64	69.23%	5	**48**	**75.00%**	**6**
9	143	**17**	**89.38%**	**2**	25	85.12%	4	25	85.12%	4	33	81.25%	5	17	89.38%	2	**17**	**89.38%**	**2**
10	145	**31**	**82.39%**	**3**	31	82.39%	6	31	82.39%	8	31	82.39%	5	31	82.39%	3	**31**	**82.39%**	**3**
11	151	**57**	**72.60%**	**5**	57	72.60%	7	57	72.60%	7	65	69.91%	7	57	72.60%	7	**57**	**72.60%**	**7**
12	149	**115**	**56.44%**	**8**	115	56.44%	11	115	56.44%	11	115	56.44%	13	115	56.44%	11	**115**	**56.44%**	**11**
13	155	**61**	**71.76%**	**5**	61	71.76%	7	61	71.76%	8	61	71.76%	5	85	64.58%	11	**61**	**71.76%**	**5**
14	138	**54**	**71.88%**	**5**	54	71.88%	6	54	71.88%	7	54	71.88%	6	54	71.88%	6	**54**	**71.88%**	**6**
15	183	**81**	**69.32%**	**6**	81	69.32%	9	81	69.32%	10	89	67.28%	10	81	69.32%	10	**81**	**69.32%**	**9**
16	127	**81**	**61.06%**	**6**	81	61.06%	9	81	61.06%	8	81	61.06%	9	81	61.06%	10	**81**	**61.06%**	**8**
17	153	**39**	**79.69%**	**3**	39	79.69%	5	39	79.69%	6	63	70.83%	6	39	79.69%	5	**39**	**79.69%**	**5**
18	187	**29**	**86.57%**	**3**	37	83.48%	8	29	86.57%	7	77	70.83%	8	37	83.48%	5	**29**	**86.57%**	**7**
19	154	**54**	**74.04%**	**5**	54	74.04%	8	54	74.04%	6	54	74.04%	7	54	74.04%	5	**54**	**74.04%**	**5**
20	145	**111**	**56.64%**	**8**	111	56.64%	11	111	56.64%	11	111	56.64%	12	111	56.64%	13	**111**	**56.64%**	**11**
21	149	**51**	**74.50%**	**5**	51	74.50%	7	51	74.50%	6	51	74.50%	5	51	74.50%	5	**51**	**74.50%**	**5**
22	162	**38**	**81.00%**	**5**	38	81.00%	5	38	81.00%	7	54	75.00%	5	38	81.00%	6	**38**	**81.00%**	**5**
23	165	**51**	**76.39%**	**5**	51	76.39%	7	51	76.39%	7	59	73.66%	6	59	73.66%	8	**51**	**76.39%**	**7**
24	164	**28**	**85.42%**	**3**	36	82.00%	6	28	85.42%	6	28	85.42%	5	28	85.42%	3	**28**	**85.42%**	**3**
25	147	**53**	**73.50%**	**4**	53	73.50%	7	53	73.50%	7	53	73.50%	8	53	73.50%	5	**53**	**73.50%**	**5**
Avg =			74.83%	4.60		74.12%	7.04		74.38%	7.12		70.89%	7.20		73.67%	6.52		74.83%	5.92
Min =			56.44%	2		56.44%	4		56.44%	4		56.44%	5		56.44%	2		56.44%	2
Max =			89.38%	8		85.12%	11		86.57%	11		85.42%	13		89.38%	13		89.38%	11
Range =			32.94%	6.00		28.68%	7.00		30.13%	7.00		28.98%	8.00		32.94%	11.00		32.94%	9.00
Std Dev =			8.53%	1.47		7.62%	1.74		7.97%	1.64		6.95%	2.25		8.90%	2.93		8.53%	2.33
Count1 =							11			9			5			14			
Count2 =																	25		9

Note: 1) S1[1,9] - meal break = randomly assigned with [ESB,LSB]=[1,9]
2) S2[1,9] - meal break = 1st available least demand hour with [ESB,LSB]=[1,9]
3) Count1 = no. of cases (out of 25) that yield the Best Heuristic Output for Ratio as well as MaxD.
(= no. of cases highlighted)
4) Count2 = no. of cases (out of 25) that the Best Heuristic Output yield the same TtlD and MaxD as that of GP Output.
(= no. of cases highlighted)

Time/Set	TtlR	GP Output [3,7]			S1[3,7]						S2[3,7]						The Best Heuristic Output [3,7]		
					Forward			Backward			Forward			Backward					
		TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD
1	158	**66**	**70.54%**	**10**	66	70.54%	11	66	70.54%	13	66	70.54%	15	66	70.54%	14	**66**	**70.54%**	**11**
2	142	**26**	**84.52%**	**8**	34	80.68%	8	34	80.68%	11	26	84.52%	8	26	84.52%	10	**26**	**84.52%**	**8**
3	164	**52**	**75.93%**	**7**	52	75.93%	7	52	75.93%	9	52	75.93%	7	52	75.93%	12	**52**	**75.93%**	**7**
4	141	**59**	**70.50%**	**6**	59	70.50%	7	59	70.50%	7	59	70.50%	8	59	70.50%	9	**59**	**70.50%**	**7**
5	141	**59**	**70.50%**	**8**	59	70.50%	8	59	70.50%	9	59	70.50%	8	59	70.50%	10	**59**	**70.50%**	**8**
6	141	**43**	**76.63%**	**4**	43	76.63%	5	43	76.63%	8	43	76.63%	5	43	76.63%	9	**43**	**76.63%**	**5**
7	176	**32**	**84.62%**	**8**	40	81.48%	9	40	81.48%	9	32	84.62%	10	32	84.62%	8	**32**	**84.62%**	**8**
8	144	**48**	**75.00%**	**8**	48	75.00%	8	48	75.00%	8	48	75.00%	9	48	75.00%	8	**48**	**75.00%**	**8**
9	143	**17**	**89.38%**	**3**	25	85.12%	4	25	85.12%	4	17	89.38%	4	17	89.38%	4	**17**	**89.38%**	**4**
10	145	**31**	**82.39%**	**4**	31	82.39%	5	31	82.39%	5	31	82.39%	5	31	82.39%	5	**31**	**82.39%**	**5**
11	151	**57**	**72.60%**	**7**	57	72.60%	10	57	72.60%	9	57	72.60%	12	57	72.60%	9	**57**	**72.60%**	**9**
12	149	**115**	**56.44%**	**11**	115	56.44%	15	115	56.44%	13	115	56.44%	18	115	56.44%	16	**115**	**56.44%**	**13**
13	155	**61**	**71.76%**	**8**	61	71.76%	8	61	71.76%	11	61	71.76%	8	61	71.76%	14	**61**	**71.76%**	**8**
14	138	**54**	**71.88%**	**7**	54	71.88%	7	54	71.88%	8	54	71.88%	7	54	71.88%	8	**54**	**71.88%**	**7**
15	183	**81**	**69.32%**	**9**	81	69.32%	13	81	69.32%	10	81	69.32%	15	81	69.32%	12	**81**	**69.32%**	**10**
16	127	**81**	**61.06%**	**7**	81	61.06%	9	81	61.06%	8	81	61.06%	12	81	61.06%	11	**81**	**61.06%**	**8**
17	153	**39**	**79.69%**	**7**	47	76.50%	7	39	79.69%	10	39	79.69%	7	39	79.69%	11	**39**	**79.69%**	**7**
18	187	**29**	**86.57%**	**6**	37	83.48%	6	37	83.48%	9	29	86.57%	6	29	86.57%	9	**29**	**86.57%**	**6**
19	154	**54**	**74.04%**	**6**	54	74.04%	12	54	74.04%	7	54	74.04%	14	54	74.04%	8	**54**	**74.04%**	**7**
20	145	**111**	**56.64%**	**10**	111	56.64%	13	111	56.64%	12	111	56.64%	16	111	56.64%	15	**111**	**56.64%**	**12**
21	149	**51**	**74.50%**	**8**	59	71.63%	10	51	74.50%	9	51	74.50%	9	51	74.50%	9	**51**	**74.50%**	**9**
22	162	**38**	**81.00%**	**7**	46	77.88%	8	46	77.88%	13	46	77.88%	7	38	81.00%	12	**38**	**81.00%**	**12**
23	165	**51**	**76.39%**	**9**	51	76.39%	9	51	76.39%	9	51	76.39%	9	51	76.39%	11	**51**	**76.39%**	**9**
24	164	**28**	**85.42%**	**6**	36	82.00%	10	44	78.85%	8	28	85.42%	11	36	82.00%	9	**28**	**85.42%**	**11**
25	147	**53**	**73.50%**	**6**	53	73.50%	9	53	73.50%	8	53	73.50%	12	53	73.50%	10	**53**	**73.50%**	**8**
	Avg =		74.83%	7.20		73.76%	8.72		73.87%	9.08		74.71%	9.68		74.69%	10.12		74.83%	8.28
	Min =		56.44%	3		56.44%	4		56.44%	4		56.44%	4		56.44%	4		56.44%	4
	Max =		89.38%	11		85.12%	15		85.12%	13		89.38%	18		89.38%	16		89.38%	13
	Range =		32.94%	8.00		28.68%	11.00		28.68%	9.00		32.94%	14.00		32.94%	12.00		32.94%	9.00
	Std Dev =		8.53%	1.89		7.47%	2.65		7.42%	2.25		8.46%	3.73		8.38%	2.82		8.53%	2.26
	Count1 =						10			12			13			7			
	Count2 =																25		10

Note: 1) S1[3,7] - meal break = randomly assigned with [ESB,LSB]=[3,7]
2) S2[3,7] - meal break = 1st available least demand hour with [ESB,LSB]=[3,7]
3) Count1 = no. of cases (out of 25) that yield the Best Heuristic Output for Ratio as well as MaxD.
(= no. of cases highlighted)
4) Count2 = no. of cases (out of 25) that the Best Heuristic Output yield the same TtlD and MaxD as that of GP Output.
(= no. of cases highlighted)

Time/Set	TtlR	GP Output [4,6]			S1[4,6]						S2[4,6]						The Best Heuristic Output [4,6]		
					Forward			Backward			Forward			Backward					
		TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD	TtlD	Ratio	MaxD
1	158	**66**	**70.54%**	**10**	66	70.54%	14	66	70.54%	12	66	70.54%	15	66	70.54%	14	**66**	**70.54%**	**12**
2	142	**42**	**77.17%**	**8**	50	73.96%	9	42	77.17%	12	50	73.96%	9	50	73.96%	13	**42**	**77.17%**	**12**
3	164	**52**	**75.93%**	**7**	52	75.93%	7	52	75.93%	9	52	75.93%	7	52	75.93%	13	**52**	**75.93%**	**7**
4	141	**59**	**70.50%**	**8**	59	70.50%	9	59	70.50%	9	59	70.50%	9	59	70.50%	10	**59**	**70.50%**	**9**
5	141	**59**	**70.50%**	**8**	59	70.50%	8	59	70.50%	9	59	70.50%	8	59	70.50%	10	**59**	**70.50%**	**8**
6	141	**43**	**76.63%**	**5**	43	76.63%	5	43	76.63%	8	43	76.63%	5	43	76.63%	9	**43**	**76.63%**	**5**
7	176	**40**	**81.48%**	**8**	48	78.57%	12	48	78.57%	9	48	78.57%	12	40	81.48%	8	**40**	**81.48%**	**8**
8	144	**56**	**72.00%**	**8**	56	72.00%	9	56	72.00%	9	56	72.00%	9	56	72.00%	9	**56**	**72.00%**	**9**
9	143	**17**	**89.38%**	**6**	25	85.12%	6	25	85.12%	6	25	85.12%	7	17	89.38%	6	**17**	**89.38%**	**6**
10	145	**31**	**82.39%**	**5**	31	82.39%	7	31	82.39%	5	39	78.80%	9	31	82.39%	5	**31**	**82.39%**	**5**
11	151	**57**	**72.60%**	**8**	65	69.91%	13	73	67.41%	14	65	69.91%	13	57	72.60%	12	**57**	**72.60%**	**12**
12	149	**115**	**56.44%**	**11**	115	56.44%	14	115	56.44%	12	115	56.44%	18	115	56.44%	16	**115**	**56.44%**	**12**
13	155	**61**	**71.76%**	**9**	61	71.76%	9	61	71.76%	14	61	71.76%	9	61	71.76%	14	**61**	**71.76%**	**9**
14	138	**54**	**71.88%**	**8**	54	71.88%	9	54	71.88%	9	54	71.88%	9	54	71.88%	9	**54**	**71.88%**	**9**
15	183	**81**	**69.32%**	**9**	81	69.32%	12	81	69.32%	10	81	69.32%	15	81	69.32%	12	**81**	**69.32%**	**10**
16	127	**81**	**61.06%**	**7**	81	61.06%	10	81	61.06%	10	89	58.80%	13	81	61.06%	12	**81**	**61.06%**	**10**
17	153	**39**	**79.69%**	**7**	47	76.50%	7	39	79.69%	10	39	79.69%	7	39	79.69%	11	**39**	**79.69%**	**7**
18	187	**45**	**80.60%**	**7**	61	75.40%	11	53	77.92%	12	53	77.92%	10	61	75.40%	13	**53**	**77.92%**	**10**
19	154	**62**	**71.30%**	**8**	54	74.04%	15	70	68.75%	9	54	74.04%	14	70	68.75%	10	**54**	**74.04%**	**14**
20	145	**111**	**56.64%**	**11**	111	56.64%	14	111	56.64%	15	111	56.64%	16	111	56.64%	15	**111**	**56.64%**	**14**
21	149	**59**	**71.63%**	**8**	83	64.22%	13	67	68.98%	10	75	66.52%	12	59	71.63%	10	**59**	**71.63%**	**10**
22	162	**54**	**75.00%**	**8**	62	72.32%	9	46	77.88%	13	70	69.83%	10	46	77.88%	13	**46**	**77.88%**	**13**
23	165	**51**	**76.39%**	**9**	59	73.66%	12	59	73.66%	11	67	71.12%	13	51	76.39%	10	**51**	**76.39%**	**10**
24	164	**36**	**82.00%**	**6**	52	75.93%	10	60	73.21%	10	52	75.93%	14	60	73.21%	11	**52**	**75.93%**	**10**
25	147	**53**	**73.50%**	**7**	53	73.50%	10	61	70.67%	10	53	73.50%	12	61	70.67%	11	**53**	**73.50%**	**10**
	Avg =		73.45%	7.84		71.95%	10.16		72.18%	10.28		71.83%	11.00		72.66%	11.04		73.33%	9.64
	Min =		56.44%	5		56.44%	5		56.44%	5		56.44%	5		56.44%	5		56.44%	5
	Max =		89.38%	11		85.12%	15		85.12%	15		85.12%	18		89.38%	16		89.38%	14
	Range =		32.94%	6.00		28.68%	10.00		28.68%	10.00		28.68%	13.00		32.94%	11.00		32.94%	9.00
	Std Dev =		7.60%	1.52		6.76%	2.75		6.99%	2.35		6.84%	3.28		7.30%	2.62		7.36%	2.50
	Count1 =						11			10			10			9			
	Count2 =																21		8

Note: 1) S1[4,6] - meal break = randomly assigned with [ESB,LSB]=[4,6]
2) S2[4,6] - meal break = 1st available least demand hour with [ESB,LSB]=[4,6]
3) Count1 = no. of cases (out of 25) that yield the Best Heuristic Output for Ratio as well as MaxD.
(= no. of cases highlighted)
4) Count2 = no. of cases (out of 25) that the Best Heuristic Output yield the same TtlD and MaxD as that of GP Output.
(= no. of cases highlighted)

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Cezary Dominiak

MULTICRITERIA DECISION AID UNDER UNCERTAINTY

INTRODUCTION

Decision making under uncertainty is a very important area of decision theory. Uncertainty implies that in certain situations a person does not have the information which quantitatively and qualitatively is appropriate to describe, prescribe or predict deterministically and numerically a system, its behavior or other characteristics [27]. Thus uncertainty relates to a state of the human mind, i.e., lack of complete knowledge about something [22].

In earlier works the term "risk" was applied to the situations in which probabilities of outcomes are known objectively. Nowadays the term "risk" means a possibility of something bad happening [5]. The term "uncertainty" is applied to the problems in which alternatives with several possible outcomes exist.

The sources of uncertainty may be divided into two main groups: internal and external. Internal sources of uncertainty are created by imprecision of human judgment as regards the specification of preferences or values or the assessment of consequences of actions [22]. In the MCDA approach we find a wide range of methods and techniques suitable to deal with uncertainty created by internal factors: sensitivity analysis (e.g. [19]), fuzzy set approach (e.g. [12; 2]), rough set approach (e.g. [8; 9]).

External uncertainty refers to lack of knowledge about the consequences of our choices [22]. For these types of problems the following methods are applied: probabilistic models and expected utility (e.g. [11; 1; 20]), pairwise comparisons based on stochastic dominance (e.g. [4; 15]). Risk measures as surrogate criteria are also applied (e.g. [16; 21; 10]). In problems where we have to take into account external uncertainty the scenario planning approach may be applied (e.g. [13; 6; 18; 23]).

1. SCENARIO PLANNING

Scenario planning was developed as a technique for facilitating the process of identifying uncertain and uncontrollable factors which may influence the consequences of decisions in the strategic management context. Scenario analysis is widely accepted as an important component of strategic planning. Scenario planning may be regarded as a process of organizational learning, distinguished by the emphasis on the explicit and ongoing consideration of multiple futures. The following five principles should guide a scenario construction:

- at least two scenarios are required to reflect uncertainty,
- each scenario must be plausible, that is, it can be seen to evolve in a logical manner from the past and present,
- each scenario must be internally consistent,
- scenarios must be relevant to the DM's concerns and must provide a useful comprehensive and challenging framework against which the DM can develop and test strategies and action plans,
- the scenarios must produce a novel perspective on the issues of concern to the DM [24].

Most users of scenario planning avoid formal evaluations, preferring to leave the selection of strategy to informal judgment [22, p. 461]. There are few papers whose authors deal with the scenario planning and multi-criteria decision analysis (e.g. [13; 6; 25; 18; 23; 22]).

In this paper we propose three multi-criteria decision aiding procedures under uncertainty based on the scenario planning approach.

2. PROPOSED METHODS FOR DECISION AIDING

2.1. Problem formulation

We consider the "traditionally understood" problem of decision making under uncertainty and therefore we assume that we don't know the probabilities of the states of nature. A discrete set of alternatives and a discrete set of scenarios have been selected for the purpose of evaluating alternatives. For single-criterion problems we can apply decision rules, but here we consider the existence of multiple criteria.

First, we define the dominance relation which can be used for pre-selection of alternatives. Next, we discuss four decision aiding methods. We propose a hierarchy and quasi-hierarchy approach, for situations when DM is able to formulate his preferences in the form of order of criteria. For cases when DM can describe weights of criteria we propose the use of the distance function. Finally an interactive approach based on the idea of IMGP [17] is proposed.

Let:

n – number of alternatives,
m – number of scenarios,
K – number of criteria.

For simplicity let us assume that the values of all criteria are maximized.

Let the vector:

$$A_i^k = [a_{i1}^k, ..., a_{im}^k] \tag{1}$$

denote values of the k-th criterion of the i-th alternative. The matrix:

$$A^k = [a_{ij}^k]_{n \times m} \tag{2}$$

consists of n vectors A_i^k; it shows the values of the k-th criterion for all alternatives in each considered scenario.

2.2. The dominance relation

The proposed dominance relation is based on the "traditional" single--criterion max-min decision rule. Let:

$$\overline{a}_i^k = \min_{j=1,...,m} a_{ij}^k \tag{3}$$

Thus $\overline{a}_i^k$ represents the worst value of the k-th criterion of the i-th alternative. A_i dominates A_j if:

$$A_i > A_j \Leftrightarrow \forall_k \overline{a}_i^k \geq \overline{a}_j^k \wedge \exists_k \overline{a}_i^k > \overline{a}_j^k \tag{4}$$

On the basis of the proposed dominance relation we can define the optimal alternative and efficient alternatives as follows:

optimal alternative $\overset{*}{A_i}$:

$$\overset{*}{A_i} \Leftrightarrow \underset{w \neq i}{\forall} A_i > A_w \tag{5}$$

efficient alternatives:

$$\widetilde{A}_i \Leftrightarrow \neg \underset{w \neq i}{\exists} A_w > A_i \tag{6}$$

The proposed dominance relation reflects the strong risk aversion approach to decision analysis. During the decision analysis we can search for efficient alternatives in the preselection of the universe of alternatives or – if we find an alternative suggested as a final decision – we can examine whether this alternative is efficient or not. Examples presented below show the use of the idea of proposed dominance relation at the first stage of the decision analysis (during the preselection of alternatives).

Example 1

Number of alternatives: n = 4
Number of scenarios: m = 4
Number of criteria: K = 2

Table 1

$\mathbf{A^1}$	S_1	S_2	S_3	S_4	MIN	
A_1	10	6	**4**	14	4	
$\mathbf{A_2}$	11	9	13	**8**	8	MAX
A_3	15	**5**	12	7	5	
$\mathbf{A_4}$	8	10	11	9	8	MAX

Table 2

$\mathbf{A^2}$	S_1	S_2	S_3	S_4	MIN	
A_1	110	**100**	120	130	100	
$\mathbf{A_2}$	130	**120**	140	150	120	MAX
A_3	130	**70**	80	110	70	
A_4	100	150	140	**30**	30	

It easy to see that alternative A_2 is optimal in the sense of the proposed dominance relation.

Example 2

Number of alternatives: n = 5
Number of scenarios: m = 4
Number of criteria: K = 3

Table 3

A^1	S_1	S_2	S_3	S_4	MIN
A_1	11	10	9	8	8
A_2	10	12	15	17	10
A_3	13	11	12	15	11
A_4	13	14	12	15	12
A_5	15	10	18	9	9

Table 4

A^2	S_1	S_2	S_3	S_4	MIN
A_1	140	160	190	180	140
A_2	160	150	175	190	150
A_3	130	160	120	200	120
A_4	200	150	145	130	130
A_5	150	110	140	130	110

Table 5

A^3	S_1	S_2	S_3	S_4	MIN
A_1	16	17	19	15	15
A_2	21	18	16	17	16
A_3	16	21	22	18	16
A_4	21	17	19	17	17
A_5	15	10	18	11	10

To simplify evaluation we show min values of each criterion for all alternatives ($\overline{a}_i^k$) in the next table:

Table 6

Criterion / Alternative	k = 1 $\overline{a}_i^1$	k = 2 $\overline{a}_i^2$	k = 3 $\overline{a}_i^3$
A_1	8	140	15
A_2	10	150	16
A_3	11	120	16
A_4	12	130	17
A_5	9	110	10

We can see that alternative A_2 dominates alternative A_1, alternative A_2 dominates alternative A_5, and alternative A_4 dominates alternative A_3. Finally we conclude that there is no optimal alternative in this case and that alternatives A_3 and A_4 are the efficient ones.

2.3. The hierarchy and quasi-hierarchy approach

Let us assume that DM is able to order criteria from the most to the least important. The lower the index of a criterion, the higher its importance:

$$k_1 \rangle k_2 \rangle \ldots\ldots \rangle k_m$$

2.3.1. The hierarchy approach

In the first step of this procedure we look for best alternatives with respect to the most important criterion (in the sense of traditional max-min rule). These alternatives are included in the first subset of alternatives $\widetilde{A}^{k_1}$. Next we consider the second criterion, obtain a subset of alternatives, and repeat the calculations.

Step 1

$$\widetilde{A}^{k_1} = \{A_i : \max_{i=1,\ldots m} \overline{a}_i^{k_1}\} \tag{7}$$

Step (t = 2,...K)

$$\widetilde{A}^{k_t} = \{\widetilde{A}^{k_{t-1}}{}_i : \max_{i=1,\ldots m} \overline{a}_i^{k_t}\} \tag{8}$$

Example 3

Example 3 shows the use of the hierarchy approach.

Number of alternatives: n = 4
Number of scenarios: m = 4
Number of criteria: K = 3

Table 7

A^1	S_1	S_2	S_3	S_4	Min
A_1	11	**10**	13	15	10
A_2	10	**8**	14	13	8
A_3	15	13	12	**10**	10
A_4	12	10	**8**	7	7

Table 8

A^2	S_1	S_2	S_3	S_4	Min
A_1	10	15	9	13	9
A_2	13	**11**	12	15	11
A_3	16	**13**	14	20	13
A_4	18	19	17	**16**	16

Table 9

A^3	S_1	S_2	S_3	S_4	Min
A_1	**10**	11	14	15	10
A_2	18	16	**12**	13	12
A_3	**14**	15	19	20	14
A_4	16	**15**	17	19	15

After the first step of procedure we select alternatives A_1 and A_3. In the second step we choose alternative A_3. Because the subset of alternatives consists of only one alternative, the procedure stops. We can see that criterion k_3 doesn't influence the final result. The alternative A_3 is suggested as the final decision.

2.3.2. Quasi-hierarchy approach

In this approach the decision maker describes the tolerance limit for each criterion. Let q_t denote the tolerance limit described for t^{th} criterion. Thus in the first step we should find the subset defined as follows:

Step 1

$$\widetilde{A}^{k_1} = \{A_i : \overline{a}_i^{k_1} \geq \max_{i=1,\dots m} \overline{a}_i^{k_1} - q_1\} \tag{9}$$

Step (t = 2,...K)

In steps $k_2,\dots,k_K$ we find the following subsets of the set of alternatives:

$$\widetilde{A}^{k_t} = \{\widetilde{A}^{k_{t-1}}{}_i : \overline{a}_i^{k_t} \geq \max_{i=1,\dots m} \overline{a}_i^{k_t} - q_t\} \tag{10}$$

Example 4

We consider data from Example 3 and tolerance limits described as follows: $q_1 = 2$, $q_2 = 3$, $q_3 = 1$. After the first step of procedure we select alternatives A_1, A_2 and A_3. In the second step we choose alternatives A_2 and A_3. Finally, taking into account the third criterion, we select the alternative A_3 which is suggested as the final decision.

2.4. The distance function

Let us assume that DM is able to describe the importance of the criteria using criteria weights $\mathbf{w_k}$, k = 1,…,K:

$$w_k \geq 0, \sum_{k=1}^{K} w_k = 1 \tag{11}$$

Let $\widehat{A}$ be the „ideal pessimistic point":

$$\widehat{A} = [\widehat{a}_k : \widehat{a}_k = \max_{i=1,\dots,m} \overline{a}_i^{\,k} ; k = 1,\dots, K] \tag{12}$$

Let D be the distance function which measures the distance between the alternative considered and the ideal pessimistic point. The function D can be defined as follows:

$$D(A_i, \widehat{A}) = \sqrt{\sum_{k=1}^{K} w_k (\overline{a}_i^k - \widehat{a}_k)^2} \tag{13}$$

The lower the value of function D, the better the evaluation of the alternative; therefore, the alternative with the minimal value of function D should be suggested as the final decision.

Example 5

Number of alternatives: $n = 4$
Number of scenarios: $m = 4$
Number of criteria: $K = 3$
Criteria weights: $w_1 = 0.4$
$w_2 = 0.5$
$w_3 = 0.1$

Table 10

A^1	S_1	S_2	S_3	S_4	Min
A_1	**11**	**10**	**13**	**15**	**10**
A_2	10	**8**	14	13	8
A_3	**15**	**13**	**12**	**10**	**10**
A_4	12	10	8	7	7

Table 11

A^2	S_1	S_2	S_3	S_4	Min
A_1	10	15	**9**	13	9
A_2	13	**11**	12	15	11
A_3	16	**13**	14	20	13
A_4	**18**	**19**	**17**	**16**	**16**

Table 12

A^3	S_1	S_2	S_3	S_4	Min
A_1	**10**	11	14	15	10
A_2	18	16	**12**	13	12
A_3	**14**	15	19	20	14
A_4	**16**	**15**	**17**	**19**	**15**

The ideal pessimistic point is equal to $\widehat{A} = [10, 16, 15]$. The calculated distances from this point are presented in Tables 13 and 14.

Table 13

$$w_k(\overline{a}_i^k - \widehat{a}_k)^2$$

	k = 1	k = 2	k = 3
A_1	0.0	24.5	2.5
A_2	1.6	12.5	0.9
A_3	0.0	4.5	0.1
A_4	3.6	0.0	0.0

Table 14

Alternative	Distance: D
A_1	5.196152
A_2	3.872983
A_3	2.144761
A_4	**1.897367**

Looking at the distances from the ideal pessimistic point obtained above it is easy to see that A_4 has the lowest value of the distance function D. Therefore, alternative $\mathbf{A_4}$ should be suggested as the proposed decision.

2.5. An interactive procedure

We propose an interactive procedure based on the idea of Interactive Multiple Goal Programming (IMGP) suggested by Spronk [17]. Some important advantages are related to the IMGP approach.

First, the DM does not have to give his preference information on an a priori basis but has to consider all kinds of choices and trade-off issues which may be relevant (see [17, p. 104]). Another important advantage of IMGP is its relatively simple and easy to understand main idea. Finally, during an interactive procedure the DM has to answer the following simple questions:

1. Is the given solution acceptable or not?
2. Which goal value needs to be improved?
3. By how much (at least) should this goal value be improved?
4. Do you accept the consequences of the proposed improvement of the value of the indicated goal variable? (see [17, p. 250]).

The proposed interactive procedure also uses a potency matrix during decision aiding process but here the potency matrix consists of three vectors: ideal optimistic point, ideal pessimistic one, and current pessimistic solution which are defined below:

Let $\overset{***}{A}$ be the "ideal optimistic point" defined as follows:

$$\overset{***}{A} = [\overset{***}{a}_k : \overset{***}{a}_k = \max_{i=1,\dots,m} \max_{j=1,\dots,n} a_{ij}^k ; k = 1,\dots,K] \tag{14}$$

Let $\widehat{A}$ be the "ideal pessimistic point":

$$\widehat{A} = [\widehat{a}_k : \widehat{a}_k = \max_{i=1,\dots,m} \overline{a}_i^k ; k = 1,\dots,K] \tag{15}$$

Let $\breve{A}$ be the "current pessimistic solution" defined as follows:

$$\breve{A} = [\breve{a}_k : \breve{a}_k = \min_{i=1,\dots,m} \overline{a}_i^k ; k = 1,\dots,K] \tag{16}$$

The Potency Matrix P is described below (where "r" is the index showing the consecutive number of the iteration):

$$P^r = \begin{bmatrix} \overset{***}{A} \\ \widehat{A} \\ \breve{A} \end{bmatrix}$$

The decision aiding procedure can be written in form of three main steps.

Step 1

Calculate the first potency matrix P^1 presented to DM.

Step 2

After the analysis of potency matrix, DM chooses the criterion according to which the value for the current solution should be improved, and decides by how much this value will be improved. Thus DM chooses criterion **k** and describes the accepted value of that criterion: $\mathbf{d_k}$

$$\breve{a}_k < d_k \le \widehat{a}_k$$

Step 3

Alternatives which don't meet conditions set up by DM in the previous step are deleted from the set of alternatives and the new potency matrix P^r is calculated. DM compares values presented in the current potency matrix with values from the previous one. DM should decide whether he accepts the consequences of his last decision (he considers the local trade-offs between criteria).

a) If DM accepts the new solution we go back directly to Step 2.

b) If DM doesn't accept the new solution then the last condition put on criteria value is omitted and the previous set of alternatives is restored. Then go to Step 2.

Stop condition

The procedure stops when there is only one alternative in the set of current alternatives and DM accepts the last solution (potency matrix).

The flow chart of the procedure is presented on the next page.

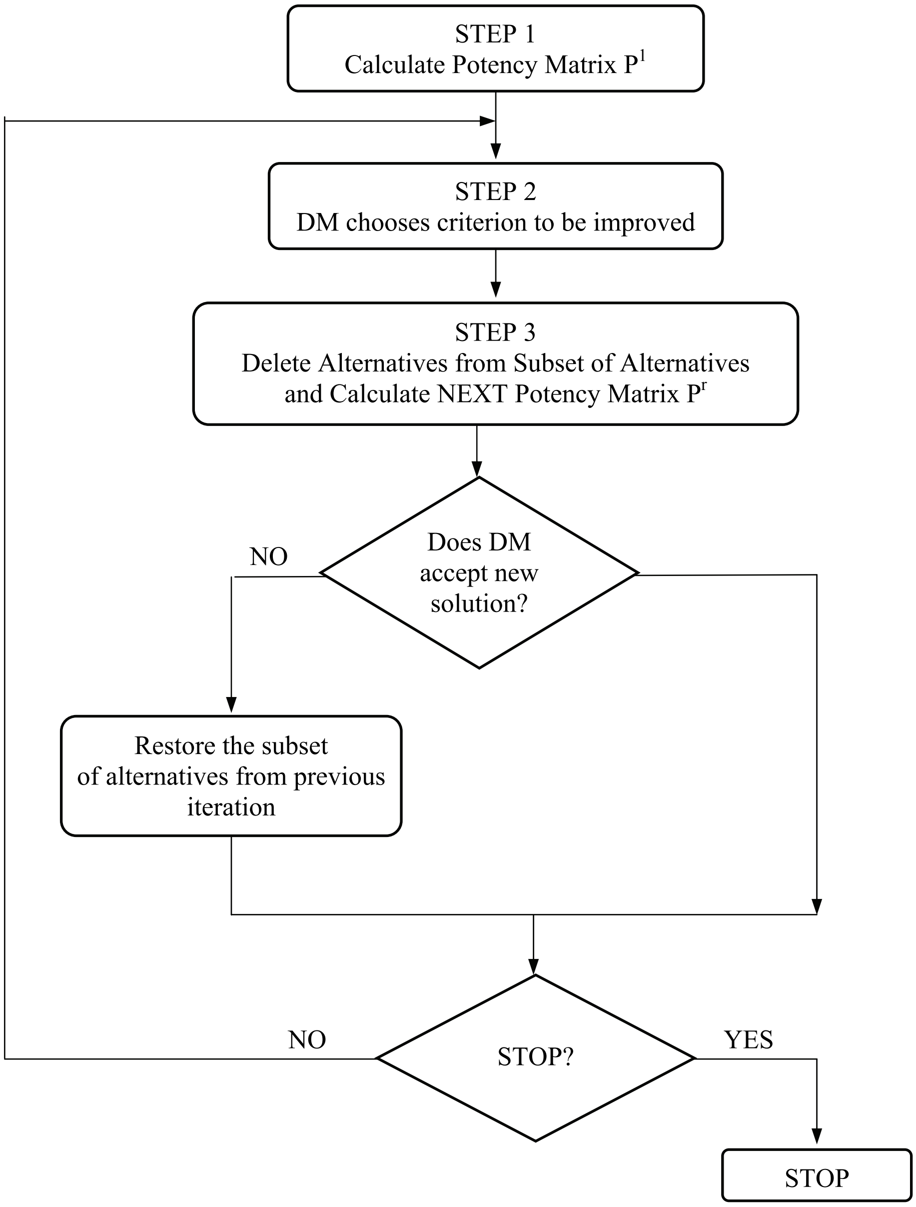

Fig. 1. The flow chart of the procedure

The next example presents the application of the proposed interactive decision aiding procedure.

Example 6

Number of alternatives: n = 4
Number of scenarios: m = 4
Number of criteria: K = 3

Table 15

A^1	S_1	S_2	S_3	S_4	min	max
A_1	7	8	10	**12**	7	12
A_2	14	**6**	11	13	6	14
A_3	11	**10**	11	**10**	**10**	11
A_4	18	15	10	9	9	**18**

Table 16

A^2	S_1	S_2	S_3	S_4	min	Max
A_1	**10**	13	14	16	**10**	**16**
A_2	**9**	11	13	**14**	9	14
A_3	12	**16**	**8**	10	**8**	**16**
A_4	**10**	11	13	**14**	**10**	14

Table 17

A^3	S_1	S_2	S_3	S_4	min	Max
A_1	21	23	4	9	**4**	**23**
A_2	20	15	16	18	**15**	20
A_3	15	13	10	11	10	15
A_4	14	15	13	14	13	15

The first calculated potency matrix is shown below:

Table 18

P^1	k = 1	k = 2	k = 3
Ideal optimistic point (max-max)	18	16	23
Ideal pessimistic point (max-min)	10	10	15
Current pessimistic solution (min-min)	6	8	4

Let us assume that the decision maker (DM) wants to improve the value of the third criterion and decides that the value of this criterion should be equal to at least 8. Then the set of alternatives is examined and as a result alternative A_1 is deleted from this set.

Table 19

A^1	S_1	S_2	S_3	S_4	min	Max
~~A_1~~	~~7~~	~~8~~	~~10~~	~~12~~	~~7~~	~~12~~
A_2	14	6	11	13	**6**	14
A_3	**11**	**10**	**11**	**10**	**10**	11
A_4	**18**	15	10	9	9	**18**

Table 20

A^2	S_1	S_2	S_3	S_4	min	max
~~A_1~~	~~10~~	~~13~~	~~14~~	~~16~~	~~10~~	~~16~~
A_2	**9**	11	13	**14**	9	14
A_3	12	**16**	**8**	10	**8**	**16**
A_4	**10**	11	13	**14**	**10**	14

Table 21

A^3	S_1	S_2	S_3	S_4	min	max
~~A_1~~	~~21~~	~~23~~	~~4~~	~~9~~	~~4~~	~~23~~
A_2	**20**	**15**	16	18	**15**	**20**
A_3	**15**	13	**10**	11	**10**	15
A_4	14	**15**	**13**	14	13	15

Thus the second potency matrix is:

Table 22

P^2	k = 1	k = 2	k = 3
Ideal optimistic point (max-max)	18	16	20
Ideal pessimistic point (max-min)	10	10	15
Current pessimistic solution (min-min)	6	8	10

Let us assume that the decision maker (DM) accepts the new solution and decides to increase the value of the second criterion to 9. Alternative A_3 is deleted from the set of alternatives. The third potency matrix is presented to DM.

Table 23

P^3	k = 1	k = 2	k = 3
Ideal optimistic point (max-max)	18	14	20
Ideal pessimistic point (max-min)	9	10	15
Current pessimistic solution (min-min)	6	9	13

DM accepts the results and decides to improve the first criterion which should be at least equal to 8 (A_2 is deleted).

Table 24

P^4	k = 1	k = 2	k = 3
Ideal optimistic point (max-max)	18	14	15
Ideal pessimistic point (max-min)	9	10	13
Current pessimistic solution (min-min)	9	10	13

Let us assume that DM does not accept the last solution. DM analyzes again the third potency matrix P^3 (A_2 is restored to set of alternatives) and decides to increase the value of the third criterion to 15. Alternative A_4 is deleted from further considerations and the potency matrix $P^{4'}$ obtained above is presented to DM.

Table 25

$P^{4'}$	k = 1	k = 2	k = 3
Ideal optimistic point (max-max)	14	14	20
Ideal pessimistic point (max-min)	6	9	15
Current pessimistic solution (min-min)	6	9	15

DM accepts this solution; the set of current alternatives consists of one element only – the procedure stops. Alternative A_2 is suggested as the final decision.

CONCLUSIONS

In this paper we discussed the problem of decision making under uncertainty. The main approaches to MCDA in uncertainty are shortly described. The scenario planning was applied as a useful technique to deal with uncertainty and the three procedures were proposed with respect to the way in which DM preferences are reflected. The hierarchy and quasi-hierarchy approach can be easily expanded to reflect the group hierarchy of the criteria.

The approach based on the distance function can be easily modified using different measures or taking into consideration the position of alternatives with respect to two reference points at the same time (e.g. ideal optimistic, ideal pessimistic).

The interactive procedure is the most flexible: it may be used without any a priori knowledge about DM's preferences and can also be applied when criteria are on the ordinal scale.

The proposed interactive procedure can be easily applied in real life problems; one of the main areas of such approach is the strategic management.

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Petr Fiala

MULTIPLE CRITERIA SUPPLIER SELECTION NETWORK MODEL*

1. SUPPLIER SELECTION PROBLEM

Supplier selection processes have received considerable attention in business (see e.g. [9]). The analysis and design of supply chains has been an active area of research (see e.g. [11]). Sourcing has come up as a very strategic issue in the management of supply chain networks in the modern era of global competition. Most production systems are organized as networks of units. Sourcing decisions have the capability of impacting the effectiveness of supply chain networks. Determining suitable suppliers in supply chain networks has become a key strategic issue. The nature of these decisions is usually complex and unstructured. The supplier selection problem is a multiple criteria problem. Many influence factors such as price, quality, flexibility, and delivery performance must be considered to determine suitable suppliers. These influence factors can be divided into quantitative and qualitative factors.

Generally, supplier selection is a multicriteria decision problem. The methods suggested in the related literatures can be classified into two categories:

– weighting models,
– mathematical programming models.

The weighting model, which focuses on commonly used evaluation criteria, includes:

– the linear scoring model (e.g. [10]),
– the Analytic Hierarchy Process (AHP) model (e.g. [1]).

The linear scoring model assigns weights and scores arbitrarily, for example, 1 for "unsatisfactory" and 5 for "outstanding". Hence, the model has an implicit and incorrect assumption: e.g., "outstanding" is five times better

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than "unsatisfactory". The problem is avoided in the AHP model by converting the priorities into the ratings with regard to each criterion using pair-wise comparisons.

Mathematical programming models are:

- goal programming or multiobjective programming (e.g. [12]),
- the linear programming or mixed integer programming with the expression of multiple objectives as constraints (e.g. [5]).

Objective function coefficients should be determined prior to making mathematical programming models. The drawback of goal programming and multiobjective programming is that it requires arbitrary aspiration levels and cannot accommodate subjective criteria.

In recent years, research on supplier selection process has highlighted the relationships that exist between companies in supply chains. Supply strategies adopt the network approach to supplier selection and focus on the coordination and integration of different supply chains. Supplier-customer relationships are changing to a cooperative form. The impact of information sharing plays a crucial role. Supplier selection process becomes a multicriteria group cooperative decision making problem. It is necessary to take in the account the network and dynamic environment.

Decision making process for supplier selection has some specifications:

- Multiple selection criteria.
- Qualitative and quantitative criteria.
- Group decision making problem.
- Cooperative behavior.
- Incomplete information.
- Networks.
- Dynamic and uncertain environment.

The proposed model respects these specifications. The approach combines the Analytic Network Process (ANP) and the Aspiration Level Oriented Procedure (ALOP). The ANP is a network generalization of AHP. The ALOP is based on goal programming approach. The GROUP-ALOP approach respects the supplier selection problem as a group decision making problem. The proposed approach can be used for dynamic environment.

2. ANP WEIGHTING MODEL

The Analytic Hierarchy Process (AHP) is the method for setting priorities [6]. A priority scale based on reference is the AHP way to standardize non--unique scales in order to combine multiple performance measures. The AHP derives ratio scale priorities by making paired comparisons of elements on a common hierarchy level by using a 1 to 9 scale of absolute numbers. The absolute number from the scale is an approximation to the ratio w_j/w_k and then is possible to derive values of w_j and w_k. The AHP method uses the general model for synthesis of the performance measures in the hierarchical structure.

$$u_i = \sum_{j=1}^{n} v_j w_{jk}$$

The Analytic Network Process (ANP) is the method [7] that makes it possible to deal systematically with all kinds of dependence and feedback in the performance system. The well-known AHP theory is a special case of the Analytic Network Process that can be very useful for incorporating linkages in the performance system.

The structure of the ANP model is described by clusters of elements connected by their dependence on one another. A cluster groups elements (success factors, managerial measures, process drivers, business units) that share a set of attributes. At least one element in each of these clusters is connected to some element in another cluster. These connections indicate the flow of influence between the elements (see Figure 1).

The clusters in the supplier selection problem can be suppliers, producers, customers, and evaluating criteria also. The connections among members of supply chain networks are material, financial and information flows.

Paired comparisons are inputs for computing a global performance of network systems. A supermatrix is a matrix of all elements by all elements. The weights from the paired comparisons are placed in the appropriate column of the supermatrix. The sum of each column corresponds to the number of comparison sets. The weights in the column corresponding to the cluster are multiplied by the weight of the cluster. Each column of the weighted supermatrix sums to one and the matrix is column stochastic. Its powers can stabilize after some iterations to limited supermatrix. The columns of each block of the matrix are identical and we can read off the global priority of units.

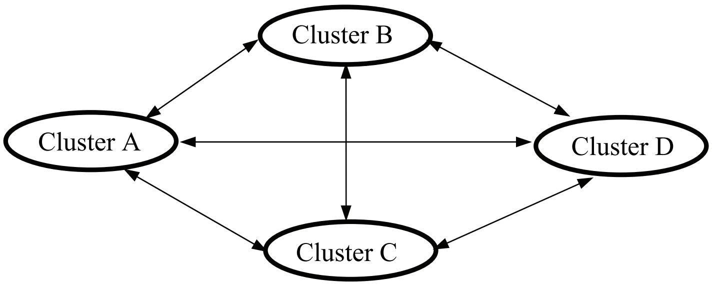

Fig. 1. Flows of influence between the elements

By ANP approach are determined weights of elements in the network model. In supplier selection problem the elements can be members of supply network, evaluating criteria, products, items etc. We made some experiments with evaluation of different supply chain structures. For computation the priorities of units we use software Super Decisions provided by Creative Decisions Foundation (see www.creativedecisions.net). We show a simple example of performance evaluation of units in supply chain structure composed from 2 suppliers, 2 producers, 2 distributors and 2 customers. The initial paired comparisons of units were implemented. On the Super Decisions Main Window (see Figure 2) are shown the structure of the system and global priorities of the units.

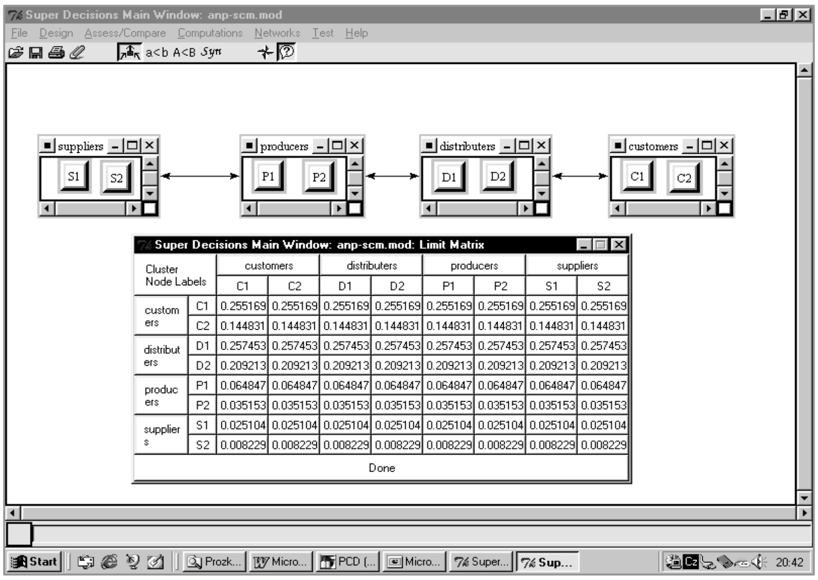

Fig. 2. Super Decisions

3. MULTI-OBJECTIVE PROGRAMMING GROUP DECISION-MAKING MODEL

Some basic ideas of formal approaches of the problem solving can be introduced to cooperative decision making. There are two aspects of the problem solving – representation and searching. The state space representation introduces the concepts of states and operators. An operator transforms one state into another state. A solution could be obtained by a search process, first applies operators to the initial state to produce new states and so on, until the goal state is produced.

Communication between suppliers and customers can be provided through information sharing (schematically see Figure 3).

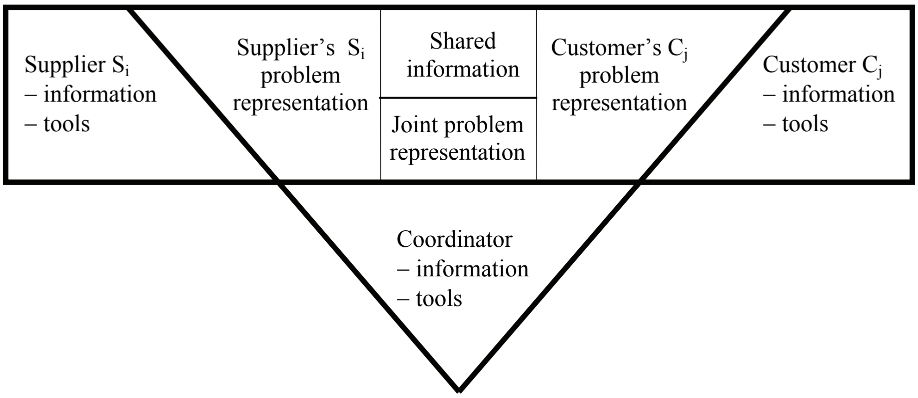

Fig. 3. Communication through information sharing

We propose a two phases' interactive approach for solving cooperative decision making problems (see [2]):

1. Finding the ideal solution for individual agents.
2. Finding a consensus for all the agents.

In the first phase every decision maker searches the ideal alternative by the assertivity principle. The general formulation of a multicriteria decision problem for an individual unit is expressed as follows:

$$\mathbf{z(x)} = (z_1(\mathbf{x}), z_2(\mathbf{x}), ..., z_k(\mathbf{x})) \rightarrow \text{"max"}, \qquad \mathbf{x} \in X$$

where X is a decision space, x is a decision alternative and $z_1, z_2, ..., z_k$ are the criteria. The decision space is defined by objective restrictions and by mutual goals of all the decision makers in the aspiration level formulation. The decision alternative **x** is transformed by the criteria to criteria values $\mathbf{z} \in Z$, where Z is a criteria space. Every decision making units has its own criteria. People appear to satisfy rather than attempting to optimize. That means substituting goals of reaching specified aspiration levels for goals of maximizing.

We denote $\mathbf{y}^{(s)}$ aspiration levels of the criteria and $\Delta\mathbf{y}^{(s)}$ changes of aspiration levels in the step s. We search alternatives for which it holds:

$$\mathbf{z}(\mathbf{x}) \geq \mathbf{y}^{(s)}, \qquad \mathbf{x} \in X$$

According to heuristic information from results of the previous condition the decision making unit changes the aspiration levels of criteria for step s + 1:

$$\mathbf{y}(s+1) = \mathbf{y}(s) + \Delta\mathbf{y}(s)$$

We can formulate the multicriteria decision problem as a state space representation. The state space corresponds with the criteria space Z, where the states are the aspiration levels of the criteria $\mathbf{y}^{(s)}$ and the operators are changes of the aspiration levels $\Delta\mathbf{y}^{(s)}$. The start state is a vector of the initial aspiration levels and the goal state is a vector of the criteria levels for the best alternative. For finding the ideal alternative we use the depth-first search method with backtracking procedure. The heuristic information is distance between an arbitrary state and the goal state.

We propose an interactive procedure ALOP (Aspiration Levels Oriented Procedure) for multiobjective linear programming problems, where the decision space X is determined by linear constraints:

$$X = \{\mathbf{x} \in R^n; A\mathbf{x} \leq b, \mathbf{x} \geq 0\}$$

and $z_i = \mathbf{c}_i\mathbf{x}$, $i = 1, 2, \ldots, k$, are linear objective functions. Then $\mathbf{z}(\mathbf{x}) = C\mathbf{x}$, where C is a coefficient matrix of objectives.

The decision alternative $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a vector of n variables. The decision maker states aspiration levels $\mathbf{y}^{(s)}$for the criteria values. There are three possibilities for aspiration levels $\mathbf{y}^{(s)}$. The problem can be feasible, infeasible or the problem has a unique nondominated solution. We verify the three possibilities by solving the problem:

$$v = \sum_{i=1}^{k} w_i^+ d_i^+ \rightarrow \max$$

$$C\mathbf{x} - \mathbf{d}^+ = \mathbf{y}^{(s)}$$

$$\mathbf{x} \in X, \mathbf{d}^+ \geq \mathbf{0}$$

If it holds:

- $v > 0$, then the problem is feasible and d_i^+ are proposed changes $\mathbf{y}^{(s)}$ of aspiration levels which achieve a nondominated solution in the next step,
- $v = 0$, then we obtained a nondominated solution,
- the problem is infeasible, then we search the nearest solution to the aspiration levels by solving the goal programming problem:

$$v = \sum_{i=1}^{k} \frac{1}{\bar{z}_i} (d_i^+ + d_i^-) \rightarrow \min$$

$$C\mathbf{x} - \mathbf{d}^+ + \mathbf{d}^- = \mathbf{y}^{(s)}$$

$$\mathbf{x} \in X, \mathbf{d}^+ \geq \mathbf{0}, \mathbf{d}^- \geq \mathbf{0}$$

The solution of the problem is feasible with changes of the aspiration levels $\Delta \mathbf{y}^{(s)} = \mathbf{d}^+ - \mathbf{d}^-$. For small changes of nondominated solutions the duality theory is applied. Dual variables to objective constraints in the problem are denoted u_i, $i = 1, 2, \ldots, k$.

If it holds:

$$\sum_{i=1}^{k} u_i \Delta y_i^{(s)} = 0$$

then for some changes $\Delta \mathbf{y}^{(s)}$ the value $v = 0$ is not changed and we obtained another nondominated solution. The decision maker can state $k - 1$ small changes of the aspiration levels $\Delta y_i^{(s)}$, $i = 1, 2, \ldots, k$, $i \neq r$, then the change of the aspiration level for criterion r is calculated from previous equation.

The decision maker chooses a forward direction or backtracking. Results of the procedure ALOP are the path of tentative aspiration levels and the ideal solution.

In the second phase a consensus could be obtained by the search process and the principle of cooperativeness is applied. The heuristic information for the decision-making unit is the distance between his proposal and the opponent's proposal. We assume that all the decision makers found their ideal alternatives. We propose an interactive procedure GROUP-ALOP for searching a consensus.

For simplicity we assume the model with one supplier and one customer:

$$z^1(\mathbf{x}) \rightarrow \text{"max"}$$

$$z^2(\mathbf{x}) \rightarrow \text{"max"}$$

$$\mathbf{x} \in X$$

The decision-making units search a consensus on a common decision space X. The decision making units change aspiration levels of the criteria y^1, y^2. The sets of feasible alternatives for the aspiration levels y^1 and y^2 are X^1 and X^2.

$$z^1(\mathbf{x}) \geq \mathbf{y}^1, \qquad \mathbf{x} \in X$$

$$z^2(\mathbf{x}) \geq \mathbf{y}^2, \qquad \mathbf{x} \in X$$

The consensus set S of the negotiations is the intersection of sets X^1 and X^2:

$$S = X^1 \cap X^2$$

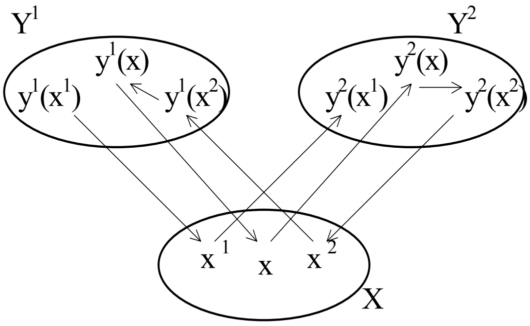

Fig. 4. Negotiation process

By changes of the aspiration levels the consensus set *S* is changed too. The decision makers search one element consensus set *S* by alternating of the consensus proposals. The image of partner's proposal can be taken as aspiration levels in one’s own criteria space. In searching for a consensus the distance between the proposals is heuristic information. The paths of the tentative aspiration levels can be used for the backtracking procedure. The forward directions can be directed by proposed new aspiration levels in step s +1:

$$\mathbf{y}^1(s+1) = (1-\alpha)\mathbf{y}^1(s) + \alpha\, \mathbf{z}^1(\mathbf{x}^2)$$

$$\mathbf{y}^2(s+1) = (1-\beta)\mathbf{y}^2(s) + \beta\, \mathbf{z}^2(\mathbf{x}^1)$$

where $\alpha, \beta \in <0,1>$ are the coefficients of cooperativeness.

Each decision maker applies cooperative strategy as long as his partner does the same. If the partner exploits the decision maker on a particular step, the decision maker then applies the exploitative strategy on the next step and continues to do so until the partner switches back to the cooperative strategy. Under these conditions, the problem stabilizes with the decision makers pursuing the mutually cooperative strategy and receiving the consensus.

The current structure is dynamic representation of results of negotiation process among units. The proposed model is discrete dynamic model and the cooperation of units is based on contracts and formal agreements achieved in negotiation process. The contracts are evaluated by multiple criteria as time, quality and costs. There are different approaches to modeling multicriteria negotiation processes, as utility concept, concept of pressure, concept of coalitions. The set of modeling concepts can be a basis for developing negotiation support.

4. SUPPLIER SELECTION DYNAMIC NETWORK MODEL

The scope of strategic fit refers to the functions and stages within a network system that coordinate strategy and target a common goal. Agile intercompany scope refers to firm's ability to achieve strategic fit when partnering with network stages that change over time. A manufacturer may interface with a different set of suppliers depending on the product. The situation in reality is much more dynamic as product life cycles get shorter and companies try to satisfy the changing needs of individual customers. The level of agility becomes more important as the competitive environment becomes more dynamic.

The proposed model is a combination of advantages of traditional approaches with adding new approaches for new specifications of supplier selection problem. The approach combines the ANP and the GROUP-ALOP. The ANP provides weights **w** in the network model. The elements can be members of supply network, evaluating criteria, products, items etc. By ANP can be evaluated qualitative criteria also. The weights **w** are used in the GROUP-ALOP approach.

Today's world is dynamic. The proposed approach can be used for this dynamic environment. The AHP and ANP have been static but for today's world analyzing is very important time dependent decision making. The DHP/DNP (Dynamic Hierarchy Process/Dynamic Network Process) methods were introduced [8]. There are two ways to study dynamic decisions:

structural, by including scenarios, and functional by explicitly involving time in the judgment process. For the functional dynamics there are analytic or numerical solutions. The basic idea with the numerical approach is to obtain the time dependent principal eigenvector by simulation. The DNP provides weights for time periods t =1,2,…,T.

The connections are time dependent. The importance of the criteria, suppliers etc. and aspiration levels changes. There are many time dependent situations in network economy. For example the dynamics of new product adoption can be expressed by S-curve (see Figure 5).

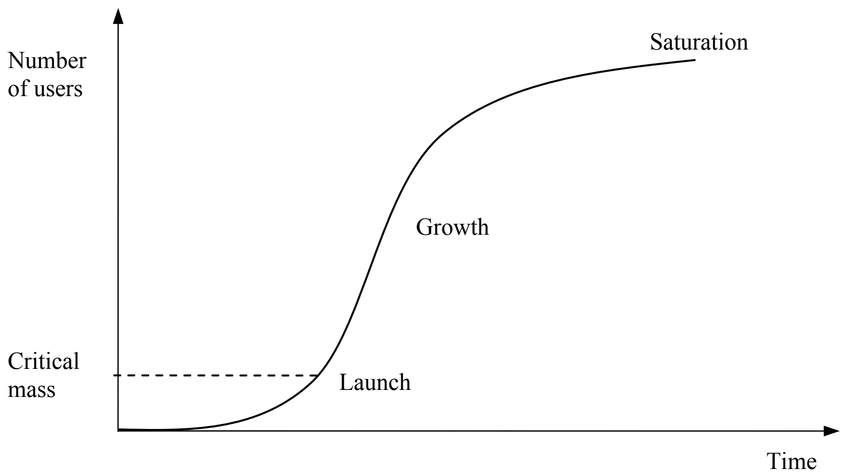

Fig. 5. Adaptation dynamics

The reality of today's networks includes features:

– large-scale nature and complexity,
– increasing congestion,
– complementarity,
– externalities,
– switching costs,
– alternative behaviors of users of the networks,
– interactions between the networks themselves.

Many of today's networks are characterized by both a large-scale nature and complexity of the network topology. Congestion is playing an increasing role in not only transportation networks but also in telecommunication networks. The crucial relationship in networks is the complementarity between the pieces of the network. Networks exhibit positive externalities. The value of a unit of the good increases with the expected number of units to be sold. Cost of switching to a different service or adopting a new technology are significant. There are various types of these costs as contracts, training and learning, data conversion, search costs etc. The decisions made by the users of the networks, in turn, affect not only the users themselves but others, as well, in terms of profits and costs, timeliness of deliveries, the quality of the environment etc. Network connections bring important effects. Networks established for the purpose of sharing or creating new information provide better, more complete information as more units join and use them. The attractiveness to users of networks increases as they increase in size.

In the supplier selection problem are important feedback and dependencies between items or suppliers:

- positive and negative feedback,
- substitution and complementarity.

The positive feedback (see Figure 6) can be expressed as: the strong will be stronger and the weak will be weaker.

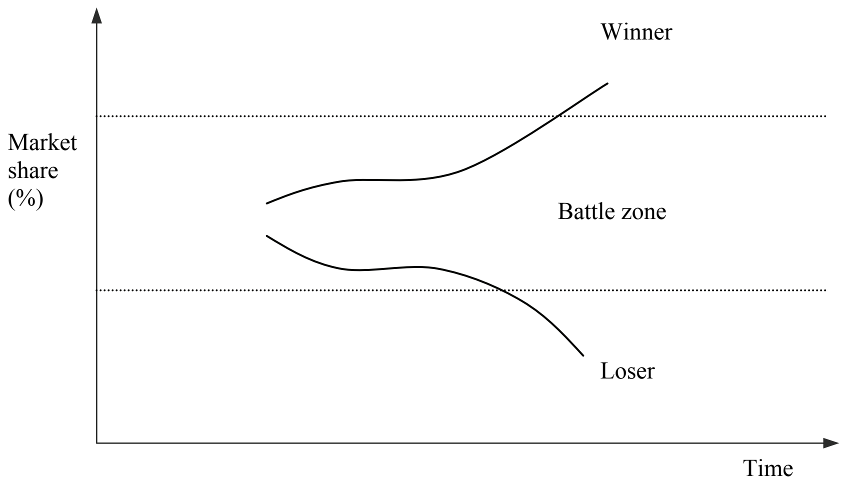

Fig. 6. Positive feedback

Example 1

We use the DNP method for an illustration of positive feedback. The time dependent comparison of two products is expressed by the S-curve:

$$a_{12}(t) = \frac{9}{1 + 7 \cdot 0{,}01^{t}}$$

The paired comparison matrix:

$$\begin{bmatrix} 1 & a_{12}(t) \\ 1/a_{12}(t) & 1 \end{bmatrix}$$

The numerical data are shown in Table 1 and plotted in Figure 7 and Figure 8.

Table 1

Dynamic comparisons

t	$a_{12}(t)$	$w_1(t)$	$w_2(t)$
0	1,13	0,53	0,47
0,1	1,66	0,62	0,38
0,2	2,38	0,7	0,3
0,3	3,26	0,77	0,23
0,4	4,27	0,81	0,19
0,5	5,29	0,84	0,16
0,6	6,24	0,86	0,14
0,7	7,04	0,87	0,13
0,8	7,65	0,88	0,12
0,9	8,10	0,89	0,11
1	8,41	0,9	0,1

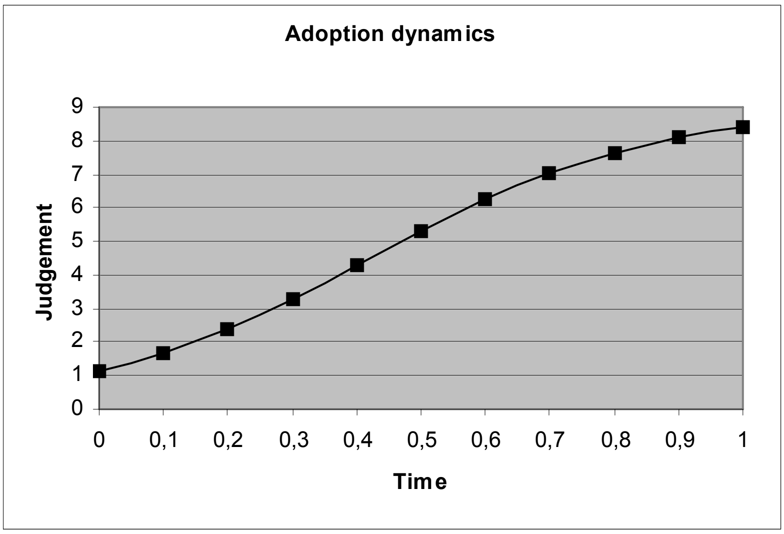

Fig. 7. Adaptation dynamics – example

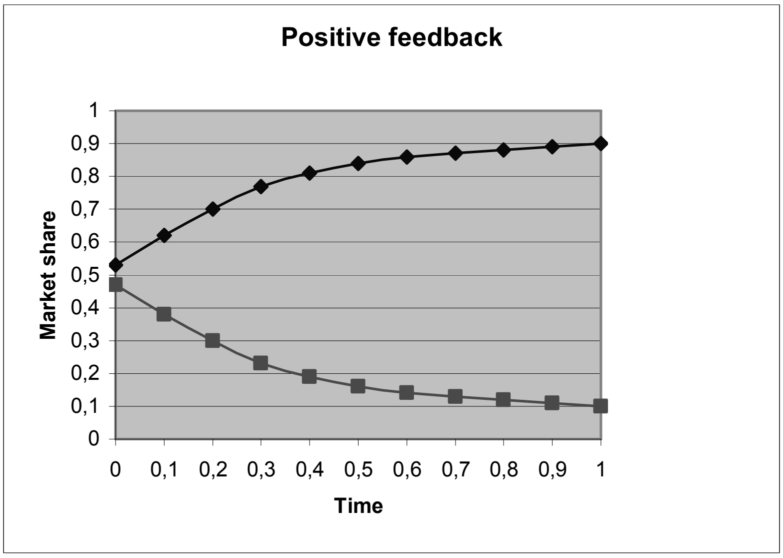

Fig. 8. Positive feedback – example

Supplier selection processes have received considerable attention in business. Sourcing has become a very strategic issue in the management of supply chain networks in the modern era of global competition. Sourcing decisions have the capability of impacting the effectiveness of supply chain networks. Determining suitable suppliers in supply chain networks has become a key strategic issue. The nature of these decisions is usually complex and unstructured. Supplier-customer relationships are changing to a cooperative form. The impact of information sharing plays a crucial role. The supplier selection problem is a multiple criteria group decision making problem. Many influence factors such as price, quality, flexibility, and delivery performance must be considered to determine suitable suppliers. These influence factors can be divided into quantitative and qualitative factors.

Two items A and B are complementary, if it holds for weights:

$$w(\{A, B\}) > w(\{A\}) + w(\{B\})$$

Two items A and B are substitute, if it holds for weights:

$$w(\{A, B\}) < w(\{A\}) + w(\{B\})$$

The combination of Dynamic Network Process and dynamic version of GROUP-ALOP seems to be the appropriate method for the specific features of the supplier selection problem in network economy. The approach combines time dependent weights **w**(t) from DNP and time dependent aspiration levels **y**(t), t =1,2,…,T, from GROUP-ALOP.

The approach can be structured in the following phases:

1. The DNP is used for comparison of importance of the suppliers, supplies, criteria etc. There are dependencies among units. Inputs are objective and subjective information of units. Outputs are weights for time periods t =1,2,…,T.
2. The GROUP-ALOP approach is applied for negotiation process between suppliers and the customer. For every time period in negotiation steps *s* the aspiration levels of criteria are changed to get a consensus. Inputs are the common decision space, criteria, and weights from the previous phase. Outputs are proposals for a consensus for time periods t =1,2,…,T.
3. Participants evaluate the proposals by own characteristics and make next proposals or determine the final values. Outputs are supplies from selected suppliers for time periods t =1,2,…,T.

The approach is illustrated in Example 2. The approach is very flexible and a simple example can clarify basic insights only.

Example 2

Assume a manufacturer produces three products (P_i, i = 1,2,3) from two key parts (A, B). The product P_1 contains one piece of the part A, the product P_2 contains one piece of the part B, and the new product P_3 contains one piece of the part A and one piece of the part B. The manufacturer looks at three suppliers, (S_j, j = 1,2,3) providing the two parts (A, B) and compares bids according two criteria, prices and reliability levels (p, r). The supplier S_1 produces parts A, the supplier S_2 produces parts B, and the supplier S_3 produces parts A and B. The supplier selection process is dynamic, in time periods (t = 1,2,3).

The relative importance of criteria (p, r) for parts (A, B) changes dramatically in time periods. The criteria are dependent each other and the parts are dependent each other for the product P_3 also. The DNP method was used for weights calculation. For simplicity we assume that weights are the same for the parts.

Table 2

Weights of criteria

t	$w_p(t)$	$w_r(t)$
1	0.8	0.2
2	0.5	0.5
3	0.3	0.7

In every time period will be in progress negotiation process with suppliers. The firm negotiates quantity, price and reliability levels of parts A, B. The weights **w** are used in the GROUP-ALOP approach. In every negotiation step s aspiration levels are changed. Results of the negotiation process are price and reliability levels for time periods.

Table 3

Negotiated final price and reliability levels

t	$p_{1A}(t)$	$p_{2B}(t)$	$p_{3A}(t)$	$p_{3B}(t)$	$r_{1A}(t)$	$r_{2B}(t)$	$r_{3A}(t)$	$r_{3B}(t)$
1	2	3	3	4	0.7	0.7	0.7	0.7
2	2.5	3.5	2	3	0.75	0.75	0.8	0.8
3	3	4	1.5	2.5	0.8	0.8	0.9	0.9

The decision set for the firm is restricted by forecasted demands $D_i(t)$, $i = 1,2,3$, $t = 1,2,3$, and capacities. Unit profits $c_i(t)$, $i = 1,2,3$, $t = 1,2,3$, are dependent on price and reliability levels of parts A, B, among others.

Table 4

Forecasted demands and unit profits

t	$D_1(t)$	$D_2(t)$	$D_3(t)$	$c_1(t)$	$c_2(t)$	$c_3(t)$
1	50	80	10	5	6	4
2	30	60	30	4	5	7
3	10	30	100	3	4	10

The production quantities $x_i(t)$, $i = 1,2,3$, $t = 1,2,3$ are bounded by forecasted demands:

$$x_i(t) \leq D_i(t) \quad i = 1,2,3, \; t = 1,2,3$$

The firm capacity makes possible to produce 100 final products in every time period:

$$x_1(t) + x_2(t) + x_3(t) \leq 100\,,\ t = 1,2,3$$

For simplicity, assume the firm evaluates the negotiation position by expected profit $z(t)$, $t = 1,2,3$,

$$z(t) = c_1(t)x_1(t) + c_2(t)x_2(t) + c_3(t)x_3(t) \rightarrow \max$$

The solution of the decision problem:

Table 5

Production quantities and profits

t	$x_1(t)$	$x_2(t)$	$x_3(t)$	$z(t)$
1	20	80	0	580
2	10	60	30	550
3	0	0	100	1000

The required supplies $q_{jA}(t)$, $q_{jB}(t)$, $i = 1,2,3$, $t = 1,2,3$, are calculated:

Table 6

Required supplies

t	$q_{1A}(t)$	$q_{2B}(t)$	$q_{3A}(t)$	$q_{3B}(t)$
1	20	80	0	0
2	10	60	30	30
3	0	0	100	100

CONCLUSIONS

Supplier selection process is a very important strategic issue. The process is very complex. There are new trends in supply process. The new very important features in supplier selection problem are network structure of suppliers and items, dynamic connections and cooperative decision making. The proposed model captures important trends in supply process. The approach combines advantages of the traditional approaches for supplier selection problems, weighting models and mathematical programming models and adds approaches for new specifications of supplier selection problem. There is a combination of Dynamic Network Process and the dynamic version of GROUP-Aspiration Levels Oriented Procedure. The approach is very flexible. The aim is not only a supplier selection but managing supplier--customer relations also. Research work continues and testing on real applications is needed.

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Josef Jablonsky

A SLACK BASED MODEL FOR MEASURING SUPER-EFFICIENCY IN DATA ENVELOPMENT ANALYSIS*

INTRODUCTION

Data envelopment analysis (DEA) is a tool for measuring the relative efficiency and comparison of decision making units (DMU). The DMUs are usually described by several inputs that are spent for production of several outputs. Let us consider the set E of n decision making units $E = \{DMU_1, DMU_2, \ldots, DMU_n\}$. Each of the units produces r outputs and spends m inputs for their production. Let us denote $\mathbf{x}^j = \{x_{ij}, i = 1,2,\ldots,m\}$ the vector of inputs and $\mathbf{y}^j = \{y_{ij}, i = 1,2,\ldots,r\}$ the vector of outputs of the DMU_j. Then **X** is the (m, n) matrix of inputs and **Y** the (r, n) matrix of outputs.

The basic principle of the DEA in evaluation of efficiency of the DMU_q, $q \in \{1,2,...,n\}$ consists in looking for a virtual unit with inputs and outputs defined as the weighted sum of inputs and outputs of the other units in the decision set – $\mathbf{X\lambda}$ a $\mathbf{Y\lambda}$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\boldsymbol{\lambda} > 0$ is the vector of weights of the DMUs. The virtual unit should be better (or at least not worse) than the analysed unit DMU_q. The problem of looking for a virtual unit can generally be formulated as a standard linear programming problem:

$$\begin{aligned} \text{minimise} \quad & \theta \\ \text{subject to} \quad & \mathbf{Y\lambda} \geq \mathbf{y}^q \qquad (1) \\ & \mathbf{X\lambda} \leq \theta \mathbf{x}^q \\ & \boldsymbol{\lambda} \geq 0 \end{aligned}$$

The DMU_q is to be considered as efficient if the virtual unit is identical with evaluated unit (virtual unit with better inputs and outputs does not exist). In this case $\mathbf{Y\lambda} = \mathbf{y}^q$, $\mathbf{X\lambda} = \mathbf{x}^q$ and minimum value of $z = \theta = 1$. Otherwise

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the DMU_q is not efficient and minimum value of $\theta < 1$ can be interpreted as the need of a proportional reduction of inputs in order to reach the efficient frontier. The presented model is input oriented model because its objective is to find a reduction rate of inputs in order to reach the efficiency. Analogously can be formulated output oriented model.

Model (1) shows just the basic philosophy of DEA models. The first DEA model was formulated in 1978 by Charnes, Cooper, Rhodes (CCR model). Its input oriented form (CCR-I) looks as follows:

$$\begin{aligned} &\text{minimise} && z = \theta - \varepsilon(\mathbf{e}^T\mathbf{s}^+ + \mathbf{e}^T\mathbf{s}^-) \\ &\text{subject to} && \mathbf{Y}\boldsymbol{\lambda} - \mathbf{s}^+ = \mathbf{y}^q \\ & && \mathbf{X}\boldsymbol{\lambda} + \mathbf{s}^- = \theta\mathbf{x}^q \\ & && \boldsymbol{\lambda}, \mathbf{s}^+, \mathbf{s}^- \geq 0 \end{aligned} \tag{2}$$

where $\mathbf{e}^T = (1,1,\ldots,1)$ and ε is a infinitesimal constant (usually 10^{-8}). Presented formulations (1) and (2) are very close each other. The variables $\mathbf{s}^+, \mathbf{s}^-$ are just slack variables expressing the difference between virtual inputs/outputs and appropriate inputs/outputs of the DMU_q. Obviously, the virtual inputs/outputs can be computed using the optimal values of variables of the model (2) as follows:

$$\mathbf{x}^{q\prime} = \mathbf{x}^q\theta^* - \mathbf{s}^-$$
$$\mathbf{y}^{q\prime} = \mathbf{y}^q + \mathbf{s}^+$$

The CCR model supposes constant returns to scale – it is supposed that a considered percentual change of inputs leads to the same percentual change of outputs. The modification of the CCR model taking into account variable returns to scale (so called BCC model) is derived from model (2) by adding the convexity constraint $\mathbf{e}^T\boldsymbol{\lambda} = 1$.

1. SUPER-EFFICIENCY MODELS

The efficiency score in standard DEA models is limited to unity (100%). Nevertheless, the number of efficient units identified by DEA models and reaching the maximum efficiency score 100% can be relatively high and especially in problems with a small number of decision making units the efficient set can contain almost all the units. In such cases it is very important to have a tool for a diversification and classification of efficient units. That is why several DEA models for classification of efficient units were formulated. In these models the efficient scores of inefficient units remain lower than 100% but the efficiency score for efficient units can be higher than 100%. Thus the efficiency score can be taken as a basis for a complete ranking of efficient units. The DEA models that relax the condition for unit efficiency are called super-efficiency models.

Basic principles of super-efficiency models are illustrated on Figure 1. It presents an example with 8 DMUs, each of them described by one input and one output. The Figure shows the BCC efficient frontier defined by units DMU_3, DMU_2, DMU_8 and DMU_6. These units are BCC efficient and their efficiency score is equal to 1 because it is not possible to find any convex combination of other units with better characteristics (lower input and higher output). The remaining four units are not efficient. The super-efficiency models are always based on removing the evaluated efficient unit from the set of units (unit DMU_2 on Figure 1). This removal leads to the modification of the efficient frontier (heavy line of Figure 1) and the super-efficiency is measured as a distance between evaluated unit (DMU_2) and a unit on the new efficient frontier (DMU*). Of course several distance measures can be used – this leads to different super-efficiency definitions.

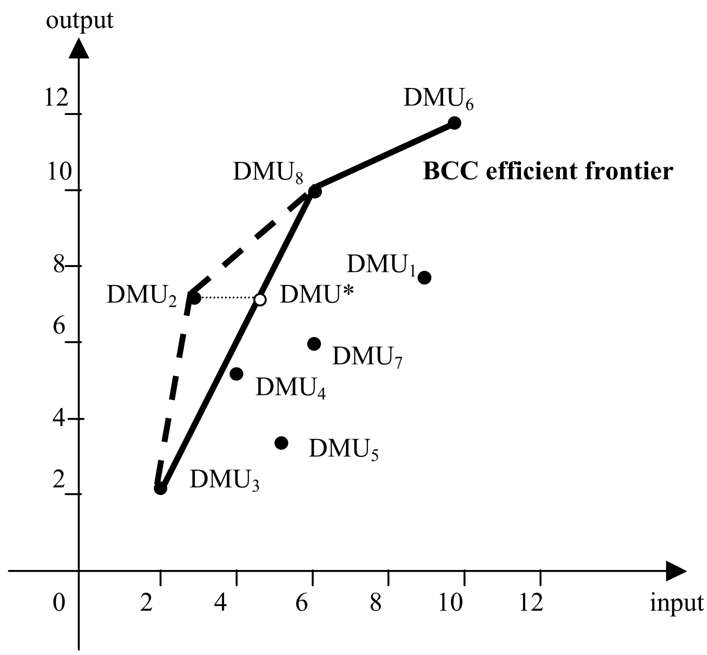

Fig. 1. A super-efficiency measure

The first super-efficiency DEA model was formulated by Andersen and Petersen [1]. Its input oriented formulation (3) is very close to the standard input oriented formulation of the CCR-I model (2). In this model the weight λ_q of the evaluated unit DMU_q is equated to zero. This cannot influence the efficiency score of the inefficient units but the efficiency score of the efficient units is not limited by unity in this case. The input oriented formulation of the Andersen and Petersen model is as follows:

minimise θ

subject to

$$\sum_{j=1,\neq q}^{n} x_{ij}\lambda_j + s_i^- = \theta x_{iq}, \qquad i = 1,2,...,m \tag{3}$$

$$\sum_{j=1,\neq q}^{n} y_{ij}\lambda_j - s_i^+ = y_{iq}, \qquad i = 1,2,...,r$$

$$\boldsymbol{\lambda}, \mathbf{s}^+, \mathbf{s}^- \geq 0$$

Tone [6] proposes a slack based measure of efficiency (SBM model) that is basis for his formulation of the super-efficiency model presented in [7]. The Tone's SBM model is formulated as follows:

minimise

$$\rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^- / x_{iq}}{1 + \frac{1}{r}\sum_{i=1}^{r} s_i^+ / y_{iq}}$$

subject to

$$\sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = x_{iq}, \qquad i = 1,2,...,m \tag{4}$$

$$\sum_{j=1}^{n} y_{ij}\lambda_j - s_i^+ = y_{iq} \qquad i = 1,2,...,r$$

$$\boldsymbol{\lambda}, \mathbf{s}^+, \mathbf{s}^- \geq 0$$

The formulation shows that the SBM model is non-radial and deals directly with slack variables. The model returns efficiency score between 0 and 1 and is equal to 1 if and only if the DMU_q is on the efficient frontier without any slacks. It is possible to prove that the efficiency score of the SBM model is always lower or equal than the efficiency score of the appropriate CCR input oriented model. The formulation of the model (4) with fractional objective function can be simply transformed into a standard problem with linear objective function.

The super-efficiency SBM model removes the evaluated unit DMU_q from the set of units (like Andersen and Petersen model) and looks for a DMU* with inputs x* and outputs y* being SBM (and CCR) efficient after this removal. It is clear that all the inputs of the unit DMU* have to be higher or equal than the inputs of the unit DMU_q and all the outputs will be lower or equal comparing to outputs of DMU_q. The super-efficiency is measured as a distance of the inputs/outputs of both the units. As a distance measure in the mathematical formulation of the super SBM model below, the variable δ is used:

minimise
$$\delta = \frac{\frac{1}{m}\sum_{i=1}^{m} x_i^*/x_{iq}}{\frac{1}{r}\sum_{i=1}^{r} y_i^*/y_{iq}} \tag{5}$$

subject to
$$\sum_{j=1,\neq q}^{n} x_{ij}\lambda_j + s_i^- = x_{iq} \qquad i = 1,2,...,m \tag{6}$$

$$\sum_{j=1,\neq q}^{n} y_{ij}\lambda_j - s_i^+ = y_{iq}, \qquad i = 1,2,...,r$$

$$x_i^* \geq x_{iq}, \qquad i = 1,2,...,m$$

$$y_i^* \leq y_{iq}, \qquad i = 1,2,...,r$$

$$\boldsymbol{\lambda}, \mathbf{s}^+, \mathbf{s}^-, \mathbf{y}^* \geq 0$$

The numerator in the ratio (5) can be interpreted as a distance of both the units in the input space and an average reduction rate of inputs of DMU* to inputs of DMU_q. The same holds for the output space in the denominator of the ratio (5). The model (5)-(6) takes into account both the inputs and outputs and measures the distance in the input and output space simultaneously. It is not a model with linear objective function but it can be simply re-formulated as a standard LP problem by means of Charnes-Cooper transformation.

Similarly to the previous model the input (output) oriented modification can be formulated. This modified model measures the distance of the DMU_q and the DMU* in the input (output) space only. The formulation of the input (output) oriented SBM model is derived from the model (5)-(6) by setting the denominator equal to 1, i.e. $y_i^* = y_{iq}$ (setting the numerator equal to 1, i.e. $x_i^* = x_{iq}$). The input oriented formulation of the super SBM model (SBM-I) is given as follows (similarly can be written output oriented form SBM-O):

minimise $$\delta_I = \frac{1}{m}\sum_{i=1}^{m} x_i^*/x_{iq} \qquad (7)$$

subject to
$$\sum_{j=1,\neq q}^{n} x_{ij}\lambda_j + s_i^- = x_{iq}, \quad i = 1,2,...,m$$

$$\sum_{j=1,\neq q}^{n} y_{ij}\lambda_j - s_i^+ = y_{iq}, \quad i = 1,2,...,r \qquad (8)$$

$$x_i^* \geq x_{iq}, \quad i = 1,2,...,m$$

$$y_i^* = y_{iq}, \quad i = 1,2,...,r$$

$$\boldsymbol{\lambda}, \mathbf{s}^+, \mathbf{s}^- \geq 0$$

In the previous models it holds $\delta^*(DMU_q) \geq 1$ ($\delta_I^*(DMU_q) \geq 1$) where δ^* (δ_I^*) are the optimal objective function values of models (5)-(6) and (7)-(8). The optimal efficiency score is greater than 1 for SBM efficient DMUs and it is possible to prove that $\delta_I^*(DMU_q) \geq \delta^*(DMU_q) > 1$ – higher value is assigned to more efficient units. All the SBM inefficient units reach in the super SBM model optimal score 1. It means that this model cannot be used for classification of inefficient units and have to be used in two steps:

- to apply the SBM model (4) in order to identify efficient units and classify inefficient units,
- to compute the super-efficiency scores by means of one of the super--efficiency SBM models – the models (5)-(6) or (7)-(8).

2. THE SBMG MODEL

The super-efficiency SBM models measure the distance of the evaluated unit DMU_q from a virtual unit DMU* under the assumption that its inputs **x*** and outputs **y*** are not better than the inputs $\mathbf{x^q}$ and outputs $\mathbf{y^q}$ of the unit DMU_q. When this assumption is not kept the super-efficiency can be measured by the following goal programming model (super SBMG model):

$$\text{minimise} \quad \rho = 1 + t\gamma + (1-t)\left(\sum_{i=1}^{m}\left[s_{1i}^{+}/x_{iq}\right] + \sum_{i=1}^{r}\left[s_{2i}^{-}/y_{iq}\right]\right)$$

$$\text{subject to} \quad \sum_{j=1,\neq q}^{n} x_{ij}\lambda_j + s_{1i}^{-} - s_{1i}^{+} = x_{iq}, \qquad i = 1, 2, ..., m \tag{9}$$

$$\sum_{j=1,\neq q}^{n} y_{ij}\lambda_j + s_{2i}^{-} - s_{2i}^{+} = y_{iq}, \qquad i = 1, 2, ..., r$$

$$s_{1i}^{+}/x_{iq} \leq \gamma, \qquad i = 1, 2, ..., m$$

$$s_{2i}^{-}/y_{iq} \leq \gamma, \qquad i = 1, 2, ..., r$$

$$t \in \langle 0,1 \rangle$$

$$\lambda_j \geq 0,\ s_{1i}^{-} \geq 0,\ s_{1i}^{+} \geq 0,\ s_{2i}^{-} \geq 0,\ s_{2i}^{+} \geq 0$$

In this model the distance between the units DMU_q a DMU* is measured by the positive and negative deviational variables – s_{1i}^{-}, s_{1i}^{+} for the inputs and s_{2i}^{-}, s_{2i}^{+} for the outputs. The minimised objective function ρ contains just the positive deviations for the inputs and the negative ones for the outputs, because the model tries to measure the distance in the undesirable way for both the groups of characteristics – for the inputs the undesirable deviations are the positive ones, for the outputs the negative ones on the contrary.

The goal programming technique can optimise the problem by using deviational variables in two basic ways – minimisation of the weighted sum of deviational variables and minimisation of the maximum deviation. That is why the objective function of the model (9) consists of two parts – the first one minimises the maximum relative deviation γ and the second one the sum of relative deviations from the inputs and outputs characteristics of the unit DMU_q. Depending on the selection of the parameter t the model either minimises the maximum deviation γ ($t = 1$) or the weighted sum of deviations ($t = 0$). By selection of parameter t between zero and one, $t \in (0,1)$, the both approaches can be combined.

The super-efficiency models have to fulfil the basic requirement – when any input or output of the evaluated unit DMU_q worsens (improves), its super-efficiency score have to decrease (increase) or at least remain without changes. The proof of this feature for Tone's models is given in [7]. The proof for the model (9) follows directly from its definition – the worsening of the i-th input (output) does not lead to the higher positive deviational variable s_{1i}^{+} (negative variable s_{1i}^{-}) and by this the super-efficiency score cannot be higher.

Let us denote ρ_0*(DMU$_q$) the super-efficiency score of the unit DMU$_q$ given by the model (9) with parameter t = 0 and ρ_1*(DMU$_q$) the score given by the model (9) with parameter t = 1. It is obvious that the score ρ_0*(DMU$_q$) is always greater or equal to the score ρ_1*(DMU$_q$). Both the characteristics are always lower than the super-efficiency score given by Andersen and Petersen model (3). The objective function of the super SBM model (5)-(6) is defined as the ratio of the average deviations. The objective function of the SBMG model (9) pro t = 0 is the sum of positive deviations of inputs and negative deviations of outputs. That is why it is possible simply to show that the following relation holds: $\delta^*(DMU_q) \geq 1 + (\rho_0^*(DMU_q) - 1)/(m + r)$, where $\delta^*(DMU_q)$ is the super-efficiency score given by the model (5)-(6).

The objective function of the model (9) for parameter t = 0 is the sum of the relative undesirable deviations. This sum can be replaced by the average of all the undesirable slacks. The results in this case can often be better explained. The model (9) makes it possible to perform a sensitivity analysis of the problem – according to the optimum dual values of the input and output constraints it is possible to find out how the changes of the input and output values of the evaluated unit influence the SBGM super-efficiency score.

3. A NUMERICAL ILLUSTRATION

The presented three concepts of super-efficiency, including our own definition (9), are illustrated on the small numerical example taken from [7]. The example considers 6 decision making units (power plants locations) with four inputs and two outputs defined as follows:

- manpower required ($\mathbf{x}^1$),
- estimated construction costs in millions of USD ($\mathbf{x}^2$),
- annual maintenance costs in millions of USD ($\mathbf{x}^3$),
- the number of villages that have to be evacuated ($\mathbf{x}^4$),,
- plant power in megawatts ($\mathbf{y}^1$),
- safety level given by an ordinal scale from 1 to 10 (higher values are better) - ($\mathbf{y}^2$).

The input and output data of the problem are given in Table 1.

Table 1

Data of the problem

	Inputs				Outputs	
	x^1	x^2	x^3	x^4	y^1	y^2
DMU_1	80	600	54	8	90	5
DMU_2	65	200	97	1	58	1
DMU_3	83	400	72	4	60	7
DMU_4	40	1000	75	7	80	10
DMU_5	52	600	20	3	72	8
DMU_6	94	700	36	5	96	6

Applying the DEA models to small set of DMUs with relation to the number of inputs and outputs of the problem can often lead to the result that all the units are efficient. This situation occurs in our illustrative example. When the decision maker wants to discriminate among the efficient units the super-efficiency DEA models can be used. The super-efficiency score computed by Andersen and Petersen model (3), Tone's SBM model (5)-(6) SBMG model (9) for t = 0 and t = 1 are presented in Table 2.

Table 2

Comparison of super-efficiency scores

	A-P θ^*	SBM δ^*	SBMG t=0, ρ_0^*	SBMG t=1, ρ_1^*
DMU_1	1.0283	1.0116	1.0275	1.0139
DMU_2	2.4167	1.4146	1.5862	1.4146
DMU_3	1.3125	1.0781	1.2976	1.1351
DMU_4	1.6250	1.1563	1.5556	1.2381
DMU_5	2.4026	1.5859	1.8454	1.4122
DMU_6	1.0628	1.0198	1.0591	1.0304

Table 2 illustrates a conclusion derived by Tone [7] that the super-efficiency score for unit DMU_q given by the A-P model (3) is always greater or equal to the score given by the SBM model, ie. $\theta^*(DMU_q) \geq \delta^*(DMU_q)$. Of course it holds $\theta^*(DMU_q) \geq \rho_0^*(DMU_q) \geq \rho_1^*(DMU_q)$.

The following Table 3 illustrates in detail the computation of the super--efficiency score for the unit DMU_5. First row of the table contains the original input and output values of this unit. The remaining rows (except row SBM) contain the virtual inputs and outputs given as follows:

$$\sum_{j=1}^{n} x_{ij}\lambda_j^*, \qquad i = 1, 2, ..., m$$

$$\sum_{j=1}^{n} y_{ij}\lambda_j^*, \qquad i = 1, 2, ..., r$$

where λ_j^* are optimum weights given by the super-efficiency models. The row SBM contains values x_i^* and y_i^* which are used for calculation of the super--efficiency measure in the model (5)-(6). The virtual units with the virtual inputs and outputs presented in Table 3 are always CCR efficient.

Table 3

Comparison of virtual units given by different models

Model	Inputs				Outputs	
	x^1	x^2	x^3	x^4	y^1	y^2
DMU_5	52.00	600.00	20.00	3.00	72.00	8.00
A-P	124.93	932.76	48.05	6.66	127.73	8.00
SBM	70.50	600.00	27.00	3.75	72.00	4.50
SBMG(t=0)	52.00	388.24	20.00	2.77	53.16	3.33
SBMG(t=1)	73.43	548.27	28.24	3.92	75.08	4.70

The super-efficiency score of the unit DMU_5 given by above used three different models is derived as follows (it is always a distance of the virtual unit and the evaluated unit):

1. *Andersen and Petersen model (3)*. The super-efficiency score $\theta^*(DMU_5)$ is derived as the maximum ratio of the virtual and the original inputs, ie. $\theta^*(DMU_5)$ = max[124.93/52, 932.76/600, 48.05/20, 6.66/3] = 2.4026. Due to the radial nature of this model the virtual unit on the efficient frontier lies often very far from the original unit and that is why the super-efficiency score can be very high. The conclusions following from this, the evaluated unit is efficient even its inputs increase $\theta^*(DMU_q)$-times, need not be always acceptable.

2. *Super SBM model (5)-(6)*. This model calculates the super-efficiency score as the ratio of two values. The numerator is the average expansion rate of inputs of the virtual unit comparing to the inputs of the evaluated unit and the denominator is the average reduction rate of outputs. The numerator in our example for the unit DMU_5 is (70.5/52 + 600/600 + 27/20 + + 3.75/3)/4 = 1.356. Similarly, the average reduction of outputs is (72/72 + 4.5/8)/2 = 0.781. Finally the super-efficiency score is $\delta^*(DMU_5)$ = = 1.356/0.781 = 1.5859. The virtual unit given by the SBM model is usually significantly closer to the original unit than in the previous model. That is why the super-efficiency score is here lower than the score given by the Andersen and Petersen model. It can be usually better explained and accepted for decision makers.
3. *Super SBMG model (9)*. The SBMG model minimises the sum of relative undesirable deviations (parameter t = 0) or the maximum relative deviation (t = 1). The undesirable deviations are positive slacks for inputs and the negative ones for outputs. The super-efficiency score in the first case is $\rho_0^*(DMU_5) = 1 + (72 - 53.16)/72 + (8 - 3.33)/8 = 1.8454$. Instead of the sum of deviations it could be possible to use their simple average. In this case the score equals to $\rho_0^*(DMU_5) = 1 + [(72 - 53.16)/72 + + (8 - 3.33)/8]/6 = 1.1409$. The minimisation of the maximum deviation leads to the optimum value $\rho_1^*(DMU_5) = 1 + (73.43 - 52)/52 = 1.4122$.

As the example shows, the ranking of the evaluated units defined by the presented super-efficiency characteristics is not always corresponding each other. Nevertheless this conclusion is typical for most multiple criteria decision making methods and corresponds to complexity of real decision problems.

CONCLUSIONS

The paper presents a new definition of super-efficiency in data envelopment analysis models. This new definition measures the distance of the evaluated real unit DMU_q and the virtual unit DMU* lying on the new efficient frontier by the positive and negative deviational variables that express the difference of the virtual inputs/outputs from the inputs/outputs of the unit DMU_q. The objective function of the super-efficiency model contains just the positive deviations for the inputs and the negative ones for the outputs. It is minimised in order to measure the distance in the undesirable way for both the groups of characteristics. Similarly to goal programming methodology either the sum of the undesirable deviation or the maximum deviation can be minimised. The proposed super-efficiency definition fulfils the basic requirement of the super-efficiency models, i.e. improving/worsening of any input or output has to lead to not worse/not better super-efficiency score. Our new definition has several advantages comparing to other ones:

- the super-efficiency model has always optimal solution, i.e. all the units receive their super-efficiency score,
- the model is non-radial – it works directly with the slacks of the inputs and outputs,
- the results of the model can be simply explained,
- the results of the model do not depend on its input or output orientation.

The results of the model were compared with other super-efficiency definitions on several numerical examples. All the comparisons show similarity of results in case the other models have a feasible solution. The future research will be concentrated on modification of the proposed model and its adaptation on specific conditions of the analysed sets of units.

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Dorota Kuchta

BICRITERIAL ROBUST APPROACH IN PROJECT MANAGEMENT

1. ROBUST APPROACH – BASIC IDEA

In recent years a new approach to decision making in the situation of uncertainty and incompleteness of information has been used more and more often. It is a so called robust approach. Generally, in robust decision making a decision has to be taken when not all the parameters of the problem are known exactly yet. The question consists in taking such a decision which will be "good enough" when all the parameters become known and definitively fixed, even in case of their unfavorable perturbations. "Good" decision means in turn such one with which the decision maker will be sufficiently satisfied or which not require any significant corrections. The approaches proposed in the literature so far and the results gained seem promising, and at the same time this domain is in its initial development stage (if compared to the progress of the research concerning the fuzzy, interval and stochastic approaches). It seems natural to try to apply this approach to optimization and management decision making in the situation of uncertainty and incompleteness of information. It is also interesting that the notion of robust solution or robust decision is not unequivocal. Different authors understand them often in quite different ways.

The general philosophy of robust decision making, or at least a philosophy which the author claims to be such, is presented in [15]. The approach can be summarized through its 3 basic stages:

1) Elimination of uncertainty – to the extent it is possible.
2) Definition of the set of satisfying solutions.
3) Selecting from the satisfying solutions the one which is least sensitive to the uncertainty which we have not been able to eliminate.

However, this description does not cover the different understanding of robust approach that can be found in the literature. They will be discussed in the next section.

2. ROBUST APPROACH – EXISTING RESULTS AND APPLICATIONS

Robust approach in discrete optimization, together with numerous results, is presented in [10]. The basic notion in the robust approach discussed in this book is that of a scenario. Each scenario corresponds to a certain possible realization of the problem parameters. Two ways of defining scenarios are possible. The first one consists in enumerating a certain finite number of various scenarios. In the second one for each parameter an uncertainty interval is defined and the set of all the scenarios is the Cartesian product of these intervals. The essence of the discussed approach consists in determining a solution which minimizes the cost function value in the worst possible case (in the worst scenario).

A slightly different robust approach can be applied to the planning and scheduling of vehicle routes. One can search for solutions which will be robust in the sense that there will be enough "space" or reserve in them to protect them against the possible lengthening of the vehicle travel or unfavourable changes in the demand or supply [9].

In [6], for the colouring problem, yet another understanding of the robust solution for discrete optimization problems is proposed: it is a solution which will remain good in changed circumstances (e.g. in the situation when new edges are added to the graph or several existing edges disappear).

The notion of robust solution in the one and multicriteria linear programming problem is understood in many ways. The authors of [5] applied also the minimal regret criterion. However, the most interesting seems to be the worst scenario approach. The worst scenario can be defined here as the occurrence of the "biggest possible" number of "unstable" coefficients which have taken on other values than expected (such an approach has been used in [3], also there it is justified from the practical point of view: in practice usually not all the problem parameters attain the unfavorable values, and limiting the number of coefficients which may vary does not mean indicating which ones will it be). Another understanding of the worst scenario in linear programming refers to the highest possible magnitude of the deviation of all the parameters of the problem. In [11] the robust approach with the worst scenario understood in the former sense has been extended to goal programming.

The authors of [8] indicate, in the context of project management, another robust approach, used also in the present paper. In this approach the decision is taken in the moment when the coefficients are not fully known, yet is only an initial decision, allowing to undertake adequate preparations. However, it is the optimal solution (in the classic understanding of this notion) which will be implemented, determined only in the moment, when all the coefficients will be known exactly. The robust solution is the one determined in the conditions of incomplete information. It should differ as little as possible from the one which will be implemented in practice, and this not so much with respect to the objective function value, but rather with respect to the details, like the decision variables values – so that the preparations undertaken in the moment when the robust solution is determined make the most sense possible from the point of view of the yet unknown optimal solution.

Robust project management seems to be a domain in the initial stage of development, and at the same time very promising and important from the practical point of view, as a good project management is something needed by most companies of today. It is a very broad domain. It would comprise the same various aspects as project management *per se*, i.e. among other the estimation of project duration and cost, scheduling, progress tracking. The present state of knowledge in robust project scheduling is presented in [1; 4; 8; 16]. A robust schedule is usually defined as one which will not differ very much from the actual schedule, which will be actually put in practice. The open question is the meaning of the expression "not differ very much" with respect to project schedules. In [4; 8] the expression means either the difference in the total duration of the project or in the planned and actual starting times of individual activities. In [1] a robust schedule is defined as one which maximised the sum of free slacks – this sum, called "robustness of a schedule", is a second criteria, applied to the schedule evaluation and selection together with the "traditional" criteria of critical path minimisation. This approach has been modified in [13]. In [16] the authors use the worst case criteria and understand the robust schedule as one which also in the worst case will have a high chance of meeting the deadline. In [4] we find also other ways of understanding the notion of a "robust schedule". For example, a robust project schedule is defined as a schedule with a sufficient quantity of in-built reserves (of time, cost, resources), which assure that the schedule will be protected against unfavorable deviations and will be able to be put into practice whatever the actual values of the individual project parameters are. The best known example of this type of schedule is the schedule constructed according to the critical chain method [7; 12].

In the next section we will propose a new robust approach to constructing robust schedules in project management. It will be a bicriterial approach.

3. A NEW ROBUST APPROACH TO PROJECT MANAGEMENT

Let us consider the following critical path problem:

$$x_n \to \min$$
$$x_j - x_i \geq d_{ij}(t) \qquad (1)$$
$$x_i \geq 0 \ (I = 1,...,n) \ (j = 1,...,n)$$

where $d_{ij}(t) = d'_{ij} + d''_{ij}t$, $t \in S = [S_1, S_2]$ are duration times of a project activities, x_i are the occurrence times of events a project network, x_n is the final event.

$d_{ij}(t) = d'_{ij} + d''_{ij}t$ will depend on a parameter only in the initial stage. Later the duration times will take on crisp values. The final solution will be the optimal solution of (1) for the final value of the parameter, but while this value is not known, a robust solution is searched for. This robust solution should be as close as possible to all the possible optimal solutions (for various values of the parameter), because all the preparations to the project execution have to start now, while the final value of the parameter is still unknown. Once it becomes known, we do not want to be forced to undertake big changes while adapting the current, transitional solution to the final one.

As mentioned before, there are many possible ways of understanding the robustness of a schedule. Here, similarly as in [1] and [13], we propose a bicriterial approach, but using two different criteria. That is why we start with two definitions:

Definition 1

A robust solution of (1) is such a solution $\mathbf{x^0} = (x_1^0, ..., x_n^0)$ which minimizes the following objective function:

$$\max_{\substack{j=1,...m; \\ t \in \langle T_{r-1}, T_r \rangle \\ r=1,...,l}} \lambda_r^j \left| x_j^0 - x_i^r(t) \right|$$

where:

– $\mathbf{x^r}(t)=\left(x_1^r(t),...,x_n^r(t)\right)$ ($t\in\left\langle T_{r-1},T_r\right\rangle$, $T_0=S_0;T_l=S_1;T_{r-1}\le T_r$ for $r=(1,...,l)$, $\langle\ \rangle$ stands for a one side or two side open interval) are all the optimal solutions of (1) for different crisp values of t (these solutions, in which the values of the decision variables are linear functions of t, can be found by means of the parametric version of simplex algorithm for the case of the parameter in the constraints right hand side coefficients). Each solution $\mathbf{x^r}(t)$ is valid in an interval $\left\langle T_{r-1},T_r\right\rangle$, whose exact form follows unequivocally from the algorithm.
– λ_r^j (j = 1,...n; r = 1,...,l) are weights chosen by the decision maker to control the importance of individual variances (several of those weights can be 0, in case the corresponding variance is if no importance) and to scale or to price them adequately.
– Signs $|\ |$ stand for the absolute value.

Thus, in the above definition we assume that a robust schedule is one in which the occurrence times of the events are as close as possible to the ones in the final solution, which is yet unknown.

Definition 2

A robust solution of (1) is such a solution $\mathbf{x^0}=\left(x_1^0,...,x_n^0\right)$ which minimizes the following objective function:

$$\max_{\substack{i,j=1,...m\\ t\in\left\langle T_{r-1},T_r\right\rangle\\ r=1,...,l}} \eta_r^{ij}\left|\left(x_j^0-x_i^0\right)-\left(x_j^r(t)-x_i^r(t)\right)\right|$$

where η_r^{ij} (i,j = 1,...,n; r = 1,...,l) are weights chosen by the decision maker to control the importance of individual variances and to scale or to price them adequately, the other notation is the same as in Definition 1.

The second definition wants the transitional, robust schedule to differ as little as possible from the final one with respect to the times between the individual events. These times are strongly linked to the actual duration times of activities (comprising free floats which can be used in the actual execution of the project without influencing the scheduled events times).

The proof of the following theorem is straightforward:

Theorem 1

A schedule satisfying to some degree Definition 1 and Definition 2 can be found by means of the following bicriterial parametric linear programming problem with *n*+2 decision variables:

$$\begin{aligned}
& y \to \min;\ z \to \min \\
& \lambda_r^j x_j - \lambda_r^j x_j^r(t) \le y \ \ (j = 1,\ldots,n) \\
& \lambda_r^j x_j - \lambda_r^j x_j^r(t) \ge -y \ \ (j = 1,\ldots,n) \\
& \eta_r^{ij} x_j - \eta_r^{ij} x_i - \eta_r^{ij}\left(x_j^r(t) - x_i^r(t)\right) \le z \ \ (i,j = 1,\ldots,n) \\
& \eta_r^{ij} x_j - \eta_r^{ij} x_i - \eta_r^{ij}\left(x_j^r(t) - x_j^r(t)\right) \ge -z \ \ (i,j = 1,\ldots,n) \\
& y \ge 0;\ z \ge 0;\ x_j \ge 0 \ \ (j = 1,\ldots,n) \\
& t \in \langle T_{r-1}, T_r \rangle,\ r = 1,\ldots l.
\end{aligned} \tag{2}$$

As $x_i^r(t)$ and $x_j^r(t) - x_i^r(t)$ (for $r = 1,\ldots l$ and $i = 1,\ldots,n$) are linear functions of *t*, the maximal and minimal values of $x_i^r(t)$ are attained in points T_{r-1} or T_r and are very easy to determine. Lets us thus denote by $x_i^{r,\min}$, $x_i^{r,\max}$ the minimum and maximum, respectively, of $x_i^r(t)$ in $\langle T_{r-1}, T_r \rangle$ and by $x_{ij}^{r,\min}$, $x_{ij}^{r,\mathrm{ma}}$ the corresponding values for functions $x_j^r(t) - x_i^r(t)$. The proof of the following lemma is straightforward.

Lemma 1

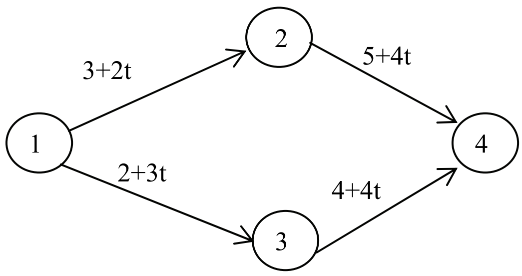
Solution of problem (2) can be found by solving the following linear programming problem:

$$\begin{aligned}
& y \to \min;\ z \to \min \\
& \lambda_r^j x_j - \lambda_r^j x_j^{r,\min} \le y \ \ (j = 1,\ldots,n) \\
& \lambda_r^j x_j - \lambda_r^j x_j^{r,\max} \ge -y \ \ (j = 1,\ldots,n) \\
& \eta_r^{ij} x_j - \eta_r^{ij} x_i - \eta_r^{ij} x_{ij}^{r,\min} \le z \ \ (i,j = 1,\ldots,n) \\
& \eta_r^{ij} x_j - \eta_r^{ij} x_i - \eta_r^{ij} x_{ij}^{r,\max} \ge -z \ \ (i,j = 1,\ldots,n) \\
& y \ge 0;\ z \ge 0;\ x_j \ge 0 \ \ (j = 1,\ldots,n) \\
& r = 1,\ldots l
\end{aligned} \tag{3}$$

Any multicriterial approach can be applied to the above problem. To adopt the easiest solution, we can to transform it to a one criterion problem, assigning weights to the objectives. We get then the objective function $\beta y + \chi z \to \min$ with the same constraints as above. Then, it will be enough to solve l one criteria linear programming problems (for each $r = 1,...l$) and to choose the best solution from the l solutions obtained.

4. EXAMPLE

Let us consider the following project network (t takes on values from the interval [0,5]):



If we solve problem (3) with the objective function $y + z \to \min$ (we assume both objectives to equally important), we get the following solution: $x_1 = 0,\ x_2 = 4,\ x_3 = 4,\ x_4 = 14,\ y = 6,\ z = 4.$

CONCLUSIONS

The robust approach to optimisation is very wide and promising. The robustness of a decision or solution, whatever definition of robustness has been selected, is an important feature and that is why the robustness should be used at least as one of the criteria in the decision making process. In this paper we propose one way of incorporating robustness into the search of project schedule. Other possibilities of using the robustness as a decision criterion are also mentioned in the paper and will be the object of further research.

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Mikuláš Luptáčik

Bernhard Böhm

MEASURING ECO-EFFICIENCY IN A LEONTIEF INPUT-OUTPUT MODEL

1. THE MACROECONOMIC PRODUCTION FUNCTION

The economic input-output model augmented by pollution generation and its abatement as introduced by Leontief [6] has often been used to analyse environmental and economic repercussions. Pollutants are considered as undesirable outputs of industrial activities. The aggregate generation of pollution is controlled by a specific given tolerance limit, an environmental standard. Abatement activities are absorbing pollution at the expense of intermediate and primary inputs.

In the context of pursuing sustainable development a new concept termed "eco-efficiency" has surfaced in business economics and in the public discussion on environmental policy. Efficiency of production as well as environmental efficiency should simultaneously be taken into account. Eco-efficiency is characterised by production of goods and services with minimal resource use and generation of waste and other emissions of pollutants.

Production efficiency is typically determined in relation to the production possibility frontier. In a multi-input, multi-output production technology distance functions can be used to characterise the efficiency of an economy [3; 12]. Given the input vector, the output distance function considers the maximal proportional expansion of the output vector, while the input distance function considers the minimal proportional contraction of the input vector, given the output vector. Using the input-output model based on make and use tables (denoted by V and U) for n industries and n commodities[1], such distance functions derive from the following linear optimisation problems:

[1] Without loss of generality we assume quadratic make and use tables.

1. Minimise the use of inputs (capital (K), labour (L)) without altering their proportions, for a given vector of final demand $\bar{y}$:

$$\begin{aligned} &\min_{\lambda} \gamma \\ &\text{subject to} \quad [V'\text{-}U]\lambda \geq \bar{y} \\ &\qquad\qquad\quad k'\lambda - \gamma\bar{K} \leq 0 \\ &\qquad\qquad\quad l'\lambda - \gamma\bar{L} \leq 0 \\ &\qquad\qquad\quad \lambda \geq 0, \gamma \geq 0 \end{aligned} \tag{1}$$

where V' denotes the transposed make matrix.

2. Maximise the proportional expansion of final demand $\bar{y}$ for given amounts of primary factors:

$$\begin{aligned} &\max_{\lambda} \alpha \\ &\text{subject to} \quad [V'\text{-}U]\lambda - \alpha\bar{y} \geq 0 \\ &\qquad\qquad\quad k'\lambda \leq \bar{K} \\ &\qquad\qquad\quad l'\lambda \leq \bar{L} \\ &\qquad\qquad\quad \lambda \geq 0, \alpha \geq 0 \end{aligned} \tag{2}$$

where (V'-U) is the net output flow matrix, k' the row vector of capital requirements, l' the row vector of labour requirements, $\bar{K}$ and $\bar{L}$ are the given upper limits of the respective primary inputs, and λ is the column vector of intensity levels of sector production. The scalar γ describes the proportional reduction factor of primary inputs and α the proportional expansion factor of final demand. Because the input-output model exhibits constant returns to scale the input distance function is the reciprocal of the output distance function. Thus in the optimum $\gamma = 1/\alpha$.

Models of this kind have been proposed by ten Raa [12]. To measure eco--efficiency these models are extended to include undesirable outputs, i.e. pollutants, produced in the economy, and abatement activities which reduce the emissions at the cost of desirable outputs and value added. Let then the production possibility frontier of the economy be determined by the input-output model, primary inputs, pollution generation and abatement, and final demand. Models (1) and (2) are appropriately amended and discussed in the next section. The degree by which a net-output vector, for given primary inputs and environmental standards, could be extended, can be considered as measure of eco-inefficiency. Equivalently, this could be also be achieved by a reduction of primary inputs for given environmental standards and given final demand. Including abatement activities in our model we take into account that desirable outputs are strongly disposable while undesirable ones are only weakly disposable meaning that their reduction can only be achieved by a reduction of desirable outputs or an increase of primary inputs.

Because the efficiency indicators derived from the distance function approach are based on optimisation without the possibility of altering the proportions of net outputs or primary inputs in the respective models, they do not imply necessarily Pareto-Koopmans efficiency. For example, it might still be possible to reduce a specific primary input without reducing any of the net-outputs if its slack variable is positive (see [7]). In this case a change in the proportion of primary inputs might be required to achieve efficiency. In view of the sectoral disaggregation of the economy section three proposes, therefore, a new "slack" – based measure of eco-efficiency which goes beyond the traditional proportional approaches to the changes in final demand or primary inputs. Thus, we are able to take into account the changes in the structure of final demand and the composition of primary inputs.

In section four we show the relationship of the macroeconomic efficiency model of section two to that of Data Envelopment Analysis (DEA), which is a widely used method for efficiency measurement. We derive the production possibility frontier from a multi-objective problem subject to the input-output model. Different DEA models can be constructed which provide eco-efficiency measures comparable to the proportional measures derived in the previous section.

Section five provides a demonstration of the viability of all methods discussed by applying them to Austrian input-output data and National Accounts Matrices including Environmental Accounts (NAMEA) data.

2. MEASURING ECO-EFFICIENCY OF AN ECONOMY

The augmented Leontief model known from the literature (e.g. [6; 7]) extends the economic system by pollution generation and abatement activities. We define emission-matrix $W = \{w_{hj}\}$ containing the amounts of pollutants h = 1,...,q generated by sector j = 1,...,n, emission-matrix $W_a = \{w_{hi}\}$ stating the emissions of pollutants h = 1,...,q generated by abatement activity i = 1,...,q, and matrix $U_a = \{u_{jh}{}^a\}$ containing the inputs of commodity j = 1,...,n into the technological process that removes pollutant h = 1,...q. The q-dimensional row vectors of capital and labour inputs into abatement activities are denoted k'_a and l'_a. Diagonal matrix $\hat{A}$ represents pollution eliminated and λ_a the abatement activity vector. With $\bar{y}$ the (n x 1) final demand vector as above and $\bar{w}$ vector of net generation of pollutants which remain untreated (i.e. pollution standards) we can apply the idea of distance function to this partitioned system. When treating the undesirable outputs like inputs, the proportional measure of eco-efficiency can be derived from the following model formulations:

1. Minimise the use of primary factors for a given level of final demand and tolerated pollution:

$$\begin{aligned} &\min_{\lambda} \gamma \\ &\text{subject to} \quad (V'-U)\,\lambda - U_a\lambda_a \geq \bar{y} \\ &\qquad W\lambda - (\hat{A} - W_a)\lambda_a \leq \bar{w} \\ &\qquad k'\,\lambda + k_a'\,\lambda_a - \bar{K}\gamma \leq 0 \\ &\qquad l'\,\lambda + l_a'\,\lambda_a \leq \bar{L}\gamma \leq 0 \\ &\qquad \lambda \geq 0;\, \lambda_a \geq 0;\; \gamma \geq 0 \end{aligned} \tag{3}$$

2. Maximise expansion of final demand for given levels of tolerated pollution and primary factors:

$$\begin{aligned} &\max_{\lambda} \alpha \\ &\text{subject to} \quad (V'-U)\,\lambda - U_a\lambda_a - \alpha\bar{y} \geq 0 \\ &\qquad W\lambda - (\hat{A} - W_a)\lambda_a \leq \bar{w} \\ &\qquad k'\,\lambda + k_a'\,\lambda_a \leq \bar{K} \\ &\qquad l'\,\lambda + l_a'\,\lambda_a \leq \bar{L} \\ &\qquad \lambda \geq 0;\, \lambda_a \geq 0;\, \alpha \geq 0 \end{aligned} \tag{4}$$

We note that due to the presence of the pollution subsystem representing undesirable outputs, the optimal values of α and γ are no longer the reciprocal of each other. However, by treating these undesirable outputs like inputs in the model, i.e. by changing the problem formulation into a proportional reduction of primary inputs and undesirable outputs for given final demand, the reciprocal property of the distance function can be re-established.

3. A SLACK-BASED MEASURE OF ECO-EFFICIENCY

To avoid the limitation of efficiency indicators assuming unchanged proportions of outputs or inputs and taking into account the similarity of undesirable outputs and inputs we turn now to a slack based measure of eco--efficiency.

The slack based measure of eco-efficiency is constructed in analogy to a measure proposed by Cooper, Seiford, and Tone [2] for data envelopment analysis. The following goal programming model can be formulated when one treats the undesirable outputs of pollutants just like inputs in a conventional definition of efficiency. We formulate it as a minimisation problem of a scalar which is unit invariant and monotone, subject to the constraints of the augmented Leontief model and the relations restricting primary input use [7]:

$$\min \left\{ \rho = \frac{1 - \frac{1}{q+2}\left(\frac{S_K}{\overline{K}} + \frac{S_L}{\overline{L}} + \sum_{h=1}^{q} \frac{S_{wh}}{\overline{w}_h} \right)}{1 + \frac{1}{n}\sum_{j=1}^{n} \frac{S_{yj}}{\overline{y}}} \right\} \tag{5}$$

subject to

$$\begin{aligned} &(V'-U)\,\lambda^* - U_a\lambda^*_a - S_y = \overline{y} \\ &W\lambda^* - (\hat{A} - W_a)\lambda^*_a + S_w = \overline{w} \\ &k'\,\lambda^* + k_a'\,\lambda^*_a + S_K = \overline{K} \\ &l'\,\lambda^* + l_a'\,\lambda^*_a + S_L = \overline{L} \\ &\lambda^* \geq 0;\ \lambda^*_a \geq 0;\ S_y \geq 0; S_w \geq 0; S_K \geq 0; S_L \geq 0 \end{aligned} \tag{6}$$

where we use n sectors and q pollutants. We denote by λ^*, λ^*_a the intensity vectors, by S_y and S_w the vectors of slack variables of sector outputs and pollutants. S_K and S_L are capital and labour slacks. $\overline{y}, \overline{w}, \overline{K}, \overline{L}$ are given values.

Evidently $0 \leq \rho \leq 1$ since slacks cannot exceed the values on the right hand side. Re-defining $s_j = tS_j$ (j = y, w, L, K), and $\lambda = t\,\lambda^*$, $\lambda_a = t\,\lambda^*_a$ this fractional program can be linearised to yield the equivalent problem:

$$\min \left\{ \tau = t - \frac{1}{q+2}\left(\frac{s_K}{\overline{K}} + \frac{s_L}{\overline{L}} + \sum_{h=1}^{q} \frac{s_{wh}}{\overline{w}_h} \right) \right\} \tag{7}$$

$$\text{subject to} \quad \begin{aligned} &(V'-U)\,\lambda - U_a\lambda_a - s_y = t\bar{y} \\ &W\lambda - (\hat{A} - W_a)\lambda_a + s_w = t\bar{w} \\ &k'\,\lambda + k_a{}'\,\lambda_a + s_K = t\bar{K} \\ &l'\,\lambda + l_a{}'\,\lambda_a + s_L = t\bar{L} \\ &t + \frac{1}{n}\sum_{j=1}^{n}\frac{s_{yj}}{\bar{y}} = 1 \\ &\lambda \geq 0;\, \lambda_a \geq 0;\, s_y \geq 0; s_w \geq 0; s_K \geq 0; s_L \geq 0 \end{aligned} \qquad (8)$$

Some numerical calculations with data of the Austrian economy will provide an opportunity to compare the different eco-efficiency measures and give an impression about their usefulness.

In the next section we return to the radial efficiency measures of section two and show that these models are closely related to data envelopment analysis models which are now widely used for efficiency measurement.

4. RELATIONS TO DATA ENVELOPMENT ANALYSIS

The most obvious relationship of the models considered so far with the standard model of DEA can be found by contemplating system (1). For simplicity of exposition we use the model without pollution and abatement activities. The extension to include the environmental components is relatively straightforward and can be found in the Appendix. The idea of DEA is to use data on m inputs and n outputs of N decision making units using the same technology to derive the efficiency frontier by the "best" producing units. The efficiency frontier is defined by the data envelope of all units considered. The envelope form of the DEA model minimises the efficiency score θ, a radial contraction of the input vector of a particular decision making unit, while remaining in the feasible input set:

$$\begin{aligned} &\min_{\mu}\ \theta \\ &\text{subject to} \quad Q\mu \geq q^0 \\ &\qquad\qquad\quad Z\mu - \theta z^0 \leq 0 \\ &\qquad\qquad\quad \mu \geq 0, \theta \geq 0 \end{aligned} \qquad (9)$$

with q^0 the (n x 1) output vector and z^0 the (m x 1) input vector of the unit "0" whose efficiency is to be investigated, Q the (n x N) matrix containing the output vectors of all N units, Z the (m x N) matrix containing the input vectors of all units, μ a (N x 1) vector of coefficients to be determined, and θ a scalar. The radial contraction of the input vector z^0 generates a projection point ($Q\mu$, $Z\mu$) on the surface of the technology set spanned by the efficient subset of the N units. The projected point is a linear combination of the observed data points of those efficient units.

Setting $Q = [V'\text{-}U]$, $Z = [k', l']'$, $q^0 = \bar{y}$, and $z^0 = (\bar{K}, \bar{L})'$ the problem (9) is equivalent to problem (1) with $\theta = \gamma$. But there is a significant difference in the economic interpretation of DEA model (9) and a general DEA model. While DEA uses inputs and outputs of different independent decision making units, the I-O model uses data of one country but disaggregated into interrelated sectors. One way to exploit the formal similarity of the problem consists in generating such levels of outputs of and inputs into sectors as can optimally be generated by the given input-output system. This will establish the production possibility set or the input requirement set depending on the formulation.

In essence we propose to formulate a multi-objective optimisation problem in which final demand for each commodity is maximised subject to restraints on the production of other outputs and required inputs, or each input is minimised for the given levels of final demand. Denoting by s the vector of n slack variables of the n goods and by s_K and s_L the slacks in the capital and labour input relation, the following model is solved j times for given nonnegative values of sector net-outputs and inputs to obtain the maximal value of each slack variable s_j for j = 1,..., n, K, L.

$$\begin{aligned} &\max s_j \\ &\text{subject to} \quad (V'-U)\,\lambda - s = \bar{y} \\ &\qquad k'\lambda + s_K = \bar{K} \\ &\qquad l'\lambda + s_L = \bar{L} \\ &\qquad s_j \geq 0 \quad \text{all} \quad j = 1,\dots,n,K,L \\ &\qquad \lambda_i \geq 0 \quad \text{all} \quad i = 1,\dots,n. \end{aligned} \tag{10}$$

For each of the n + 2 solutions of (10) the values of the net-output column vector y* are then obtained by $y^* = \bar{y} + s$ and those of the inputs by $K^* = \bar{K} - s_K$ and $L^* = \bar{L} - s_L$. These sets of values are arranged row-wise in a pay-off matrix with the maximal (or minimal) values appearing in the main diagonal while the off-diagonal elements provide the levels of other sector net--outputs (or inputs) compatible with the individually optimised one. Thus, each column of the pay-off matrix yields an efficient solution, i.e. characterises a potential efficient point which can be generated by the economic system. In other words each column represents a ficticious decision making unit. Each of these points is constructed independently of the other points but taking account of the entire systems relations. The independence derives from the fact that each hypothetical experiment to find the maximal net output of a particular sector (or a particular minimal input) is conducted independently from that of another sector output or input, although all are using the same technology. In this way the experiments can be taken to generate data equivalent to those of hypothetical firms with different input and output characteristics, which all use the same production technique. The whole set of such efficient solutions can, therefore, establish the frontier of the production possibility set (or the input requirement set). Thus, the efficient envelope of the economy is defined by:

$$\begin{bmatrix} Q^* \\ -- \\ Z^* \end{bmatrix} = \begin{bmatrix} y_{11}^* & y_{12}^* & \cdots & y_{1,n+2}^* \\ \vdots & \vdots & & \vdots \\ y_{n1}^* & y_{n2}^* & \cdots & y_{n,n+2}^* \\ -- & -- & -- & --- \\ K_1^* & K_2^* & \cdots & K_{n+2}^* \\ L_1^* & L_2^* & \cdots & L_{n+2}^* \end{bmatrix} \tag{11}$$

This efficient frontier constitutes the standard envelope (a notion introduced by Golany and Roll [4]) for the DEA model measuring the efficiency of the economy given by the actual output and input data (q^0, z^0). For this purpose we solve the following problem (12):

$$\begin{aligned} &\min_{\mu} \theta \\ &\text{subject to} \quad Q^* \mu \geq q^0 \\ &\qquad\qquad\quad Z^* \mu - \theta z^0 \leq 0 \\ &\qquad\qquad\quad \mu \geq 0, \theta \geq 0 \end{aligned} \tag{12}$$

The efficiency score, the scalar θ gives us the proportion of all inputs of the economy which must be sufficient – compared to the production frontier – to achieve the given output levels. In other words (1-θ) describes the necessary reduction of all inputs of the economy to achieve the efficiency frontier. Therefore θ describes the efficiency of the economy.

The vector μ provides the weighting pattern in the construction of the projection point on the efficient surface derived from a radial input contraction. It informs about the weight a particular artificial decision making unit (i.e. a particular efficient solution as described by (11)) has in the projection of the given state of the economy to the efficient frontier. All those units $i = 1,...,N$ (here $N = n + 2$) with $\mu_i > 0$ form the "peer group" defining the efficient production level for the economy under investigation.

The extension of the model to include the environment is straightforward, remembering that quantities of undesirable outputs (pollutants) are treated like inputs in view of them being minimised. A short derivation is given in the Appendix.

According to Korhonen and Luptáčik [5] four different DEA models can be constructed: The first model (A) is based on the idea of presenting all outputs as a weighted sum, but using negative weights for undesirable outputs. Here, the efficiency is measured by proportional reduction of inputs only. In model (B) undesirable outputs are taken as inputs. Efficiency is measured by a proportional simultaneous reduction of inputs and undesirable outputs. If efficiency is measured by the ratio of the weighted sum of desirable outputs minus inputs to that of undesirable outputs we obtain model (C). Model (D) is an output oriented model where efficiency is measured by proportional improvements of outputs and constitutes the reciprocal formulation of model (B). It has been shown that the eco-efficient frontier is independent of the specific model used.

Model (3) which measures efficiency only by proportional reduction of inputs can, therefore, be considered equivalent to model (A). Changing the second constraint of the input oriented model (3) to:

$$W\lambda - (\hat{A} - W_a)\lambda_a - \gamma\overline{w} \leq 0 \tag{13}$$

i.e. treating the undesirable outputs exactly like primary inputs, implies DEA--model (B), while keeping constraint (13) but replacing the primary input constraints by:

$$\begin{aligned} k'\lambda + k_a'\lambda_a &\leq \overline{K} \\ l'\lambda + l_a'\lambda_a &\leq \overline{L} \end{aligned} \tag{14}$$

leads to DEA model (C). Model (4) is seen to correspond to model (D) because of efficiency measured by the proportional increase of outputs.

These different model versions for eco-efficiency measurement permit decompositions of the efficiency score according to desirable outputs, undesirable outputs, and inputs.

5. AN EMPIRICAL DEMONSTRATION FOR THE AUSTRIAN ECONOMY

The following models use a highly aggregated version of the Austrian input-output table of 1995 [10] and NAMEA data [11; 13] for air and water pollution. The empirical examples are calculated for five sectors, two pollutants and two primary inputs as follows (the numbering of intensity and slack variables follows the item numbers):

Sectors:
1. Agriculture, forestry, mining (mill. ATS).
2. Industrial production (mill. ATS).
3. Electricity, gas, water, construction (mill. ATS).
4. Trade, transport and communication (mill. ATS).
5. Other public and private services (mill. ATS).

Pollutants:
6. Air pollutant (NO_x, tons per year) (Source: [11]).
7. Water pollutant (P, tons per year) (Source: [13]).

Primary Inputs:
8. Labour (total employment, 1000 persons).
9. Capital (gross capital stock, 1995, nominal, mill. ATS) (Source: [1]).

5.1. Proportional and slack based eco-efficiency measures

A first experiment employs the simple models (1) and (2) with levels of capital and labour corresponding to a 5% underutilisation of both of these inputs. As expected the proportional efficiency measure α yields \$1.05, (output could be expanded by 5% proportionally) and the minimum γ equals 0.952, the reciprocal value of α. The λ values are the same for all sectors, i.e. $\lambda_i = 1.05$ for model (1) and equal to one for model (2) (i.e. the same output can be produced by a 4.76189% reduction of both inputs).

Expanding the model for pollutants and abatement we repeat the exercise with the same levels of inputs as before. Model (3) yields a minimum value of γ equal to 0.95194 with $\lambda_i = 0.999$ for $i = 1,...,5$ and $\lambda_6 = 0.99867$, $\lambda_7 = 0.90727$. We observe a slight change compared to the simple model for the intensities of the output sectors, but quite different values for the abatement intensities λ_6 and λ_7. Calculating the output oriented model (4) the maximum $\alpha = 1.04899$ is not the reciprocal value of min γ. Here again the intensities are almost the same for the outputs but different for the pollutants ($\lambda_i = 1.0494$ for $i = 1,...,5$ and $\lambda_6 = 1.137$, $\lambda_7 = 1.1007$). We observe that the efficiency measure of the extended model gives a proportional factor of expansion or reduction of outputs (respectively inputs) while intensities reveal disproportionate abatement activities.

Let us now compare these results with the slack based efficiency measure (7) under the same assumptions on primary inputs. Minimisation of τ yields a value of 0.409853 with a $t = 0.823$. All output slacks except that of sector one are zero, but pollution slacks and the capital slack are positive. The following table states the results including the optimal intensity values:

Table 1

λ1 = 1. 017 840 7	s1 = 25366.251
λ2 = 0.834 212 68	s2 = s3 = s4 = s5 = s8 = 0
λ3 = 0.832 229 32	s6 = 54206.968
λ4 = 0.831 069 16	s7 = 831.700 32
λ5 = 0.827 642 68	s9 = 75486.407
λ6 = 2. 454 075 3	t = 0.823 188 57
λ7 = 1. 995 758 5	τ = 0.409 853 52

The (inefficient) economy produces too much of agricultural output thereby generating more pollution of both kinds requiring higher abatement intensities (λ_6, λ_7). The positive pollution slacks indicate that too much undesirable outputs are generated while the capital slack shows that capital utilisation is by 0.69% only. The limiting primary factor is labour which is fully utilized.

5.2. The empirical eco-efficiency analysis with DEA

To construct the envelope as described in section 4 the nine problems (10) are solved. The resulting pay-off matrix is given below:

Table 2

Pay-off table

		y1	y2	y3	y4	y5
max	y1	71.409204	1037.514	287.531	638.251	918.239
max	y2	28.693	1217.22728	287.531	638.251	918.239
max	y3	28.693	1037.514	424.7812	638.251	918.239
max	y4	28.693	1037.514	287.531	755.2308	918.239
max	y5	28.693	1037.514	287.531	638.251	1011.103
min	Poll1	28.693	1037.514	287.531	638.251	918.239
min	Poll2	28.693	1037.514	287.531	638.251	918.239
min	K	28.693	1037.514	287.531	638.251	918.239
min	L	28.693	1037.514	287.531	638.251	918.239

		Poll1	Poll2	Capital	Labour
max	y1	65.850	1010.34	10618.188	4123.830
max	y2	65.850	1010.34	10705.312	4123.830
max	y3	65.850	1010.34	10785.810	4123.830
max	y4	65.850	1010.34	10704.571	4123.830
max	y5	65.850	1010.34	10840.940	4052.4773
min	Poll1	0	1010.34	10458.756	3962.061
min	Poll2	65.850	0	10403.477	3947.831
min	K	65.850	1010.34	10324.70548	4123.830
min	L	65.850	1010.34	10840.940	3927.553

Using this pay-off table for the same experiment as above (i.e. with 5% capital and labour surplus) the DEA model with pollutants (cf. Appendix (17)) is solved yielding a minimum θ value of 95.33 for the economy, while all other artificial units are 100% efficient as they should be from the construction of the efficiency frontier. The projection of the economy to the efficient frontier is performed by using the following coefficients:

$\mu_1 = 0.0279$, $\mu_2 = 0.2399$, $\mu_3 = 0.0871$, $\mu_4 = 0.2267$, $\mu_5 = 0.4109$,

$\mu_6 = 0.0074$, $\mu_7 = 0.0074$, $\mu_8 = \mu_9 = 0$.

The θ value indicates the inefficiency in the use of primary factors and excess pollution. In other words, both primary factors and both pollution levels should be reduced by 4.7% in order for the economy to become efficient. This is DEA-model B of Korhonen and Luptáčik [5].

The following properties can be proved (see [9]). Solving the modified model (3) with additional constraint (13), i.e. considering the proportional reduction in both, primary inputs and undesirable outputs, the value of the efficiency score γ is exactly equal to θ of the previous DEA-model. If we calculate the output oriented model (DEA-model D in [5]) we obtain the efficiency score of 1.04899 which is exactly the reciprocal of the input oriented value θ. For given levels of primary factors and net-pollution the net output (i.e. final demand) of all sectors could be increased by 4.9% to make the economy efficient. The same efficiency score follows from model (4) as can be seen from the calculated value of α in the previous subsection.

CONCLUSIONS

The purpose of this paper was to present new alternative measures for eco-efficiency. We basically distinguished two concepts: The first was based on a linear programming input-output model which is able to provide optimal intensity levels for production and abatement activities. The other was based on the construction of a production possibility frontier by multiple objective optimisation. Then, using data envelopment analysis the eco-efficiency of an economy related to this hypothetical frontier was estimated. The results of the DEA application show the potential improvements of eco-efficiency with particular outputs, primary inputs, and undesirable outputs. An analysis of the relationships between the concepts shows an equivalence of radial efficiency measures. However, because of their different model structures useful additional insights and interpretations of the same criterion of performance can be obtained.

APPENDIX

The extension of the simple input-output-DEA model (12) to incorporate the environmental aspects can be achieved along the following lines. First, problem (10) is re-written to incorporate abatement activities and pollution generation using the notation of section 2. Combining the k given primary inputs in (k x 1) vector $\bar{z}$ we have to solve $n + q + k$ problems (n sectors, k primary inputs, q pollutants):

$$
\begin{aligned}
&\max s_i \\
&\text{subject to} \quad (V'-U)\,\lambda - U_a\lambda_a - s_1 = \bar{y} \\
&\qquad W\lambda - (\hat{A} - W_a)\lambda_a + s_2 = \bar{w} \\
&\qquad v'\,\lambda + v_a{}'\lambda_a + s_3 = \bar{z} \\
&\qquad s_j \geq 0, \quad j = 1,2,3 \\
&\qquad \lambda \geq 0;\ \lambda_a \geq 0; \\
&\qquad i = 1,\ldots,n,n+1,\ldots,n+q,n+q+1,\ldots,n+q+k
\end{aligned}
\tag{15}
$$

where s_1 and λ are (n x 1) vectors, s_2 and λ_a are (q x 1) vectors, and s_3 is a (k x 1) vector. After obtaining the n +q + k solutions the (n + q + k) x (n + q + k) pay-off matrix is constructed with the submatrices of output (Q_1^*), pollution (Q_2^*) and input values (Z^*) calculated from $y^* = \bar{y} + s_1$, $w^* = \bar{w} - s_2$ and $z^* = \bar{z} - s_3$ of all solutions. The efficient envelope is then given by:

$$
\begin{bmatrix} Q_1^* \\ -- \\ Q_2^* \\ -- \\ Z^* \end{bmatrix} = \begin{bmatrix} y_1^* & y_2^* & \cdots & y_{n+q+k}^* \\ -- & -- & -- & --- \\ w_1^* & w_2^* & \cdots & w_{n+q+k}^* \\ -- & -- & -- & --- \\ z_1^* & z_2^* & \cdots & z_{n+q+k}^* \end{bmatrix}
\tag{16}
$$

to be incorporated in the extended DEA model (17) (denoting observed desirable outputs by q_1^0, undesirable ones by q_2^0 and inputs by z^0:

$$
\begin{aligned}
&\min_{\mu}\ \theta \\
&\text{subject to} \quad Q_1^*\mu \geq q_1^0 \\
&\qquad Q_2^*\mu - \theta q_2^0 \leq 0 \\
&\qquad Z^*\mu - \theta z^0 \leq 0 \\
&\qquad \mu \geq 0, \theta \geq 0
\end{aligned}
\tag{17}
$$

The weighting vector μ now has dimension N = (n + q + k). Its positive elements determine the efficient production or reference set onto which the economy with performance (q_1^0, q_2^0, z^0) is projected when it is inefficient.

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Kaisa Miettinen

IND-NIMBUS FOR DEMANDING INTERACTIVE MULTIOBJECTIVE OPTIMIZATION*

INTRODUCTION

In multiobjective optimization, we handle optimization problems with more than one objective function. Because the objectives are typically conflicting, they cannot reach their individual optima simultaneously but the aim is to find the best compromise. As compromises, we define mathematically equivalent nondominated or Pareto optimal solutions. To be able to select the best possible compromise, we usually need the input of a human expert, a decision maker who can express preference information related to the problem.

A widely used approach for solving multiobjective optimization problems is to use interactive methods, where the decision maker can iteratively direct the solution process and, at the same time, learn about the interdependencies among the objectives in the problem to be solved (see e.g. [2; 9; 10; 14; 26; 29] and references therein). Interactive methods differ from each other, for example, by what kind of information is to be given to the decision maker about the problem, what kind of preference information is asked from the decision maker and how the preference information is used for directing the solution process. In some interactive methods, the decision maker can even change her/his mind while learning. Besides giving the decision maker a possibility to learn about the problem, interactive methods are usually computationally efficient because only such Pareto optimal solutions are generated that the decision maker is interested in. For these reasons, interactive methods can be expected to give very satisfactory solutions. This naturally necessitates that the decision maker has enough time and interest to take part in the iterative solution process.

Unfortunately, software for solving multiobjective optimization problems involving continuous variables is not easy to find. The task gets even more difficult if the problem is nonlinear. WWW-NIMBUS [17], an interactive software system operating on the Internet, was developed in 1995 to answer this need. WWW-NIMBUS (available at http://nimbus.it.jyu.fi/) has changed quite

*The implementation of IND-NIMBUS was realized in a project supported by the Finnish Funding Agency for Technology and Innovation. The author wishes to thank Dr. Marko M. Mäkelä for his share in developing NIMBUS as well as the other members of the team that worked in the above-mentioned project.

a lot during the years but it can still be used free of charge for teaching and academic research proposes. WWW-NIMBUS can be used for solving nonlinear and even nondifferentiable and nonconvex multiobjective optimization problems. Because the Internet is easily accessible, the system is automatically available to large numbers of people. In 1995, the first version of WWW-NIMBUS was the first interactive multiobjective optimization software operating on the Internet. Even now, it continues to be a unique software system.

WWW-NIMBUS is based on the principles of centralized computing and distributed interface. This means that all the calculations take place in a server computer (at the University of Jyväskylä) and the user interface is the browser of each individual user. In this way, the system sets no requirements on the user's computer and the operating system used and/or compilers available play no role. There is nothing to be installed and the latest version of the system is always available. Furthermore, the World-Wide Web (WWW) provides a convenient and graphical user interface with visualization possibilities.

Operating via the Internet is convenient with academic problems but when the problem involves computationally expensive function evaluations and/or the functions values come from a simulation or modelling tool, another approach is needed. For this purpose, IND-NIMBUS (INDustrial NIMBUS), a software operating in MS-Windows and Linux operating systems has been developed.

The NIMBUS method is the core of both WWW-NIMBUS and IND-NIMBUS. NIMBUS (Nondifferentiable Interactive Multiobjective BUndle-based optimization System) is an interactive method where preference information is acquired from the decision maker in the form of a classification of the objective functions. The method has been applied, for example, in structural design problems [21], in the optimal control problems of the continuous casting of steel [22] and in the optimal shape design of paper machine headboxes [7]. Results with both small-scale and large-scale problems give evidence of the reliability and efficiency of the method. Different versions of NIMBUS are described in [14; 15; 16]. Here we concentrate on the latest, so-called synchronous, version [19].

In NIMBUS, the decision maker can iteratively learn about the problem and can conveniently direct the solution process. NIMBUS has been designed to be easy to use and, unlike many interactive methods, it does not require consistent information from the decision maker. Furthermore, the information handled is straightforward. The objective function values have a direct meaning to the decision maker and no artificial concepts are needed.

In this paper, we briefly introduce the synchronous NIMBUS method in Section 1 and its implementation, IND-NIMBUS in Section 2. Finally, we conclude in Section 3.

1. NIMBUS METHOD

Multiobjective optimization problems to be considered are of the form

$$\begin{array}{rl} \text{minimize/maximize} & \{f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x})\} \\ \text{subject to} & \mathbf{x} \in S \end{array}$$

with k *objective functions* $f_i : \mathbf{R}^n \to \mathbf{R}$ to be optimized simultaneously. Each of them is either to be minimized or maximized. The *decision vector* $\mathbf{x}$ belongs to the (nonempty) *feasible set* S. The feasible region may consist of linear and nonlinear inequality and equality constraints as well as box constraints for the variables. The images of the feasible decision vectors are called feasible *objective vectors*. Without loss of generality we assume in this section that all the objective functions are to be minimized.

The idea of the interactive NIMBUS method is to move around the set of Pareto optimal solutions, where the value of an objective function can only be improved by allowing at least one of the others to impair (see e.g. [14)]. In order to help the decision maker in getting an impression of what is possible to achieve, we need information about the ranges of the feasible objective vectors in the Pareto optimal set. We refer to the vector containing the best values of each objective function as the *ideal objective vector* $\mathbf{z}^\star \in \mathbf{R}^k$. The vector with the worst values is a so-called *nadir objective vector* $\mathbf{z}^{\mathrm{nad}} \in \mathbf{R}^k$. Unfortunately, it is difficult to obtain but can be approximated (see e.g. [14] and references therein). In what follows, we assume that we have approximations of ideal and nadir values available for each objective function.

In the interactive solution process with NIMBUS, the decision maker can at each iteration indicate what kind of a solution would be more satisfactory than the current one with the help of a classification. Thus, the user can evaluate the problem to be solved and adapt one's preferences during the solution process in an iterative and flexible way. Let $\mathbf{x}^h$ stand for the Pareto optimal decision vector at the iteration h. We show the objective function values calculated at this point to the decision maker. Then the decision maker is asked to classify the objective functions into up to five classes for objective functions f_i whose values

- should be decreased ($i \in I^<$),
- should be decreased till some aspiration level $\bar{z}_i^h < f_i(\mathbf{x}^h)$ ($i \in I^\leq$),
- are satisfactory at the moment ($i \in I^=$),
- are allowed to increase till some upper bound $\varepsilon_i^h > f_i(\mathbf{x}^h)$ ($i \in I^>$),

– are allowed to change freely ($i \in I^{\diamond}$).

The difference between the first two classes is that the objective functions in the first class are to be minimized as far as possible but the functions in the second class only till the aspiration level specified. The decision maker is asked to specify the aspiration levels and upper bounds, if needed. Since improvement in the Pareto optimal set in any objective function value is possible only by allowing impairment in some other objective function, the classification is feasible only if $I^{<} \cup I^{\leq} \neq \emptyset$ and $I^{>} \cup I^{\diamond} \neq \emptyset$.

After the classification, a *subproblem* [19] is formed based on the information specified as:

$$\begin{array}{ll} \text{minimize} & \max\limits_{\substack{i \in I^{<} \\ j \in I^{\leq}}} \left[\dfrac{f_i(\mathbf{x}) - z_i^{\star}}{z_i^{\mathrm{nad}} - z_i^{\star\star}} \dfrac{f_j(\mathbf{x}) - \hat{z}_j}{z_j^{\mathrm{nad}} - z_j^{\star\star}} \right] + \rho \sum_{i=1}^{k} \dfrac{f_i(\mathbf{x})}{z_i^{\mathrm{nad}} - z_i^{\star\star}} \\ \text{subject to} & f_i(\mathbf{x}) \leq f_i(\mathbf{x}^c) \ \text{ for all } \ i \in I^{<} \cup I^{\leq} \cup I^{=}, \\ & f_i(\mathbf{x}) \leq \varepsilon_i \ \text{ for all } \ i \in I^{\geq} \\ & \mathbf{x} \in S \end{array} \tag{1}$$

where a so-called augmentation coefficient $\rho > 0$ is a relatively small scalar. The weighting coefficients $1/(z_j^{\mathrm{nad}} - z_j^{\star\star})$ have proven to facilitate capturing the preferences of the decision maker well. They also increase computational efficiency [20].

As shown in [18], different subproblems may lead to different solutions even though they are based on the same preference information. Usually method developers select one subproblem, which means that they select the solution to be generated. Yet, there is no general way how to identify the best solution without involving the decision maker.

In the synchronous version of NIMBUS [19], there are three subproblems available in addition to (1). This means that if the decision maker wants so, (s)he can see up to four different solutions after one classification. In other words, by classifying the objective functions once, the decision maker can get a better picture of different Pareto optimal solutions satisfying the preference information specified. Besides, the method developers do not have to make the choice related to the subproblem. Based on the experiments and comparison of different subproblems [18], we have selected subproblems extracted from the STOM, the GUESS and the reference point methods. They all involve reference point information that can be derived from the classification if we know the ranges of the Pareto optimal set. In this case, we set the components of a reference point $\bar{\mathbf{z}}$ as $\bar{z}_i = z_i^{\star}$ for $i \in I^{<}$, $\bar{z}_i = \hat{z}_i$ for $i \in I^{\leq}$, $\bar{z}_i = f_i(\mathbf{x}^c)$ for $i \in I^{=}$, $\bar{z}_i = \varepsilon_i$ for $i \in I^{\geq}$ and $\bar{z}_i = z_i^{\mathrm{nad}}$ for $i \in I^{\diamond}$.

The subproblem coming from the satisficing trade-off method (STOM) [24] has the form:

$$\begin{array}{ll} \text{minimize} & \max\limits_{i=1,\dots,k} \left[\dfrac{f_i(\mathbf{x}) - z_i^{\star\star}}{\bar{z}_i - z_i^{\star\star}} \right] + \rho \sum\limits_{i=1}^{k} \dfrac{f_i(\mathbf{x})}{\bar{z}_i - z_i^{\star\star}} \\ \text{subject to} & \mathbf{x} \in S \end{array} \tag{2}$$

where the aspiration levels $\bar{z}_i$ must be strictly higher than the corresponding components of the utopian objective vector $z_i^{\star\star}$.

Among the different achievement (scalarizing) functions that can be used in the reference point method [27], we use a basic formulation in the subproblem:

$$\begin{array}{ll} \text{minimize} & \max\limits_{i=1,\dots,k} \left[\dfrac{f_i(\mathbf{x}) - \bar{z}_i}{z_i^{\text{nad}} - z_i^{\star\star}} \right] + \rho \sum\limits_{i=1}^{k} \dfrac{f_i(\mathbf{x})}{z_i^{\text{nad}} - z_i^{\star\star}} \\ \text{subject to} & \mathbf{x} \in S \end{array} \tag{3}$$

Finally, the last subproblem originates from the GUESS method [1]:

$$\begin{array}{ll} \text{minimize} & \max\limits_{i \notin I^{\diamond}} \left[\dfrac{f_i(\mathbf{x}) - z_i^{\text{nad}}}{z_i^{\text{nad}} - \bar{z}_i} \right] + \rho \sum\limits_{i=1}^{k} \dfrac{f_i(\mathbf{x})}{z_i^{\text{nad}} - \bar{z}_i} \\ \text{subject to} & \mathbf{x} \in S \end{array} \tag{4}$$

The solutions of subproblems (1)-(4) are Pareto optimal [19]. In order to guarantee Pareto optimality, we include in subproblem (4) an augmentation term that was not used in the original formulation. We also need to make some minor changes in the scalarizing function because our reference point originates from classification and we must avoid dividing by zero. Thus, for $i \in I^{\diamond}$, we replace the denominator in the sum term by $z_i^{\text{nad}} - z_i^{\star\star}$. Notice that the aspiration levels $\bar{z}_i$ have to be strictly lower than the components of the nadir objective vector z_i^{nad}. Even though this subproblem relies significantly on the nadir objective vector, which has to be estimated, its success does not heavily depend on the correctness of the estimate (see [18]).

In NIMBUS, the decision maker can also explore a desired number of intermediate solutions between any two solutions. This means that steps of equal length are taken in the decision space between the two selected solutions and the corresponding objective vectors are used as reference points in subproblem (3). In this way, new Pareto optimal solutions are generated. Note that the solutions generated using different subproblems or as intermediate solutions are not all necessarily different [18]. In this case, we only show the different ones to the decision maker.

Next we can formulate the synchronous NIMBUS algorithm. Due to several subproblems and intermediate solutions, the amount of (Pareto optimal) solutions generated increases and, thus, a flexible data management is important. For this reason, we provide a possibility for the decision maker to save solutions in a database. We denote the set of saved solutions by A. At first, we set $A = \emptyset$. The starting point of the solution process can come from the decision maker or it can be some neutral compromise [28] between the objectives. To get the neutral compromise solution we set $\bar{z}_i = (z_i^{\mathrm{nad}} + z_i^{\star})/2$ for all $i = 1, \ldots, k$ and solve subproblem (3).

The steps of the NIMBUS algorithm are the following:

1. Generate a Pareto optimal starting point.

2. Ask the decision maker to classify the objective functions at the current solution and to specify aspiration levels and upper bounds (if needed in the classification specified).

3. Ask the decision maker to select the maximum number of different solutions to be generated (between one and four) and solve as many subproblems (among (1)-(4)).

4. Present the different new solutions obtained to the decision maker.

5. If the decision maker wants to save one or more of the new solutions to A, include it/them to A.

6. If the decision maker does not want to see intermediate solutions between any two solutions, go to step 8. Otherwise, ask the decision maker to select the two solutions from among the new solutions or the solutions in A. Ask the number of the intermediate solutions from the decision maker.

7. Generate the desired number of intermediate solutions and project them to the Pareto optimal set. Go to step 4.

8. Ask the decision maker to choose the most preferred one among the new and/or the intermediate solutions or the solutions in A. Denote it as the current solution. If the decision maker wants to continue, go to step 2. Otherwise, stop.

The algorithm is terminated if the decision maker does not want to decrease any objective value or is not willing to let any objective value increase. Otherwise, the search continues iteratively by moving around the Pareto optimal set.

Versatility is the most important characteristic of the NIMBUS algorithm. The decision maker has a variety of strategies available for directing the interactive solution process. Flexibility in the algorithm means that the decision maker is free to change one's mind while (s)he learns more about the problem.

The synchronous NIMBUS method has been used for solving complex problems related to the designing of paper machines [11] and planning paper making processes [6]. The experts who have acted as decision makers have obtained a better picture of the possibilities of the problem without having to input too much preference information.

2. IND-NIMBUS SYSTEM

The IND-NIMBUS system is capable of solving nonlinear multiobjective optimization problems involving even nondifferentiable and nonconvex functions where the variables can be continuous or integer-valued. It can be used in both MS-Windows and Linux operating systems and it has been designed so that it can be connected to different modelling and simulation tools. In this way, it can be used when solving complicated real-world problems that do not have explicit function formulations but function values come, for example, from the solution of a system of partial differential equations.

The basic setting in IND-NIMBUS is the same as in WWW-NIMBUS but the user interface is completely different. The IND-NIMBUS user interface has been built with the wxPython toolset and there exists a high-level NIMBUS GUI framework where different parts of the user interface have been isolated as independent components [25].

After the optimization problem has been specified, the system generates a Pareto optimal neutral compromise solution as a starting point or the solution given by the decision maker is projected onto the Pareto optimal set. This point is the basis of the first classification. This solution is shown to the decision maker as a bar chart where each bar stands for one of the objective functions. The classification of the objective functions can be carried out by indicating desirable values in a bar chart with a mouse. The bars include information about the current objective values with colours as well as the estimated ranges of each objective function in the Pareto optimal set as the end points of the bars. The less colour one can see in the bar, the closer the current value is to the ideal one and, thus, the better it is.

An example of the classification phase is given in Figure 1 with a problem involving five objective functions, where all the others are to be minimized but the third one is to be maximized. The decision maker can classify the functions either by clicking with the mouse or by specifying desirable values for the objective functions in the boxes next to the bars. If the decision maker wants the function to get as good a value as possible, it is possible to click the big circle next to the ideal value. On the other hand, the function can change freely if it is not classified at all or if the circle next to the nadir value is clicked.

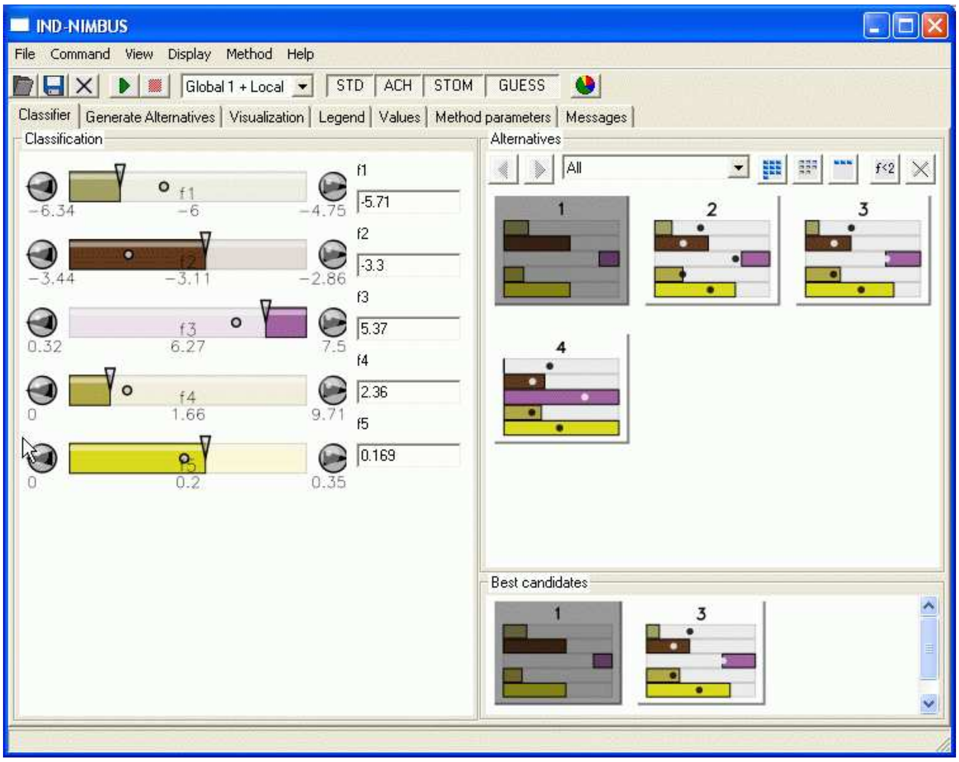

Fig. 1. Classification window

After the classification, the user selects the maximum number (between one and four) of different new solutions to be generated. The system produces them by solving different (single objective) subproblems using some of the underlying optimizers. The user can select the optimizer for each iteration individually. If the user wishes to use a computationally efficient local solver, it is possible to use the proximal bundle method [13]. This method can solve even nondifferentiable problems but it assumes the objective and the constraint functions to be locally Lipschitz continuous and it needs (sub)gradient information.

If the user prefers global optimization, (s)he can select between two variants of an evolutionary algorithm. In this case, the problem to be solved may contain also integer-valued variables. The two variants use different constraint-handling techniques. One of them is based on adaptive penalties [4] and the other is a method of parameter free penalties [3]. For further details, see [23]. All the optimizers contain technical parameters and the user can change the default values, if necessary.

If the decision maker has decided to use a global solver and (sub)gradient information is available, then the system uses a hybrid solver. This means, that the local solver is started from the final solution of the global solver. Evolutionary algorithms cannot guarantee optimality, but the solution of this hybrid solver is at least locally optimal.

The system shows all the solutions that have been generated during the solution process. An example can be seen in the top right part of Figure 1. Any of them can be hidden if it is not interesting. Furthermore, the decision maker can save any particularly promising or interesting solutions in the set of 'best candidates'. This set can be seen in the bottom right part of Figure 1. In this way, (s)he can comfortably return to previous solutions if they turn out to be interesting. The decision maker can select any of the solutions generated as a starting point of a new classification.

As mentioned in the NIMBUS algorithm, it is also possible to generate a desired number of solutions between any existing solutions. An example of this is given in Figure 2. Here, the decision maker has asked for three new solutions to be generated and they can be seen in the bottom left part of the figure (labelled as output) between the two selected end point solutions (labelled as input in the figure).

The comparison task between any set of solutions is facilitated by using visualizations of the alternatives. The decision maker can compare visually all the solutions generated, the solutions of the previous iteration, the best candidate solutions or a set of any selected solutions. The decision maker can select between bar charts, 3-dimensional bars, value paths, spider-web charts, multiway dot plots, whisker plots and petal diagrams in both absolute and relative scales. It is also possible to filter out some of the solutions by specifying upper or lower bounds or intervals for objective functions. A selection of different visualizations is given in Figure 3.

IND-NIMBUS has been used in solving problems related to process simulation in pulp and paper processes [5; 6] where it has been connected to the BALAS process simulator (http://www.vtt.fi/pro/balas). It has also been used in optimizing paper making in solving problems related to controlling the whole paper machine (as an entireness consisting of many different parts) [12]. In addition, IND-NIMBUS has been connected with the Numerrin software (http://www.numerola.fi/englanti/numerrin02.html) for model based simu-

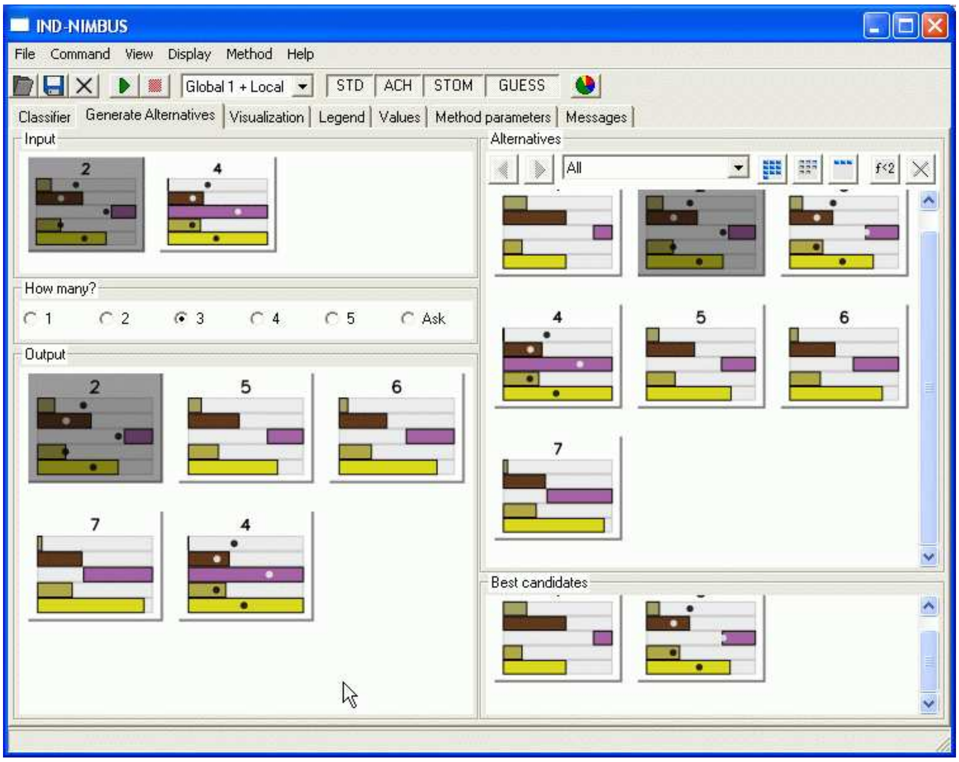

Fig. 2. Intermediate solutions

lation and with this combined tool a design problem related to ultrasonic transducers has been solved [8].

The decision makers involved in the solution processes have found IND-NIMBUS very useful and easy to use. In their opinion, the information exchanged has been understandable and the user interface convenient to use. They have expressed their appreciation to the versatile possibilities of expressing preference information while solving the problem and they have found very satisfactory solutions.

CONCLUSIONS

We have described the main features of a new software system for solving demanding nonlinear multiobjective optimization problems, IND-NIMBUS. It can be connected with different modelling and simulation tools so that problems

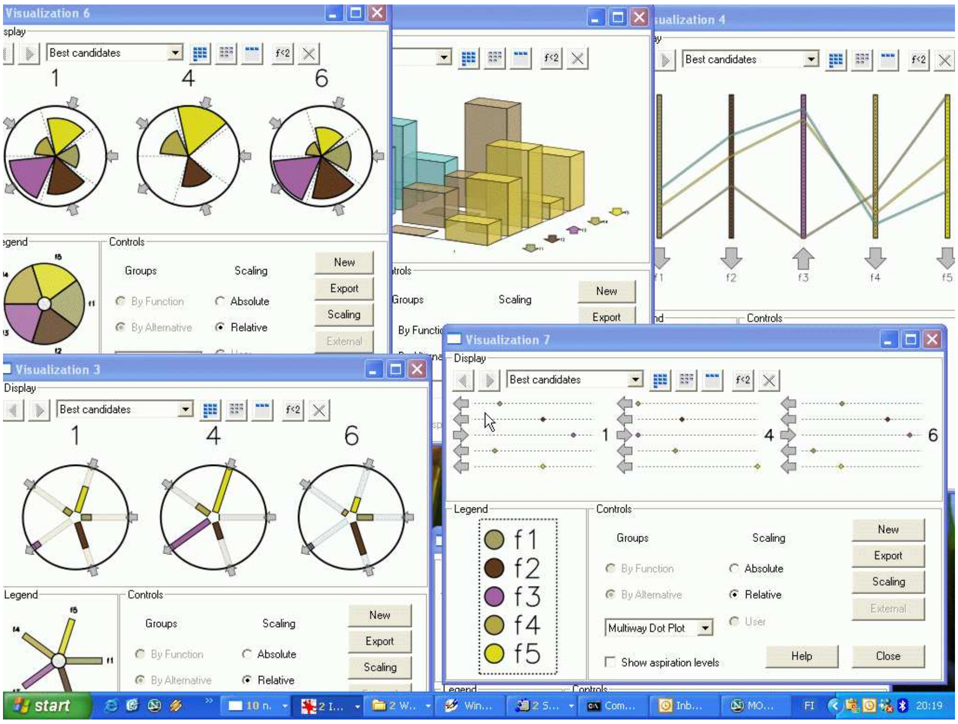

Fig. 3. Different visualizations for three solutions

where function evaluations necessitate, for example, the solution of a system of partial differential equations can be conveniently handled. We have also described the underlying interactive NIMBUS method briefly.

The further development of IND-NIMBUS continues. For example, special emphasis has to be given to the computational efficiency so that solving complex problems will not take too much time. Otherwise, the interactive character of the solution process suffers.

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Sigitas Mitkus

GRAPHICAL RISK ALLOCATION MODEL IN CONSTRUCTION CONTRACTS FOR CHANGES IN MARKET PRICES

INTRODUCTION

One of the main purposes of construction contracts is to allocate risk and liability clearly, comprehensibly and unambiguously. Otherwise, arguments, including legal ones, between contract parties are inevitable, which always leads to extra expenses incurred by both parties [7].

The aim of this article is to analyse the allocation of risk between the participants of the construction process (the contractor and the client) when implementing construction projects, to analyse any constraints when allocating risks provided for in the Civil Code of the Republic of Lithuania (hereinafter referred to as the CC) and in other effective legal regulatory acts of the Republic of Lithuania, to determine to what extent the risk allocation depends on the terms and conditions of the construction contract and the behaviour of the parties to the construction contract when implementing the construction project.

Adequate and reasonable allocation of risk and liability in construction contracts makes the construction process more efficient, and a reasonable price is set in calls for construction contract bids. For example, C. Charoenngam and C.-Y. Yeh have found [2] that in East Asian countries if the client sets contract conditions in public procurement procedures for construction works contracts so that the greater part of the risk and liability is transferred to the contractor, the contract value increases dramatically. Significant price increases, due to inadequate risk and liability allocation between the client and the contractor, were also noticed in the USA and Canada [10]. The allocation of risk and liability between the contractor and the client has direct impact on the costs of construction projects [4]. In case of transferring the risks of performing construction works from the client to the contractor, two scenarios undesirable for the client are possible [9]:

1) seeking to insure oneself from risk consequences, the contractor increases the bid price,
2) the contractor does not increase the bid price and has financial problems in case of risk materialisation.

Neither scenario described above is desirable for the contractor. In the former case they will pay a higher price, and in the latter, they would have to implement the construction project cooperating with the partner experiencing financial problems, which also threatens the success of the project.

The risk is especially high when the object of the construction contract consists of complex buildings. If the risk is materialised during the construction process, it has great impact on the costs, duration and quality of the construction project. On the other hand, it is impossible to avoid all risks when executing construction contracts. Most often risk can only be distributed between the parties and managed [6]. Risk costs are always assumed by one or both contract parties.

When implementing construction projects, it is not unusual that disagreements and disputes concerning liability arise. It is possible to eliminate such ambiguities and disputes by setting the terms and conditions of liability allocation between the parties in a construction contract as clearly as possible. S.C. Ward, C.B. Chapman and B. Curtis [9] consider that for the proper distribution of risks between parties to construction contracts, the following two conditions are crucial:
1) trust between the contracting parties,
2) clear mutual understanding of all possible risks and their possible impact.

General terms and conditions of a construction contract with regard to the risk and liability allocation is the most important part of the construction contract documentation. For example, contract conditions providing for the contractor's liability for risks that are totally beyond their control or vice versa constitute a potential source of claims and arguments in many construction projects [4].

The scholarly works referred to above focus on the agreement between the parties to the construction contract (the contractor and the client) as regards the risk and liability allocation. Those works do not analyse the impact of legal norms regulating construction contracts on the risk and liability allocation.

The costs of construction works are influenced by various factors. Some factors depend exclusively on the contractor implementing the construction project while others are closely related to the sociocultural, economic, technical and political environment of the project location (global risk factors) [1].

The new Civil Code of the Republic of Lithuania, effective 1 July 2001, contains much stricter regulations concerning the issues of risk allocation and liability. The CC norms normally regulate the above issues by applying imperative legal norms. For that reason the afore-mentioned risk and liability issues are to a great extent conditioned by the CC legal norms and the contracting parties cannot change those provisions by mutual agreement. This is why this article focuses mainly on the legal aspects of the risk and liability allocation.

The aim of this article is to analyse how risk is allocated between parties to a construction contract in the case of a fixed price contract depending on the behaviour of the contract parties and taking into account the CC provisions.

It is universally assumed that fixed price contracts are most appropriate when the risk of price increase is low [3]. Although high risk usually exists when implementing construction projects (executing construction contracts), fixed value contracts are still quite common.

The fact of drawing a fixed price construction contract cannot be interpreted in such a manner that the contract value may be changed by any means. When executing construction contracts (including fixed value contracts), there may be situations when the value of the construction object increases significantly independently of the contractor's will, who did not and could not foresee such a price increase. Such cases include force majeure circumstances, modifications to the construction design project, major changes on the markets of construction products and labour force as well as the appearance of unforeseen additional works.

1. ALLOCATION OF RISK FOR MARKET PRICE CHANGES

The value of implementing a construction project (a construction contract) may change for the reasons beyond the control of the contractor. In most cases such changes are influenced by changes of market prices (of construction materials, labour force, mechanisms).

The risk allocation between the parties to the construction contract in such case is regulated by the construction contract and the CC. The above risk allocation also depends on the behaviour of the parties to the contract.

The CC analysis shows that when the market prices change or when for some other reasons not depending on the contractor the factual value of the construction works under the construction contract increases, two alternatives are possible (par. 2 of Art. 6.685 of the CC):

1) the factual value of the construction works increases by less than 15% (Figure 1, 1-2),
2) the factual value of the construction works increases by over 15% (Figure 1, 1-3).

In the former case the risk of changes in market prices is borne by the contractor unless otherwise provided for in the construction contract. In such case one must follow Articles 6.189 and 6.200 of the CC providing that the contract is binding for both parties and they must execute it even if they receive less profit than expected or even if they incur losses when executing the contract. In this respect the CC reiterates the provisions of the international principles of drawing commercial contracts (UNIDTROIT) and stipulates the principle of *pacta sunt servanda* demanding the adherence to and execution of contracts despite any possible difficulties.

The lawmaker considers that due to the increase of the factual construction work value by over 15% (the latter case) the execution of the contract becomes much more complicated for the contractor and in such case the requirement to follow the principle of *pacta sunt servanda* and to unconditionally fulfil the contract would breach the principles of reason, honesty and justice stipulated in the CC as well as the balance of the interests of the contracting parties. The above circumstance essentially changes the balance of contractual obligations (Art. 6.204 of the CC) and in such case another important principle of the law must be followed, which is *rebus sic stantibus*, implying that the contract must be fulfilled taking into consideration any changed circumstances. In such case the construction contractor has the right to demand a recalculation of the value of the construction contract. The contractor may exercise this right only while the contract is not yet fulfilled and only given the following conditions (Figure 1, 3-5):

1. Such circumstances (market price increase) arise or the contractor becomes aware of them after the contract is concluded. If such circumstances already existed, i.e. market prices were already higher, and the contractor was (or should have been) aware of that, the contractor must be deemed to have assumed all the related risk and not to be able to demand to adjust the contract value.
2. The contractor could not possibly foresee such circumstances. This means that the contractor would not be able to refer to such circumstances if the market price increase (e.g. due to inflation, tax reforms etc.) was officially forecasted.

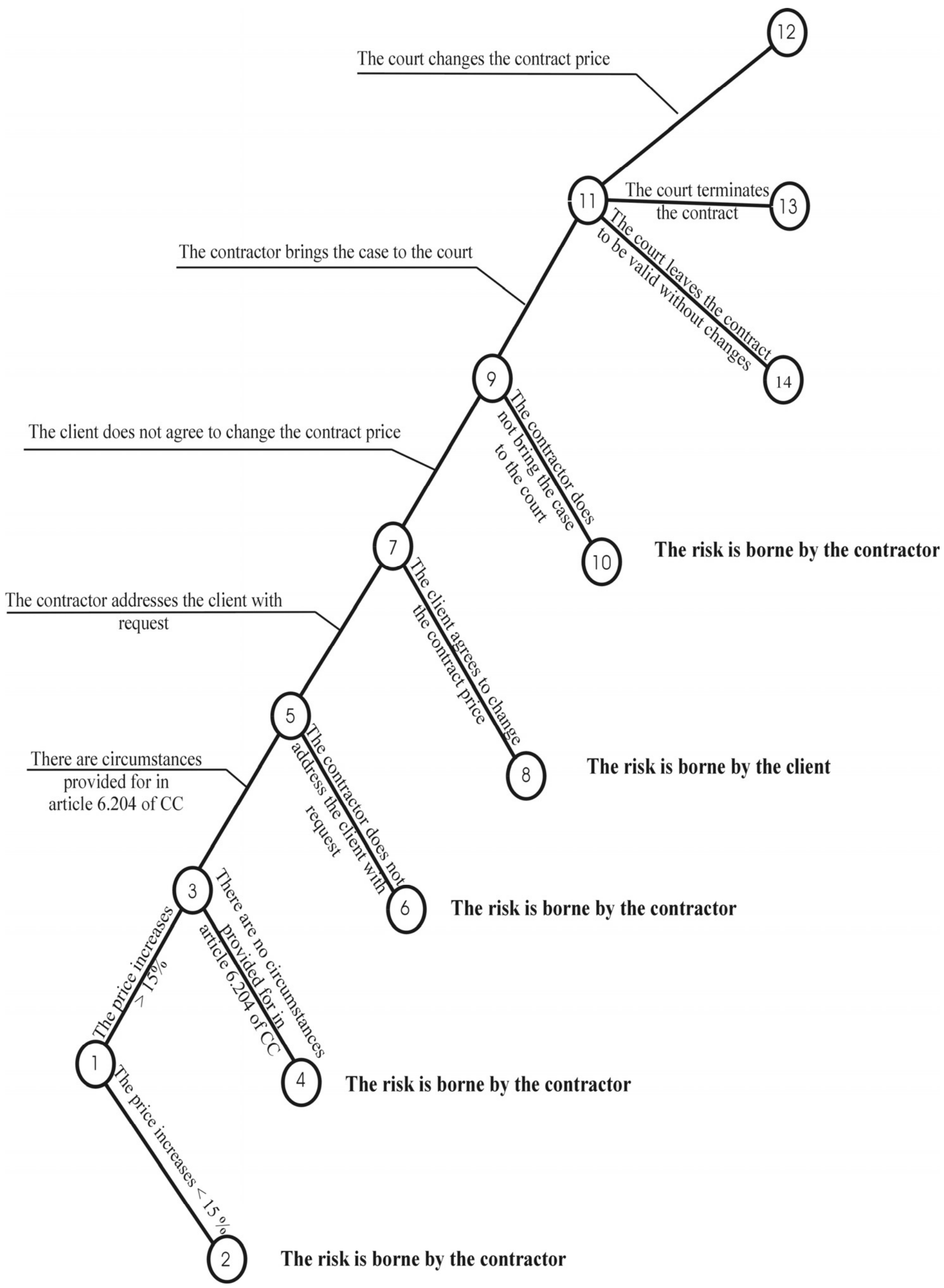

Fig. 1. A tree of variants of risk allocation for changes in the market prices

3. The contractor cannot control those circumstances, i.e. the circumstances are objective and independent of the contractor's will. Normally contractors cannot control market prices but if they can control any price component (e. g., when they manufacture certain products or provide a service), they cannot refer to the change of the market price of such a component and demand to recalculate the contract value.
4. The contractor did not assume the risk of the price increase. The contractor may assume the risk by explicitly stating so in the contract, or such risk assumption may be implied.

The CC does not prohibit providing also for other risk allocation conditions in a construction contract, for example, to include a provision of another percent of the price increase given which the contractor acquires the right to demand to recalculate the contract value. The contract may also contain provisions relating to pricing principles to apply in case of any changes in market prices of components of the construction work par value.

Given all the above circumstances, the contractor acquires the right to address the customer with the request to recalculate the contract value (Figure 1, 5-7). This means that even in case of the aforementioned circumstances, the contractor has no right to suspend the contract execution. In practice the CC provisions oblige the contract parties to initiate negotiations that from the procedural viewpoint would be treated as the mandatory pre-trial procedure of settling arguments. The contractor's request to adjust the contract value must be motivated and presented immediately. If the contractor does not file such request or does so when it is too late (Figure 1, 5-6), they bear all the costs of the increased value of the construction works.

If the contractor addresses the customer with the request to recalculate the contract value, two situations are possible:

1) the contractor and the customer agree on the contract value (Figure 1, 7-8),
2) the contractor and the customer do not agree on the contract value (Figure 1, 7-9).

If the contractor and the customer agree on the contract value, the contract is continued with the recalculated (adjusted) contract value valid. If the contractor and the customer fail to agree on the contract value, two situations are possible:

1) the contractor or the constructor brings the case to court (Figure 1, 9-11),
2) neither the contractor nor the customer brings the case to court (Figure 1, 9-10).

If the contract parties do not bring the case to court, the contractor must execute the construction contract with the contract value unchanged. Such conclusion can be drawn because there is no legal ground to terminate the contract, and the contractor may not unilaterally change the contract value.

If the parties go to court, they can appeal to the court with one of the following two claims: to amend the contract (the contract value) or to terminate the contract.

The court may settle the argument between the parties in one of the following three ways:

1) to terminate the contract and to set the date and terms and conditions of the contract termination (Figure 1, 11-12),
2) to adjust the contract value (Figure 1, 11-13),
3) to leave the contract unchanged (Figure 1, 11-14).

2. SPECIFIC FEATURES OF CHANGES IN MARKET PRICE RISK ALLOCATION IN PUBLIC PROCUREMENT AGREEMENTS ON CONSTRUCTION WORKS

Although the Civil Code is the major source regulating terms of the construction contracts, Article 6.382 of the present act of law provides that "provisions of the Code shall be applied to the public procurement agreements to the extent that other acts of law do not provide otherwise". Definition of the public procurement agreement provided in Article 6.380 of the Civil Code shows that procurement of construction works (drawing of construction contracts) is also attributed to the public procurement agreements.

The main specific feature of construction works procurement is use of public funds (government, municipality, state funds etc.) or any other funds equivalent to such funds for said construction works procurement. The use of other funds instead of private funds is determined by the fact that the procurement procedure is regulated in detail in the Law on Public Procurement and other subordinate legislation.

Taking into consideration the fact that the owner of funds disposed in the process of construction works procurement is not considered a subject of the construction works procurement, the state, by establishing the legal regulatory measures, limits the right of the said subject to make discounts for the contractor and regulates its conduct in detail.

Accordingly when implementing the construction contracts there exist probabilities of change in the integral elements of the market price (rate) of construction works or constructions costs (construction products, labour force, mechanisms etc.). Therefore the Law on Public Procurement regulates conduct of parties in such situations.

The Law on Public Procurement (Par. 3, Art. 15) imposes rather rigid requirements on the terms when drawing up an agreement: "When drawing a procurement agreement the winner's bid and procurement terms established in the procurement documents or bid shall not be subject to any amendments". Interpretation of the above-mentioned provision of the Law should be expanded: the bid and the agreement terms may be amended, however, such amendments should be beneficial to the procuring organisation. This interpretation is based on the fact that this provision is established in favour of the procuring organisation and it has a right to give its consent to amendment of the said terms of the agreement.

Paragraph 6 of Article 15 of the Law on Public Procurement establishes mandatory terms of the procurement agreement. Certain terms of the above--mentioned Paragraph of the said Article are related to pricing. Item 3 of Paragraph 6 of Article 15 provides that price or pricing regulations must be specified in the agreement. It means that the law does not require a unilateral specifying of the agreed price, but permits the price establishment by using the pricing regulations. Use of the pricing regulations (instead of price) is rather convenient in such cases when, for instance, the precise scope of construction works is not known.

Item 4 of Paragraph 5 of Article 15 of the Law on Public Procurement provides that the agreement shall include a provision on "inflation-related indexation of price or charges if the term of the agreement exceeds one year". The above-mentioned provision of the Law on Public Procurement allows us to make the following conclusions:

1. Agreement on the inflation-related indexation of price and charges is an obligation of the parties to the agreement. Listing of said provisions in the procurement document would be the most rational solution for the procuring organisation. In that case the above-mentioned provisions shall be also applicable to the construction contract. And conversely, if the parties fail to reach consent on the present terms, in a sense they may "be deadlocked", as the Law does not provide any other method for resolving such a problem. The obligation to agree on the inflation-related indexation of price and charges does not mean that indexation of price

is mandatory, i.e., a zero indexation coefficient may be applied. Such cases may occur when the procuring organisation includes a provision in the tender documents that a contractor will accept any and all inflation--related risk, and in accordance with the Law on Public Procurement such provision will be also applicable to the construction contract.

2. The provision will be applicable in such cases only if the term of the agreement exceeds one year. In fact said term (a one-year period) is applicable to the term specified in the construction contract instead of the entire term of implementation of the agreement. This conclusion can be drawn considering that a legitimately signed construction contract is binding upon the parties to the contract. Therefore the contractor has to perform the construction works within the period specified in the contract, and the procuring organisation is not held liable for compensation of any losses it incurs as a result of faulty implementation of the contract. The following is an exception to this rule: cases when the term of the contract was extended as a result of circumstances beyond the control of the contractor and the contractor is not responsible (force majeure circumstances, suspension of the construction works resulting from the customer's fault etc.), and if the contractor did not accept the risk of occurrence of such circumstances in the contract.

Item 5 of Paragraph 6 of Article 15 of the Law on Public Procurement establishes an obligation to include in the procurement agreement a provision on change in price and charges resulting from changes in taxes. Naturally, taxes are levied by the state and parties to the contract may not impact imposing of taxes. Following the law principles of prudence, justice and fairness, subject to changes in taxes that do not have any direct impact on cost price of construction works, contract price should correspondingly change. Value Added Tax (and also other taxes depending on turnover) as well as employees' social insurance instalments made by the Company should be attributed to the above taxes. The contract price should not be adjusted subject to change in the profit tax rate, individual income tax rate, etc. Changes in the above-mentioned tax rates do not have any direct impact on costs of construction works.

On 25 February 2003, the Director of the Public Procurement Agency of the Republic of Lithuania by his decree approved the Pricing Methodology of the Public Procurement Agreements (hereinafter referred to as the Methodology) where the above-mentioned provisions of the Law on Public Procurement are regulated in detail.

The Methodology provides that the following methods of price calculation may be established:

1) fixed price establishment,
2) fixed charge establishment,
3) charge base establishment,
4) partial coverage of the contract implementation costs,
5) coverage of contract implementation costs.

According to the price calculation methodology selected, risk resulting from changes in market price (rate) is differently allocated between the parties to the contract.

If a fixed price is specified in the contract, the contractor undertakes to pay the prices for all types of works performed in accordance with the contract. In this case risk of change in the market price (rate) is allocated to the contractor. Therefore the Methodology establishes limitations, i.e., it specifies when this price calculation method may be applied for a contract. The present method may be selected if two requirements listed below are met:

1) the customer can specify precisely the scope of work in the procurement documents,
2) when submitting the bid, the contractor has adequate potential to estimate and assess costs of the contract implementation and may accept the risk of such costs.

Analysis of the present terms enables us to conclude that a customer who decided to select said method of price establishment has to assess its potential both to precisely determine and estimate the scope of the requested work, and to objectively assess costs of the contract implementation by the contractor.

If a fixed charge is established in the Pricing Methodology, the final price to be paid by the customer to the contractor will depend on the scope of work carried out in the process of implementation of the contract. Therefore in this case the volume of risk allocated to the contractor is lower than in the case of the fixed price. In this case the contractor does not accept the risk resulting from increase in scope of construction works. It accepts only the market price risk (construction products, labour force etc.).

The fixed fee may be established prior to commencement of procurement if:

1) the customer is unable to identify the exact scope of work,
2) when preparing the bid, the contractor has adequate potential to estimate and determine the contract implementation costs per measurement unit of the procurement object and may accept the risk of the contract implementation costs per measurement unit of the procurement object.

The Methodology recommends establishment of a fixed charge in case of drawing up a long-term contract on the basis of which the scope of work to be carried out depends on circumstances that are difficult to forecast: a) at the moment of procurement and b) on the interim results of the contract fulfilment.

Price calculation methods specified in the Methodology such as the charge base establishment or the partial coverage of the contract implementation costs when carrying out the construction works may not be applied, therefore they will not be analysed in this article.

If the coverage of contract implementation costs is specified in the Pricing Methodology, the price to be paid to the contractor will be calculated by aggregating the amount of costs directly connected with the implementation of the contract actually incurred by the contractor, with the amount of profit specified in the Pricing Methodology and proposed by the contractor.

In this case, the entire price fluctuation risk in the price market (and also the risk of fulfilment of the scope of works of the contract) will be accepted by the customer. Such method of price calculation may not be applied when during the process of bid preparation the contractor has no potential to estimate and assess the contract implementation costs in advance, and in the process of the contract implementation may not accept risk of the contract implementation costs. The Methodology recommends using such price establishment method when new or non-standard works are performed, for instance, works related to research activities or implementation of sophisticated innovative projects. If this price establishment method is selected, the entire risk of price and scope of works is allocated to the customer, i.e., state or municipality institution and funds of the state, municipality budget and other funds equivalent to them are exposed to risk. The Methodology establishes an imperative clause that the present price establishment method may be used only in the exceptional cases where it is not possible to apply other methods of price calculation. Price calculation methods are compared in Table 1 below.

Table 1

Price calculation methods and risk allocation in the process of public procurement

No	Price establishment method	Risk allocation principles	Cases of application
1.	Fixed price establishment	Entire risk allocated to the contractor	1) The scope of works to be done can be precisely estimated 2) The contractor can estimate costs and accept the risk
2.	Fixed charge establishment	1) Risk related to scope is allocated to customer 2) Market price (market rate) risk is allocated to contractor	1) The scope of works to be done cannot be precisely estimated 2) The contractor can estimate costs per measurement unit of the object
3.	Coverage of contract implementation costs	Entire risk is allocated to the customer	1) New or non-standard works are performed 2) Other methods of price establishment may not be applied

The contractor's risk (and in some cases the customer's risk) resulting from the market price fluctuation may be reduced by applying price adjustment specified in the Methodology. One of the price adjustment methods specifies that the contract price may be increased (or reduced) taking into consideration certain external macroeconomic factors (inflation etc.). Such price adjustment should be applied to long-term agreements and also in an unstable economic environment.

CONCLUSIONS

Adequate and reasonable allocation of risk and liability in construction contracts has great impact on the value, quality and efficiency of the construction project. The allocation of risks and liabilities between the parties to construction contracts depends on the terms and conditions of the contract, behaviour of the parties to the construction contract, the CC and legal norms stipulated in other sources of the construction law.

The allocation of risk and liability between parties to construction contracts can be analysed using trees of the risk allocation variants.

The article presents tree of variants of the risk allocation for changes in market prices.

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Maciej Nowak

AN INTERACTIVE PROCEDURE FOR PROJECT SELECTION

INTRODUCTION

Profitable investments lead to the growth and prosperity of an economic organization. Various objectives are usually taken into account when an investment project is analyzed. Economic desirability is undoubtedly of primarily importance. Various methods and techniques are used for evaluating investment projects [20; 21]. Net present value (NPV), internal rate of return (IRR), profitability index (PI), payback period (PP) and other measures are usually employed when financial analysis of a project is performed. In many cases, however, investor's considerations are not limited to economic desirability. Usually objectives reflecting technical, environmental, social, and/or political factors are also taken into account. As the decision maker tries to maximize or minimize outcomes associated with each objective depending on its nature, a multiple criteria decision making problem arises.

Criteria for project comparison are often of different nature. While financial criteria are quantitative, others are often of qualitative nature. If, for example, an engineering project is considered, various technical factors of qualitative kind, including the level of technological novelty, compatibility with existing facilities, reliability and technical service, are taken into account. A similar situation takes place when social and environmental consequences are examined. While some criteria are quantitative (the volume of pollutants, the area of degraded land etc.), others are qualitative (changes in landscape, changes in the way of life of the neighboring population etc.).

When faced with the decision of selecting an engineering, construction or R&D project, the decision maker has also to face uncertainty. Project evaluation involves prediction of future outcomes. In the real world, however, not all predictions are known with certainty. Even experts are sometimes wrong in their assessments. In addition, various experts often differ in their opinions on the same project. Thus, risk associated with at least some objectives has to be considered when projects are evaluated.

A number of procedures for evaluating engineering, construction, and R&D projects have been proposed in recent years. A wide survey of quantitative techniques for R&D project selection and resource allocation is given by Heidenberger and Stummer [7]. Most procedures listed in their paper can be applied for evaluation of construction and engineering projects as well. Analytical Hierarchy Process (AHP), proposed by Saaty [22], is one of the most widely employed techniques. The main idea of AHP is to exploit the results of the decision maker's subjective evaluations formulated for each pair of projects and for each criterion. Lootsma et al. [12] use a procedure similar to AHP for ranking non-nuclear energy research programs. Ferrari [4] presents a choice method, which accounts for the different nature of the two types of agents involved in the decision-making process: technicians and politicians. The method enables calculating the weights of the elements in each hierarchy level with respect to the elements in the next upper level through procedures different from those of traditional analytic hierarchy process, in that it accounts for the dependence of the weights of the viewpoints on the project attributes. Kearns [9] uses AHP approach for economic evaluation of information technology investments. In his work multi-objective technique is used to reflect both tangible and intangible benefits, link the investment to business strategies, increase management participation in the evaluation process, and provide important features of portfolio selection.

Techniques based on the utility function constitute another group of multi-criteria methods employed in project selection problems. This approach is based on the assumption that each decision maker attempts to maximize some utility function aggregating evaluation criteria. In this case the main problem is to estimate the utility function. Multi-attribute utility analysis is used, for example, by Moselhi and Deb [15], who treat uncertainty in a similar way to that used in the PERT technique. In this procedure the total expected utility is calculated by multiplying three matrices: utility matrix, objective matrix, and scaling matrix. Wong et al. [25] incorporate fuzzy analysis into multi-attribute utility theory. Their procedure uses stochastic dominance rules for ordering projects.

Numerous techniques based on the outranking relation have been also proposed. Martel and D'Avignon [13] consider a case study where each project is evaluated by experts according to a set of criteria. These evaluations lead to distributive evaluation, i.e. to the calculation of the distribution of the anticipated performance of each project with respect to each attribute. The problem is solved by establishing a confidence index, which is based on probabilities that every project is as good as another. Multiple-criteria decision-aiding procedures ELECTRE and PROMETHEE are used in project evaluation problems by Pin-Yu et al. [19], Costa et al. [2], Mavrotas et al. [14], Al-Rashdan et al. [1], and Goumas et al. [5].

Goal programming approach is also successively employed in project selection. This technique attempts to find a solution that is as close as possible to the goals specified by the decision maker. The goal programming concept is used, for example, by Santhanam and Kyprasis [23], Lee and Kim [11], de Oliveira et al. [3].

The solution of a multiple criteria decision making problem is possible if the decision maker is able to provide information about his or her preferences with respect to the set of objectives under consideration. Procedures listed above assume that preference information is collected prior to calculating the final solution. The analysis is therefore based on an a priori basis. In many situations, however, the decision maker is unable or unwilling to provide all required information at the same time. A methodology known as interactive approach is very useful in such cases. This technique assumes that the decision maker is able to provide preference information with respect to a given solution or a given set of solutions (local preference information). Two main advantages are usually mentioned for employing interactive techniques. First, such methods need much less information on the decision maker's preferences. Second, since the decision maker is closely involved in all phases of the problem solving process, he or she puts much reliance in the generated solution, and as a result, the final solution has a better chance of being implemented. Numerous interactive techniques have been proposed in recent years. Most of them are applicable in circumstances of certainty, although methods devised for the case of risk are also proposed. The INSDECM technique presented in [16], combines interactive approach and risk analysis based on stochastic dominance and mean-risk analysis.

The aim of this paper is to propose a comprehensive methodology for project selection problems that enables handling both quantitative and qualitative criteria and takes into account risk associated with each objective. While simulation technique is used for generating distributional evaluations with respect to quantitative measures, expert assessments are taken into account when projects are appraised in relation to qualitative criteria. The interactive procedure INSDECM is employed for generating the final solution. In previous work this technique was applied for cardinal data only. This study presents the way in which INSDECM can be utilized when both cardinal and ordinal variables are considered. The methodology, which essentially combines stochastic dominance technique and interactive approach, is described first, followed by an illustrative example.

1. METHODOLOGY

The decision situation considered in this paper may be conceived as a problem (A, X, E) where A is a finite set of alternative projects a_i, $i = 1, 2, ..., m$; X is a finite set of criteria X_k, $k = 1, 2, ..., n$; and E is a set of evaluations of projects with respect to criteria:

$$E = \begin{bmatrix} X_{11} & \cdots & X_{1k} & \cdots & X_{1n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ X_{i1} & \cdots & X_{ik} & \cdots & X_{in} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ X_{m1} & \cdots & X_{mk} & \cdots & X_{mn} \end{bmatrix}$$

In this paper the author assumes that two techniques are used for generating project evaluations: simulation and expert assessments. The former is employed when projects are evaluated with respect to financial measures, while the latter is used for obtaining evaluations related to qualitative criteria. In both cases the results can be transformed into probability distributions.

The procedure consists of four major steps. First, the set of criteria is defined. Next, evaluations of projects with respect to the criteria are generated. In the third step project evaluations are compared with respect to the criteria. Finally, interactive technique is employed for selection of the most desirable project. The steps required to perform the analysis are described below.

Step 1. Identification of criteria

The selection of the criteria is of crucial importance. According to Keeney and Raiffa [10] the set of criteria should be complete, operational, decomposable, non-redundant, and minimal. Completeness means that all important aspects of the problem are covered. The set of criteria is operational if it can be meaningfully used in the analysis. When the set of criteria is decomposable, the evaluation process can be simplified by breaking it down into parts. Non-redundancy means that duplicate counting of impacts is avoided. Finally, when the set of criteria is minimal, the solution of the problem is easier as its dimension is as small as possible. It is quite clear that criterion with respect to which all project outcomes are the same can be discarded, as it does not influence the decision maker's choice.

Step 2. Generation of project evaluations

Economic desirability analysis of a project involves prediction of future outcomes. In the real world, however, predictions are not known with certainty. Thus, risk associated with each project has to be taken into account. Simulation technique is an efficient and flexible tool for doing this. Various risk factors can be taken into account in a simulation model. For example, when a construction or manufacturing project is analyzed, uncertainties related to availability of resources, market prices, or demand can be considered. On the other hand, in projects with R&D elements activity durations are much more sensitive to incorrect evaluation. In such cases simulation may provide the dates of the milestones of the project, which determine the set of cash-flows during the life cycle of the project. One of the most important elements of simulation modeling is identifying appropriate probability distributions for input data. Usually, this requires analyzing empirical or historical data and fitting these data to distributions. Sometimes, however, such data are not available and an appropriate distribution has to be selected according to the decision maker's judgment. Once the simulation model is built, verified, and validated, it can be used for generating probability distributions of output variables.

If experts are asked to assess projects with respect to qualitative criteria, distributional evaluations can be constructed in a similar way. Let's assume that each project a_i is evaluated by l experts with respect to criterion X_k on a specified scale. Such scale can be defined, for example, as a 10-point one, with 1 assigned to the least desirable and 10 to the most desirable output. As a result, l evaluations are obtained for each project. Assuming equal probabilities of each assessment, a distribution evaluation is achieved. Such distribution, however, differs from the one obtained in simulation, as qualitative criteria are measured on ordinal scale. As a result, the rules for comparing such distributions are different from the ones used for real-value outcomes, i.e. outcomes measured on cardinal scale, such as return, net profit, or volume of pollutions.

Step 3. Comparing projects with respect to criteria

Once evaluations of the projects are obtained, relations between projects with respect to criteria can be analyzed. Two methods are usually used for comparing uncertain outcomes: mean-risk analysis and stochastic dominance. The former is based on two criteria: one measuring expected outcome and another representing variability of outcomes. In stochastic dominance approach random variables are compared by pointwise comparison of their distribution functions. In this paper both techniques are used. While stochastic dominance is employed for constructing rankings of projects with respect to each criterion, mean-risk technique is used when a final solution is chosen.

Let's assume that criteria are defined so that larger values are preferred to smaller ones. Let $F_{ik}(x)$ and $F_{jk}(x)$ be right-continuous cumulative distribution functions representing evaluations of a_i and a_j respectively over criterion X_k:

$$F_{ik}(x) = \Pr\left(X_{ik} \le x\right)$$
$$F_{jk}(x) = \Pr\left(X_{jk} \le x\right)$$

Definitions of the first and second degree stochastic dominance relations are as follows:

Definition 1 (FSD – *First Degree Stochastic Dominance*)

X_{ik} dominates X_{jk} by FSD rule ($X_{ik} \succ_{\mathrm{FSD}} X_{jk}$) if and only if :

$$F_{ik}(x) \neq F_{jk}(x) \text{ and } H_1(x) = F_{ik}(x) - F_{jk}(x) \leq 0 \text{ for } x \in R$$

Definition 2 (SSD – *Second Degree Stochastic Dominance*)

F_{ik} dominates F_{jk} by SSD rule (F_{ik} SSD F_{jk}) if and only if :

$$F_{ik}(x) \neq F_{jk}(x) \text{ and } H_2(x) = \int_{-\infty}^{x} H_1(y)dy \leq 0 \text{ for } x \in R$$

Hadar and Russel [6] have shown that the FSD rule is equivalent to the expected utility maximization rule for all decision makers preferring larger outcomes, while the SSD rule is equivalent to the expected utility maximization rule for risk-averse decision makers preferring larger outcomes.

The rules defined above can be applied to real-value outcomes, such as income, wealth, or rates of return, but fail to provide ranking of preferences among variables of ordinal nature. Stochastic dominance rules that can be applied in such situations were proposed by Spector et al. [24]. They distinguish between two separate ordinal measurements:

1. The alternative outcomes can only be ranked in order of preference.
2. In addition to ranking, it is also possible to rank the differences between alternative outcomes.

Let's assume that a random variable X_{ik} is defined by $(e_{k1},\dots,e_{kz},p_{ik1},\dots,p_{ikz})$, where $e_{k1},\dots,e_{kz}$ are z real numbers, such that $e_{kl} < e_{kl+1}$ for all $l = 1, \dots, t-1$, and $p_{ik1},\dots,p_{ikz}$ are the probability measures. The variable X_{jk} is defined analogously with $p_{jk1},\dots,p_{jkz}$ replacing $p_{ik1},\dots,p_{ikz}$.

If the outcomes can be ranked in order of preferences, i.e. the decision maker prefers e_{kl+1} over e_{kl} for all $l = 1, \dots, z-1$ then Ordinal First Degree Stochastic Dominance (OFSD) rule can be used:

Definition 3 (OFSD – Ordinal First Degree Stochastic Dominance)

X_{ik} dominates X_{jk} by OFSD rule ($X_{ik} \succ_{\mathrm{OFSD}} X_{jk}$) if and only if :

$$\sum_{l=1}^{s} p_{ikl} \leq \sum_{l=1}^{s} p_{jkl} \text{ for all } s = 1,\dots,z$$

Let's assume that the decision maker adds additional information and indicates that the outcome is improved more by switching from e_{kl} to e_{kl+1} than from e_{kl+1} to e_{kl+2} for all $l = 1, \dots, z-2$. In such case Ordinal Second Degree Stochastic Dominance (OSSD) rule can be employed:

Definition 4 (OSSD – *Ordinal Second Degree Stochastic Dominance*)

X_{ik} dominates X_{jk} by OSSD rule ($X_{ik} \succ_{\text{OSSD}} X_{jk}$) if and only if :

$$\sum_{r=1}^{s}\sum_{l=1}^{r} p_{ikl} \le \sum_{r=1}^{s}\sum_{l=1}^{r} p_{jkl} \quad \text{for all } s = 1,\ldots,z$$

Spector et al. [24] have shown that the OFSD rule is equivalent to the expected utility rule for all decision makers preferring larger outcomes, while the OSSD rule, to the expected utility rule for risk-averse decision makers preferring larger outcomes.

The procedure proposed in this paper assumes that the decision maker is risk averse. Such assumption is usually made in finance and corresponds to the results of experiments of Kahneman and Tversky [8] showing that decision makers are usually risk averse in relation to criteria defined in the domain of gains. As a result, FSD / SSD rules are used for modeling decision maker's preferences with respect to real-valued criteria, while OFSD / OSSD rules, in the case of qualitative variables measured on ordinal scale. Thus, the third step of the procedure involves determination of stochastic dominance relation for each pair of projects and for each criterion.

Step 4. Final solution selection

The last step of the procedure involves final selection and is realized in two steps. First, efficient projects are identified; then the final solution is selected in a dialog procedure. We assume that project a_i is efficient if and only if for no other project a_j the following condition is fulfilled:

$$\forall k = 1,\ldots,n \quad X_{jk} \succ_{\text{SD}} X_{ik}$$

where $\succ_{\text{SD}}$ stands for a stochastic dominance relation (FSD/SSD/OFSD/OSSD). Thus we assume that project a_i is efficient if there is no other project that dominates a_i according to stochastic dominance rules with respect to all criteria.

Efficient projects can be identified by pairwise comparisons. Let A' be the set of efficient projects. The generation of A' proceeds as follows:

1. Let $A' = A$.
2. Let $i = 2$.
3. Let $j = 1$.
4. If $a_j \notin A'$ then go to 7.
5. If $X_{jk} \succ_{\text{SD}} X_{ik}$ for all $k = 1, 2, ..., n$, $A' = A' \setminus \{a_i\}$, go to 8.
6. If $X_k^i \succ_{\text{SD}} X_k^j$ for all $k = 1, 2, ..., n$, $A' = A' \setminus \{a_j\}$.
7. If $j < i - 1$, let $j = j + 1$, go to 4.
8. If $i < m$, let $i = i + 1$, go to 3.
9. End of the procedure.

Interactive approach is employed for selection of the most desirable project. The proposed technique is based on the main ideas of the INSDECM procedure [16]. Each iteration includes the following phases:

- presentation of the data,
- asking the decision maker to provide preference information in the form of aspiration levels based on expected outcome measures (mean) or variability of outcomes measures,
- identification of projects satisfying restrictions.

It is assumed that the decision maker is able to specify the method of data presentation. For each criterion he or she may choose one or more scalar measures to be presented to him or her. Both expected outcome measures (mean, median, mode) and variability measures (standard deviation, semi-deviation, probability of getting outcomes not greater or not less than target value) can be chosen. It is also assumed that the decision maker defines additional requirements specifying minimum or maximum acceptable values of those scalar measures. Unfortunately such constraints are in general not consistent with stochastic dominance rules (for details see [17; 18]). Such situation takes place if the following conditions are fulfilled simultaneously:

- the evaluation of a_i with respect to X_k does not satisfy the constraint,
- the evaluation of a_j with respect to X_k satisfies the constraint,
- $X_{ik} \succ_{\text{SD}} X_{jk}$.

We propose to verify whether a constraint defined by the decision maker is consistent with stochastic dominance rules and to suggest methods of re-defining constraint if inconsistency is found for any pair of projects. Let's assume that inconsistency has been verified for projects a_i and a_j. Inconsistent constraint should be redefined in a way that results in accepting or rejecting both a_i and a_j. The former can be achieved by making the constraint less restricted, the latter, by relaxing it.

The procedure operates as follows:

1. Let $l = 1$, $B_l = A'$.
2. Rank projects $a_i \in B_l$ according to stochastic dominance rules with respect to criterion X_k, $k = 1, ..., n$.
3. If l is equal to 1, go to 6, else go to 4.
4. Present the best and the worst projects with respect to X_k, for $k = 1, ..., n$ and the values of the corresponding scalar measures to the decision maker.
5. Ask the decision maker whether he or she accepts the move from B_{l-1} to B_l. If the answer is YES, go to 6, else set $l = l - 1$ and go to 3.
6. Present the list of considered projects to the decision maker. If the decision maker is able to choose the final solution then the procedure ends, else go to 7.
7. Present the best and the worst projects with respect to X_k, for $k = 1, ..., n$ and the values of the corresponding scalar measures to the decision maker.

8. Ask the decision maker to define a new constraint.
9. Verify the consistency of the constraint defined by the decision maker with stochastic dominance rules. If inconsistency is found, go to 10, else go to 11.
10. Present to the decision maker the ways in which the restriction can be redefined and ask him or her to choose one of the suggestions. If he or she does not accept any proposal, go to 8; else replace the restriction by the accepted proposal and go to 11.
11. Generate B_{l+1} – the set of projects $a_i \in B_l$ satisfying the considered restriction. If $B_{l+1} = \varnothing$, notify the decision maker and go to 8, else set $l = l + 1$ and go to 2.

The procedure iterates until the decision maker is able to accept one of the considered projects as the final solution. Although the procedure does not limit the number of scalar measures to be presented, the decision maker is usually not able to analyze too many of them. If the number of criteria is large then it is practical to limit the number of the measures for each criterion to one. Usually, central tendency measures provide beneficial information. Measures based on probability of getting outcomes above or below the specified target value are also interesting, as they are intuitively comprehensible for the decision maker.

2. ILLUSTRATIVE EXAMPLE

To illustrate the procedure let us consider a manufacturing company operating in a growth market. The management board decided to purchase a new production facility to increase production capacity. Ten alternative projects are considered. All proposals are viable: that is, the output from any of these alternatives meets product specification. The decision for selecting a project has to be made based on net present value for each project, in addition to three other objectives identified in Step 1 below. The economic life for all projects is assumed to be 5 years. Based on past experience and data provided by the manufacturers of facilities, analysts have determined the probability distributions for:

- initial investments (triangular distributions),
- salvage values (uniform distributions),
- production costs per unit (triangular distributions),
- fixed costs (triangular distributions),
- demand (normal distributions),
- market prices (triangular distributions).

Production capacities for each project and for each year have also been specified. The data are shown in Tables 1 and 2.

Table 1

Distributions for project characteristics
(a – minimum, b – maximum, m – most likely)

	Year	Project 1	2	3	4	5	6	7	8	9	10
Initial investment EUR·10^3 (triangular distr.)		a = 480 m = 540 b = 580	a = 390 m = 490 b = 580	a = 470 m = 550 b = 610	a = 370 m = 480 b = 560	a = 390 m = 460 b = 510	a = 410 m = 490 b = 520	a = 380 m = 450 b = 530	a = 500 m = 570 b = 650	a = 570 m = 490 b = 440	a = 360 m = 470 b = 540
Prod. capacity units·10^3 (normal distribution)	1	1200	1150	1350	1050	1000	1050	950	1400	1100	1025
	2	1200	1150	1350	1050	1000	1050	950	1400	1100	1025
	3	1200	1150	1350	1050	1000	1050	950	1400	1100	1025
	4	1100	1050	1250	950	900	950	950	1300	1000	950
	5	1000	950	1150	850	800	850	900	1200	900	900
Production costs per unit EUR (triangular distr.)	1	a = 4.75 m = 5.00 b = 5.38	a = 4.80 m = 5.05 b = 5.43	a = 4.81 m = 5.06 b = 5.44	a = 4.76 m = 5.01 b = 5.39	a = 4.82 m = 5.07 b = 5.45	a = 4.79 m = 5.04 b = 5.42	a = 4.74 m = 4.99 b = 5.36	a = 4.72 m = 4.97 b = 5.34	a = 4.78 m = 5.03 b = 5.41	a = 4.77 m = 5.02 b = 5.40
	2	a = 4.85 m = 5.10 b = 5.20	a = 4.89 m = 5.15 b = 5.54	a = 4.90 m = 5.16 b = 5.55	a = 4.85 m = 5.11 b = 5.49	a = 4.91 m = 5.17 b = 5.56	a = 4.88 m = 5.14 b = 5.53	a = 4.84 m = 5.09 b = 5.47	a = 4.82 m = 5.07 b = 5.45	a = 4.87 m = 5.13 b = 5.51	a = 4.86 m = 5.12 b = 5.50
	3	a = 4.94 m = 5.20 b = 5.59	a = 4.99 m = 5.25 b = 5.64	a = 5.00 m = 5.26 b = 5.65	a = 4.95 m = 5.21 b = 5.60	a = 5.01 m = 5.27 b = 5.67	a = 4.98 m = 5.24 b = 5.63	a = 4.93 m = 5.19 b = 5.58	a = 4.91 m = 5.17 b = 5.56	a = 4.97 m = 5.23 b = 5.62	a = 4.96 m = 5.22 b = 5.61
	4	a = 5.04 m = 5.30 b = 5.70	a = 5.09 m = 5.36 b = 5.76	a = 5.10 m = 5.37 b = 5.77	a = 5.04 m = 5.31 b = 5.71	a = 5.11 m = 5.38 b = 5.78	a = 5.07 m = 5.34 b = 5.74	a = 5.03 m = 5.29 b = 5.69	a = 5.01 m = 5.27 b = 5.67	a = 5.06 m = 5.33 b = 5.73	a = 5.05 m = 5.33 b = 5.72
	5	a = 5.14 m = 5.41 b = 5.82	a = 5.20 m = 5.47 b = 5.88	a = 5.21 m = 5.48 b = 5.89	a = 5.15 m = 5.42 b = 5.83	a = 5.22 m = 5.49 b = 5.90	a = 5.18 m = 5.45 b = 5.86	a = 5.13 m = 5.40 b = 5.81	a = 5.11 m = 5.38 b = 5.78	a = 5.17 m = 5.44 b = 5.85	a = 5.16 m = 5.43 b = 5.84
Salvage value EUR·10^3 (uniform distr.)		a = 250 b = 325	a = 200 b = 275	a = 225 b = 325	a = 175 b = 250	a = 125 b = 200	a = 175 b = 225	a = 100 b = 150	a = 250 b = 350	a = 175 b = 275	a = 150 b = 225

Table 2

Fixed costs and market predictions
(a – minimum, b – maximum, m – most likely, μ – mean, σ – standard deviation)

Year	Fixed costs EUR·10^3 (triangular distr.)	Demand units·10^3 (normal distr.)	Market price EUR (triangular distr.)
1	a = 270.00 m = 300.00 b = 322.50	μ = 950.00 σ = 92.60	a = 6.00 m = 6.20 b = 6.35
2	a = 283.50 m = 315.00 b = 338.63	μ = 976.00 σ = 104.90	a = 5.95 m = 6.15 b = 6.30
3	a = 297.68 m = 330.75 b = 355.65	μ = 1008.00 σ = 115.90	a = 5.75 m = 6.10 b = 6.25
4	a = 312.56 m = 347.29 b = 373.34	μ = 1020.00 σ = 117.40	a = 5.75 m = 6.05 b = 6.20
5	a = 328.19 m = 364.65 b = 392.00	μ = 1023.00 σ = 117.60	a = 5.73 m = 6.00 b = 6.20

Step 1. Identification of criteria

The decision maker decided to consider the following criteria:

X_1 – net present value,
X_2 – reliability and technical service,
X_3 – technical novelty,
X_4 – compatibility with existing facilities.

Step 2. Generation of project evaluations

Simulation technique has been applied for generating distributional evaluations of projects with respect to attribute X_1. Table 3 presents results of simulation experiments.

Table 3

Results of simulation experiments

Project	Mean	Standard deviation
1	1413.84	265.94
2	1183.06	269.73
3	1139.12	277.16
4	1244.56	260.44
5	979.66	237.21
6	1137.93	244.98
7	1208.61	234.54
8	1432.72	283.48
9	1211.81	256.91
10	1226.72	256.32

Expert assessments are used for constructing distributional evaluations of the projects with respect to criteria X_2, X_3, X_4. Ten analysts assessed each proposal on the scale from 1 to 10. Tables 4, 5 and 6 present expert assessments.

Table 4

Analysts' evaluations with respect to X_2 (reliability and technical service)
(number of experts assigning the specified criteria value for a project)

Evaluation	Project									
	1	2	3	4	5	6	7	8	9	10
1			2		3					
2		4	2		1		1			
3		3	3	1	2		3		1	
4		2	2	2	2		1	1		
5	1	1	1	3	2		1	1	1	
6	2			2		1	3	2	1	2
7	2			1		4	1	2	3	2
8	3			1		2		3	2	3
9	2					1		1	2	2
10						2				1

Table 5

Analysts' evaluations with respect to X_3 (technical novelty)
(number of experts assigning the specified criteria value for a project)

Evaluation	Project									
	1	2	3	4	5	6	7	8	9	10
1										1
2	2			1						2
3	4	2		2				1	2	3
4	2	2		3	1	1		1	4	3
5	2	1			2	4	1	1	2	1
6		5	4	2	2	2	3	2	2	
7			2	2	3	2	1			
8			4		1		2			
9					1	1	1	2		
10							2	3		

Table 6

Analysts' evaluations with respect to X_4 (compatibility with existing facilities) (number of experts assigning the specified criteria value for a project)

Evaluation	Project									
	1	2	3	4	5	6	7	8	9	10
1		3								
2		3			1		3			
3		1			2		1	1	4	
4		1	4		3	1	1	2	3	
5		2			1	2	4	3		1
6	1		5		3	2	1	1		3
7	2		1	3		2			1	1
8	3			5		1		2	2	3
9	2			1		2		1		2
10	2			1						

Step 3. Comparing projects with respect to criteria

FSD/SSD rules are applied for comparing projects with respect to criterion X_1, while OFSD/OSSD rules are employed when projects are analyzed with respect to criteria X_2, X_3, and X_4. Tables 7, 8, 9 and 10 show relations between projects.

Table 7

FSD/SSD relations between project evaluations with respect to criterion X_1

	$X_{1\,1}$	$X_{2\,1}$	$X_{3\,1}$	$X_{4\,1}$	$X_{5\,1}$	$X_{6\,1}$	$X_{7\,1}$	$X_{8\,1}$	$X_{9\,1}$	$X_{10\,1}$
$X_{1\,1}$		FSD	FSD	FSD	FSD	FSD	FSD		FSD	FSD
$X_{2\,1}$			SSD		FSD					
$X_{3\,1}$										
$X_{4\,1}$		SSD	SSD		FSD					
$X_{5\,1}$										
$X_{6\,1}$					FSD					
$X_{7\,1}$		SSD	SSD		FSD	SSD				
$X_{8\,1}$		FSD	FSD	FSD	FSD	FSD	FSD		FSD	FSD
$X_{9\,1}$		SSD	FSD		FSD					
$X_{10\,1}$		SSD	SSD		FSD					

Table 8

OFSD/OSSD relations between project evaluations with respect to criterion X_2

	$X_{1\,2}$	$X_{2\,2}$	$X_{3\,2}$	$X_{4\,2}$	$X_{5\,2}$	$X_{6\,2}$	$X_{7\,2}$	$X_{8\,2}$	$X_{9\,2}$	$X_{10\,2}$
$X_{1\,2}$		OFSD	OFSD	OFSD	OFSD		OFSD	OFSD	OFSD	
$X_{2\,2}$			OFSD		OSSD					
$X_{3\,2}$										
$X_{4\,2}$		OFSD	OFSD		OFSD		OFSD			
$X_{5\,2}$										
$X_{6\,2}$	OFSD	OFSD	OFSD	OFSD	OFSD		OFSD	OFSD	OFSD	OSSD
$X_{7\,2}$		OFSD	OFSD		OFSD					
$X_{8\,2}$		OFSD	OFSD	OFSD	OFSD		OFSD			
$X_{9\,2}$		OFSD	OFSD	OFSD	OFSD		OFSD			
$X_{10\,2}$	OFSD	OFSD	OFSD	OFSD	OFSD		OFSD	OFSD	OFSD	

Table 9

OFSD/OSSD relations between project evaluations with respect to criterion X_3

	$X_{1\,3}$	$X_{2\,3}$	$X_{3\,3}$	$X_{4\,3}$	$X_{5\,3}$	$X_{6\,3}$	$X_{7\,3}$	$X_{8\,3}$	$X_{9\,3}$	$X_{10\,3}$
$X_{1\,3}$										
$X_{2\,3}$	OFSD			OSSD						OFSD
$X_{3\,3}$	OFSD	OFSD		OFSD	OSSD	OSSD			OFSD	OFSD
$X_{4\,3}$	OFSD									OFSD
$X_{5\,3}$	OFSD	OFSD		OFSD		OFSD			OFSD	OFSD
$X_{6\,3}$	OFSD	OFSD		OFSD					OFSD	OFSD
$X_{7\,3}$	OFSD	OFSD		OFSD	OFSD	OFSD		OSSD	OFSD	OFSD
$X_{8\,3}$	OFSD	OFSD		OFSD						OFSD
$X_{9\,3}$	OFSD	OSSD		OFSD						OFSD
$X_{10\,3}$	OFSD									

Table 10

OFSD/OSSD relations between project evaluations with respect to criterion X_4

	$X_{1\,4}$	$X_{2\,4}$	$X_{3\,4}$	$X_{4\,4}$	$X_{5\,4}$	$X_{6\,4}$	$X_{7\,4}$	$X_{8\,4}$	$X_{9\,4}$	$X_{10\,4}$
$X_{1\,4}$		OFSD	OFSD		OFSD	OFSD	OFSD	OFSD	OFSD	OFSD
$X_{2\,4}$										
$X_{3\,4}$		OFSD			OFSD		OFSD		OSSD	
$X_{4\,4}$		OFSD	OFSD		OFSD	OFSD	OFSD	OFSD	OFSD	OFSD
$X_{5\,4}$		OFSD					OSSD			
$X_{6\,4}$		OFSD	OFSD		OFSD		OFSD	OFSD	OFSD	
$X_{7\,4}$		OFSD								
$X_{8\,4}$		OFSD			OFSD		OFSD		OFSD	
$X_{9\,4}$		OFSD					OSSD			
$X_{10\,4}$		OFSD	OFSD		OFSD	OFSD	OFSD	OFSD	OFSD	

Step 4. Generating the final solution

Project a_2 is not efficient – its evaluations are dominated by the corresponding evaluations of a_7, a_8, and a_9 with respect to all attributes:

$$X_{71} \succ_{\text{SSD}} X_{21} \text{ and } X_{72} \succ_{\text{OFSD}} X_{22} \text{ and } X_{73} \succ_{\text{OFSD}} X_{23} \text{ and } X_{74} \succ_{\text{OFSD}} X_{24}$$

$$X_{81} \succ_{\text{FSD}} X_{21} \text{ and } X_{82} \succ_{\text{OFSD}} X_{22} \text{ and } X_{83} \succ_{\text{OFSD}} X_{23} \text{ and } X_{84} \succ_{\text{OFSD}} X_{24}$$

$$X_{91} \succ_{\text{SSD}} X_{21} \text{ and } X_{92} \succ_{\text{OFSD}} X_{22} \text{ and } X_{93} \succ_{\text{OSSD}} X_{23} \text{ and } X_{94} \succ_{\text{OFSD}} X_{24}$$

All other projects are efficient:

$$A' = \{ a_1, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10} \}$$

Let's assume that the decision maker has chosen the mean to be presented during the dialog procedure. The final solution is generated as follows:

Iteration 1

1. $l = 1$, $B_1 = \{ a_1, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10} \}$.
2. Projects are ranked according to stochastic dominance rules with respect to each criterion (Table 11).

Table 11

Rankings of projects with respect to the criteria

X_1	X_2	X_3	X_4
a_1, a_8	a_6	a_3, a_7	a_1, a_4
a_4, a_7, a_9, a_{10}	a_{10}	a_5, a_8	a_{10}
a_3, a_6	a_1	a_6	a_6
a_5	a_8, a_9	a_9	a_3, a_8
	a_4	a_4	a_5, a_9
	a_7	a_{10}	a_7
	a_3, a_5	a_1	

6. Projects $a_i \in B_1$ are presented to the decision maker. He states that he is not able to select the final solution yet.
7. The best and the worst projects with respect to the criteria are presented to the decision maker (Table 12).

Table 12

Iteration 1 – the best and the worst alternatives with respect to the criteria

	X_1		X_2		X_3		X_4	
	project	mean	project	mean	project	mean	project	mean
best	a_1	1413.84	a_6	7.9	a_3	7.0	a_1	8.2
	a_8	1432.72			a_7	7.5	a_4	8.0
worst	a_5	979.66	a_3	2.8	a_1	3.4	a_7	3.9
			a_5	2.9				

8. The decision maker formulates the following constraint:
The average evaluation with respect to X_1 not less than 1000: $\mu_{i\,1} \geq 1000$
9. The constraint is consistent with stochastic dominance rules – the only action that does not satisfy it is a_5; the evaluation of a_5 with respect to X_1 does not dominate the corresponding evaluation of any other $a_i \in B_1$.
11. $B_2 = \{ a_1, a_3, a_4, a_6, a_7, a_8, a_9, a_{10} \}$; $l = 2$.

Iteration 2

2. Projects $a_j \in B_2$ are ranked according to stochastic dominance rules with respect to each criterion.
4. The best and the worst projects with respect to the criteria are presented to the decision maker (Table 13).

Table 13

Iteration 2 – the best and the worst alternatives with respect to the criteria

	X_1		X_2		X_3		X_4	
	project	mean	project	mean	project	mean	project	mean
best	a_1	1413.84	a_6	7.9	a_3	7.0	a_1	8.2
	a_8	1432.72			a_7	7.5	a_4	8.0
worst	a_3	1139.12	a_3	2.8	a_1	3.4	a_7	3.9
	a_6	1137.93						

5. The decision maker accepts the move from B_1 to B_2.
6. Projects $a_i \in B_2$ are presented to the decision maker. He states that he is not able to select the final solution yet.
7. The best and the worst projects with respect to the criteria are presented to the decision maker (Table 13).
8. The decision maker formulates the following constraint:
The probability that the evaluation with respect to X_4 is not less than 7 is at least equal to 0.2: $\Pr(X_{i\,4} \geq 7) \geq 0.2$
9. The constraint is not consistent with stochastic dominance rules for pair (a_3, a_9):

$$\Pr(X_{3\,4} \geq 7) = 0.1$$
$$\Pr(X_{9\,4} \geq 7) = 0.3$$
$$X_{3\,4} \text{ OSSD } X_{9\,4}$$

10. The ways in which the restriction can be redefined are presented to the decision maker:
(1) $\Pr(X_{i\,4} \geq 7) \geq 0.1$ (2) $\Pr(X_{i\,4} \geq 6) \geq 0.2$
(3) $\Pr(X_{i\,4} \geq 7) \geq 0.31$ (4) $\Pr(X_{i\,4} \geq 9) \geq 0.2$
Proposals (1) and (2) accept both a_3 and a_9, while proposals (3) and (4) eliminate each of them. The decision maker accepts proposal (1).
11. $B_3 = \{ a_1, a_3, a_4, a_6, a_8, a_9, a_{10} \}$; $l = 3$.

Iteration 3

2. Projects $a_j \in B_3$ are ranked according to stochastic dominance rules with respect to each criterion.
4. The best and the worst projects with respect to the criteria are presented to the decision maker (Table 14).

Table 14

Iteration 3 – the best and the worst alternatives with respect to the criteria

	X_1		X_2		X_3		X_4	
	project	mean	project	mean	project	mean	project	mean
best	a_1	1413.84	a_6	7.9	a_3	7.0	a_1	8.2
	a_8	1432.72			a_8	7.2	a_4	8.0
worst	a_3	1139.12	a_3	2.8	a_1	3.4	a_9	4.7
	a_6	1137.93						

5. The decision maker accepts the move from B_2 to B_3.
6. Projects $a_i \in B_3$ are presented to the decision maker. He states that he is not able to select the final solution yet.
7. The best and the worst projects with respect to the criteria are presented to the decision maker (Table 14).
8. The decision maker formulates the following constraint:
   The average evaluation with respect to X_2 is not less than 6: $\mu_{i\,2} \geq 6$
9. The constraint is consistent with stochastic dominance rules – actions a_3 and a_4 do not satisfy such restriction; evaluation of a_3 and a_5 with respect to X_2 do not dominate the corresponding evaluations of a_1, a_6, a_8, a_9, a_{10}.
11. $B_4 = \{ a_1, a_6, a_8, a_9, a_{10} \}$; $l = 4$.

Iteration 4

2. Projects $a_j \in B_4$ are ranked according to stochastic dominance rules with respect to each criterion.
4. The best and the worst projects with respect to the criteria are presented to the decision maker (Table 15).

Table 15

Iteration 4 – the best and the worst alternatives with respect to the criteria

	X_1		X_2		X_3		X_4	
	project	mean	project	mean	project	mean	project	mean
best	a_1	1413.84	a_6	7.9	a_6	5.9	a_1	8.2
	a_8	1432.72			a_8	7.2		
worst	a_6	1137.93	a_8	6.8	a_1	3.4	a_9	4.7
	a_9	1211.81	a_9	6.9				
	a_{10}	1226.72						

5. The decision maker accepts the move from B_3 to B_4.
6. Projects $a_i \in B_4$ are presented to the decision maker. He states that he is not able to select the final solution yet.
7. The best and the worst projects with respect to the criteria are presented to the decision maker (Table 15).
8. The decision maker formulates the following constraint:
   The probability that the evaluation with respect to X_3 is not less than 5 is at least equal to 0.5: $\Pr(X_{i\,3} \geq 5) \geq 0.5$
9. The constraint is consistent with stochastic dominance rules – actions a_1 and a_{10} do not satisfy such restriction; evaluations of a_1 and a_{10} with respect to X_3 do not dominate corresponding evaluations of a_6, a_8, a_9.

11. $B_5 = \{ a_6, a_8, a_9 \}$; $l = 5$.

Iteration 5

2. Projects $a_j \in B_5$ are ranked according to stochastic dominance rules with respect to each criterion.
4. The best and the worst projects with respect to the criteria are presented to the decision maker (Table 16).

Table 16

Iteration 5 – the best and the worst alternatives with respect to the criteria

	X_1		X_2		X_3		X_4	
	project	mean	project	mean	project	mean	project	mean
best	a_8	1432.72	a_6	7.9	a_6	5.9	a_6	6.6
					a_8	7.2		
worst	a_6	1137.93	a_8	6.8	a_9	5.4	a_9	4.7
	a_9	1211.81	a_9	6.9				

5. The decision maker accepts the move from B_4 to B_5.
6. Projects $a_i \in B_5$ are presented to the decision maker. He select a_8 as the final solution.

CONCLUSIONS

Selection of a new project or a group of projects undoubtedly constitutes one of the main management functions required to ensure business survival. Such decision must usually be made using more than one criterion. Technical factors, environmental effects, social issues, and financial desirability are taken into account in the project evaluation process. Moreover, selection is made in a risky environment. In this study a new methodology for this problem has been proposed. It uses two approaches: stochastic dominance and interactive approach. The former is widely used for comparing uncertain prospects, the latter is a multiple criteria approach that is probably most often used in real-world applications. These two concepts have been combined in the INSDECM procedure.

The technique presented in this paper uses data of various types. Simulation technique and expert assessments are used for evaluating projects with respect to criteria. Therefore, both quantitative and qualitative criteria can be taken into account. Although our example illustrates an implementation for selecting a production facility, the present method can also be used to provide a similar support for selection of R&D projects, real estate projects, or marketing strategies.

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Włodzimierz Ogryczak

EQUITY, FAIRNESS AND MULTICRITERIA OPTIMIZATION*

INTRODUCTION

Equity or fairness issues appear in many decision models of Operations Research. Especially models dealing with allocation of resources try to achieve some fairness of allocation patterns [9]. More generally, the models related to the evaluation of various systems which serve many users and the quality of service for every individual user defines the criteria. This applies among others to networking where a central issue is how to allocate bandwidth to flows efficiently and fairly [1]. The issue of equity is widely recognized in location analysis of public services, where the clients of a system are entitled to fair treatment according to community regulations. In such problems, the decisions often concern the placement of a service center or other facility in a position so that the users are treated in an equitable way, relative to certain criteria [13]. Moreover, uniform individual outcomes may be associated with some events rather than physical users, like in many dynamic optimization problems where uniform individual criteria represent a similar event in various periods and all they are equally important.

Fairness is, essentially, an abstract socio-political concept that implies impartiality, justice and equity [23]. Nevertheless, fairness was usually quantified with the so-called inequality measures to be minimized [18]. Unfortunately, direct minimization of typical inequality measures (especially relative ones) contradicts the maximization of individual outcomes and it may lead to inferior decisions. Recently, several research publications relating the fairness and equity concepts to the multiple criteria optimization methodology have appeared [4; 7; 9; 10; 13]. Finally, the novel and distinct mathematical approach denoted by equitable efficiency has been developed to provide solutions to these examples of multiple criteria optimization [6]. The concept of equitably efficient solution is a specific refinement of the Pareto-optimality. This paper deals with generation techniques for equitably efficient solutions to multiple criteria optimization problems.

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1. EQUITY AND FAIRNESS

The generic decision problem, we consider, may be stated as follows. There is given a set I of m services (users, clients). There is also given a set Q of feasible decisions. For each service $i \in I$ a function $f_i(\mathbf{x})$ of the decision $\mathbf{x}$ has been defined. This function, called the individual objective function, measures the outcome (effect) $y_i = f_i(\mathbf{x})$ of the decision for service i. An outcome usually expresses the service quality. However, outcomes can be measured (modeled) as service time, service costs, service delays as well as in a more subjective way. In typical formulations a larger value of the outcome means a better effect (higher service quality or client satisfaction). Otherwise, the outcomes can be replaced with their complements to some large number. Therefore, without loss of generality, we can assume that each individual outcome y_i is to be maximized which results in a multiple criteria maximization model.

$$\max \{\mathbf{f}(\mathbf{x}) \; : \; \mathbf{x} \in Q\} \tag{1}$$

where:

$\mathbf{f}(\mathbf{x})$	–	is a vector-function that maps the decision space $X = R^n$ into the criterion space $Y = R^m$,
$Q \subset X$	–	denotes the feasible set,
$\mathbf{x} \in X$	–	denotes the vector of decision variables.

Model (1) only specifies that we are interested in maximization of all objective functions f_i for $i \in I = \{1, 2, \ldots, m\}$. In order to make it operational, one needs to assume some solution concept specifying what it means to maximize multiple objective functions.

Typical solution concepts for multiple criteria problems are defined by aggregation (or utility) functions $g : Y \rightarrow R$ to be maximized. Thus the multiple criteria problem (1) is replaced with the maximization problem:

$$\max \{g(\mathbf{f}(\mathbf{x})) \; : \; \mathbf{x} \in Q\} \tag{2}$$

In order to guarantee the consistency of the aggregated problem (2) with the maximization of all individual objective functions in the original multiple criteria problem (or Pareto-optimality of the solution), the aggregation function must be strictly increasing with respect to every coordinate, i.e., for all $i \in I$,

$$g(y_1, \ldots, y_{i-1}, y'_i, y_{i+1}, \ldots, y_m) < g(y_1, y_2, \ldots, y_m) \tag{3}$$

whenever $y'_i < y_i$.

The simplest aggregation functions commonly used for the multiple criteria problem (1) are defined as the sum of outcomes:

$$g(\mathbf{y}) = \sum_{i=1}^{m} y_i \tag{4}$$

or the worst outcome:

$$g(\mathbf{y}) = \min_{i=1,\dots,m} y_i \tag{5}$$

The sum (4) is a strictly increasing function while the minimum (5) is only nondecreasing. Therefore, the aggregation (2) using the sum of outcomes always generates a Pareto-optimal solution while the maximization of the worst outcome may need some additional refinement.

Equity is, essentially, an abstract socio-political concept, but it is usually quantified with the so-called inequality measures to be minimized. Inequality measures were primarily studied in economics [18] while recently they become very popular tools in Operations Research. For instance, Marsh and Schilling [10] describe twenty different measures proposed in the literature to gauge the level of equity in facility location alternatives. The simplest inequality measures are based on the absolute measurement of the spread of outcomes, like the mean (absolute) difference:

$$D(\mathbf{y}) = \frac{1}{2m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} |y_i - y_j| \tag{6}$$

or the maximum (absolute) difference:

$$R(\mathbf{y}) = \frac{1}{2} \max_{i,j=1,\dots,m} |y_i - y_j| \tag{7}$$

In most application frameworks better intuitive appeal may have inequality measures related to deviations from the mean outcome like the mean (absolute) deviation:

$$\delta(\mathbf{y}) = \frac{1}{2m} \sum_{i=1}^{m} |y_i - \mu(\mathbf{y})| \tag{8}$$

In economics one usually considers relative inequality measures normalized by mean outcome. Among many inequality measures perhaps the most commonly accepted by economists is the Gini coefficient, which is the relative mean difference. One can easily notice that direct minimization of typical inequality measures (especially the relative ones) may contradict the optimization of individual outcomes. As pointed out by Erkut [2], it is rather a common flaw of all the relative inequality measures that while moving away from the spatial units to be serviced one gets better values of the measure as the relative distances become closer to one-another. As an extreme, one may consider an unconstrained continuous (single-facility) location problem and find that the facility located at (or near) infinity will provide (almost) perfectly equal service (in fact, rather lack of service) to all the spatial units. Unfortunately, these flaws of the inequality measure minimization remains also valid when the inequality measure is added as an additional criterion [13].

In order to guarantee fairness (equitability) of the solution concept (2), additional requirements on the class of aggregation (utility) functions may be introduced. In particular, the aggregation function must be additionally symmetric (impartial), i.e. for any permutation τ of I,

$$g(y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(m)}) = g(y_1, y_2, \ldots, y_m) \tag{9}$$

as well as be equitable (to satisfy the principle of transfers)

$$g(y_1, \ldots, y_{i'} - \varepsilon, \ldots, y_{i''} + \varepsilon, \ldots, y_m) > g(y_1, y_2, \ldots, y_m) \tag{10}$$

for any $0 < \varepsilon < y_{i'} - y_{i''}$. In the case of an aggregation function satisfying all the requirements (3), (9) and (10), we call the corresponding problem (2) a *fair (equitable) aggregation* of problem (1). Every optimal solution to the fair aggregation (2) of a multiple criteria problem (1) defines some fair (equitable) solution.

Note that symmetric functions satisfying the requirement

$$g(y_1, \ldots, y_{i'} - \varepsilon, \ldots, y_{i''} + \varepsilon, \ldots, y_m) \geqslant g(y_1, y_2, \ldots, y_m) \tag{11}$$

for $0 < \varepsilon < y_{i'} - y_{i''}$ are called (weakly) Schur-concave [11] while the stronger requirement of equitability (10), we consider, is related to strictly Schur-concave functions. In other words, an aggregation (2) is fair if it is defined by a strictly increasing and strictly Schur-concave function g.

Note that both the simplest aggregation functions, the sum (4) and the minimum (5), are symmetric and satisfy the requirement (11), although they do not satisfy the equitability requirement (10). Hence, they are Schur-concave but not strictly Schur-concave. To guarantee the fairness of solutions, some enforcement of concave properties is required.

For any strictly concave, increasing utility function $s : R \to R$, the function

$$g(\mathbf{y}) = \sum_{i=1}^{m} s(y_i) \tag{12}$$

is a strictly monotonic and strictly Schur-concave function [11]. This defines a family of the fair aggregations according to the following proposition [12].

Proposition 1 For any strictly convex, increasing function $s : R \rightarrow R$, the optimal solution of the problem

$$\max \{\sum_{i=1}^{m} s(f_i(\mathbf{x})) \ : \ \mathbf{x} \in Q\} \qquad (13)$$

is a fair solution for decision problem (1).

Various concave functions utility s can be used to define fair aggregations (13) and the resulting fair solution concepts. In the case of the outcomes restricted to positive values, one may use logarithmic function thus resulting in the so-called proportional fairness model [5]. A parametric class of utility functions:

$$s(y_i, \alpha) = \begin{cases} y_i^{1-\alpha}/(1-\alpha) & \text{if } \alpha \neq 1 \\ \log(y_i) & \text{if } \alpha = 1 \end{cases}$$

may be used for this purpose generating various solution concepts for $\alpha \geqslant 0$. In particular, for $\alpha = 0$ one gets the total output maximization which is the only linear criterion within the entire class. For $\alpha = 1$, it represents the Proportional Fairness approach [5] that maximizes the sum of logarithms of the flows while with α tending to the infinity it converges to the lexicographic max-min optimization which represents the Rawlsian [17] concept of justice. However, every such approach requires to build (or to guess) a utility function prior to the analysis and later it gives only one possible compromise solution. It is very difficult to identify and formalize the preferences at the beginning of the decision process. Moreover, apart from the trivial case of the total output maximization all the utility functions that really take into account any fairness preferences are nonlinear. Many decisions models considered with fair outcomes are originally LP or MILP models. Nonlinear objective functions applied to such models may results in computationally hard optimization problems. In the following, we shall describe an approach that allows to search for such compromise solutions with multiple linear criteria rather than the use nonlinear objective functions.

2. ORDERED OUTCOMES

Multiple criteria optimization defines the dominance relation by the standard vector inequality. The theory of majorization [11] includes the results which allow us to express the relation of fair (equitable) dominance as a vector inequality on the cumulative ordered outcomes [6]. This can be mathematically formalized as follows. First, introduce the ordering map $\Theta : R^m \rightarrow R^m$ such that $\Theta(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \ldots, \theta_m(\mathbf{y}))$, where $\theta_1(\mathbf{y}) \leqslant \theta_2(\mathbf{y}) \leqslant \cdots \leqslant \theta_m(\mathbf{y})$ and there exists a permutation τ of set I such that $\theta_i(\mathbf{y}) = y_{\tau(i)}$ for $i = 1, \ldots, m$. Next, apply to ordered outcomes $\Theta(\mathbf{y})$, a linear cumulative map thus resulting in the cumulative ordering map $\bar{\Theta}(\mathbf{y}) = (\bar{\theta}_1(\mathbf{y}), \bar{\theta}_2(\mathbf{y}), \ldots, \bar{\theta}_m(\mathbf{y}))$ defined as:

$$\bar{\theta}_i(\mathbf{y}) = \sum_{j=1}^{i} \theta_j(\mathbf{y}) \quad \text{for } i = 1, \ldots, m \tag{14}$$

The coefficients of vector $\bar{\Theta}(\mathbf{y})$ express, respectively: the smallest outcome, the total of the two smallest outcomes, the total of the three smallest outcomes etc.

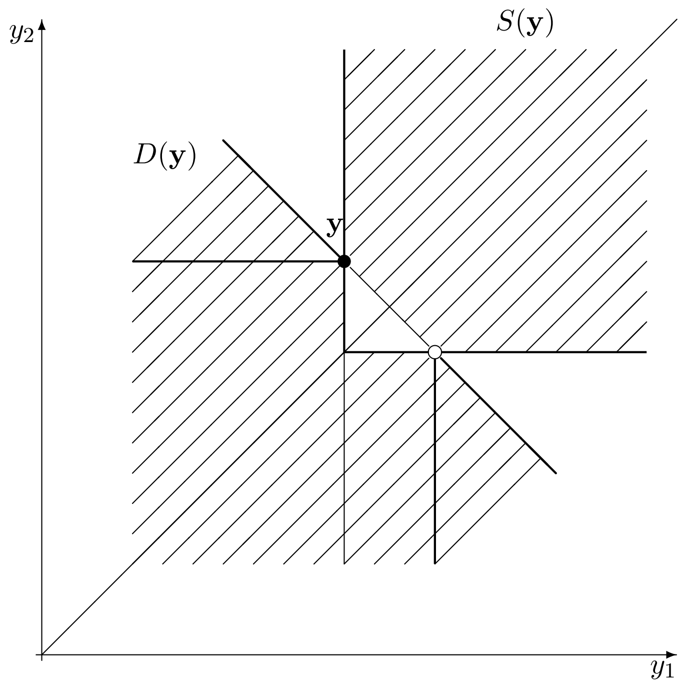

Fig. 1. Structure of the equitable dominance

Note that fair solutions to problem (1) can be expressed as Pareto-optimal solutions for the multiple criteria problem with objectives $\bar{\Theta}(\mathbf{f}(\mathbf{x}))$:

$$\max \{(\bar{\theta}_1(\mathbf{f}(\mathbf{x})), \bar{\theta}_2(\mathbf{f}(\mathbf{x})), \ldots, \bar{\theta}_m(\mathbf{f}(\mathbf{x}))) \ : \ \mathbf{x} \in Q\} \tag{15}$$

Proposition 2 A feasible solution $\mathbf{x} \in Q$ is a fair solution of the problem (1), iff it is a Pareto-optimal solution of the multiple criteria problem (15).

Proposition 2 provides the relationship between fair solutions and the standard Pareto-optimality. One may notice that the set $D(\mathbf{y})$ of directions leading to outcome vectors being dominated by a given $\mathbf{y}$ is, in general, not a cone and it is not convex. Although, when we consider the set $S(\mathbf{y})$ of directions leading to outcome vectors dominating given $\mathbf{y}$ we get a convex set. Figure 1 shows both $S(\mathbf{y})$ and $D(\mathbf{y})$ fixed at $\mathbf{y}$.

Hence, the multiple criteria problem (15) may serve as a source of fair solution concepts. Although the definitions of quantities $\bar{\theta}_k(\mathbf{y})$, used as criteria in (15), are very complicated, the quantities themselves can be modeled with simple auxiliary variables and constraints. It is commonly known that the smallest outcome may be defined by the following optimization: $\bar{\theta}_1(\mathbf{y}) = \max \{t \;:\; t \leqslant y_i \quad \text{for } i = 1, \ldots, m\}$, where t is an unrestricted variable. It turns out that this can be generalized to provide an effective modeling technique for quantities $\bar{\theta}_k(\mathbf{y})$ with arbitrary k [16]. Let us notice that for any given vector $\mathbf{y}$, the quantity $\bar{\theta}_k(\mathbf{y})$ is defined by the following LP:

$$\begin{aligned} \bar{\theta}_k(\mathbf{y}) \quad = \quad & \min \sum_{i=1}^{m} y_i u_{ki} \\ & \text{s.t.} \quad \sum_{i=1}^{m} u_{ki} = k, \; 0 \leqslant u_{ki} \leqslant 1 \quad \text{for } i = 1, \ldots, m \end{aligned} \tag{16}$$

Exactly, the above problem is an LP for a given outcome vector $\mathbf{y}$ while it begins nonlinear for a variable $\mathbf{y}$. This difficulty can be overcome by taking advantages of the LP dual to (16):

$$\begin{aligned} \bar{\theta}_k(\mathbf{y}) \quad = \quad & \max \; kt_k - \sum_{i=1}^{m} d_{ik} \\ & \text{s.t.} \quad t_k - y_i \leqslant d_{ik}, \; d_{ik} \geqslant 0 \quad \text{for } i = 1, \ldots, m \end{aligned} \tag{17}$$

where t_k is an unrestricted variable while nonnegative variables d_{ik} represent, for several outcome values y_i, their downside deviations from the value of t [16].

3. MULTICRITERIA APPROACHES

Proposition 2 allows one to generate equitably efficient solutions of (1) as efficient solutions of multicriteria problem:

$$\max\ (\eta_1, \eta_2, \ldots, \eta_m) \tag{18}$$
$$\text{subject to} \quad \mathbf{x} \in Q$$
$$\eta_k = kt_k - \sum_{i=1}^{m} d_{ik} \quad \text{for } k = 1, \ldots, m \tag{19}$$
$$t_k - d_{ik} \leqslant f_i(\mathbf{x}), \quad d_{ik} \geqslant 0 \quad \text{for } i, k = 1, \ldots, m \tag{20}$$

The aggregation maximizing the sum of outcomes, corresponds to maximization of the last (m-th) objective (η_m) in problem (18)-(20). Similar, the maximin scalarization corresponds to maximization of the first objective (η_1). For modeling various fair preferences one may use some combinations the criteria. In particular, for the weighted sum $\sum_{i=1}^{m} w_i\eta_i$ on gets equivalent combination of the cumulative ordered outcomes $\bar{\theta}_i(\mathbf{y})$:

$$\sum_{i=1}^{m} w_i\bar{\theta}_i(\mathbf{y}) \tag{21}$$

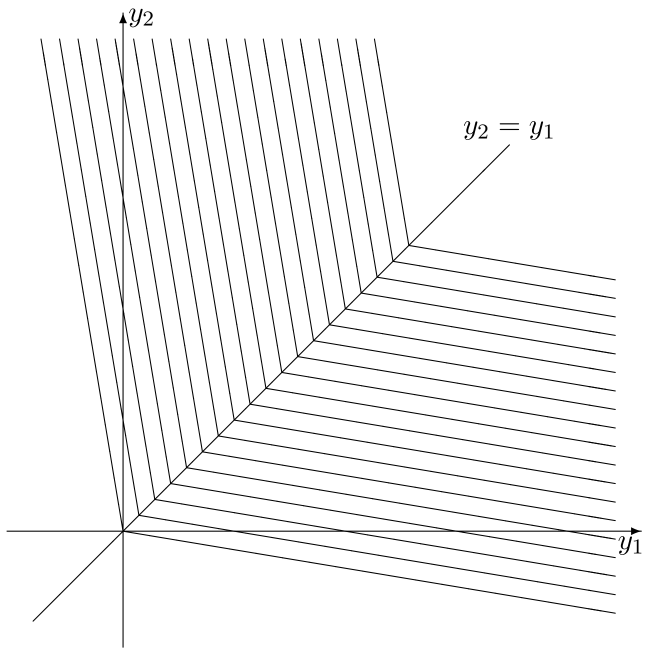

Fig. 2. Isoline contours for an equitable OWA aggregation

Note that, due to the definition of map $\bar{\Theta}$ with (14), the above function can be expressed in the form with weights $v_i = \sum_{j=i}^{m} w_j$ $(i = 1, \ldots, m)$ allocated to coordinates of the ordered outcome vector. Such an approach to aggregation of outcomes was introduced by Yager [22] as the so-called Ordered Weighted Averaging (OWA). When applying OWA to problem (1) we get:

$$\max \{\sum_{i=1}^{m} v_i \theta_i(\mathbf{f}(\mathbf{x})) \ : \ \mathbf{x} \in Q\} \tag{22}$$

The OWA aggregation is obviously a piece wise linear function since it remains linear within every area of the fixed order of arguments. If weights v_i are strictly decreasing and positive, i.e. $v_1 > v_2 > \cdots > v_{m-1} > v_m > 0$, then each optimal solution of the OWA problem (22) is a fair solution of (1).

While equal weights define the linear aggregation, several decreasing sequences of weights lead to various strictly Schur-concave and strictly monotonic aggregation functions. Thus, the monotonic OWA aggregations provide a family of piece wise linear aggregations filling out the space between the piece wise linear aggregation functions (4) and (5) as shown in Fig. 3.

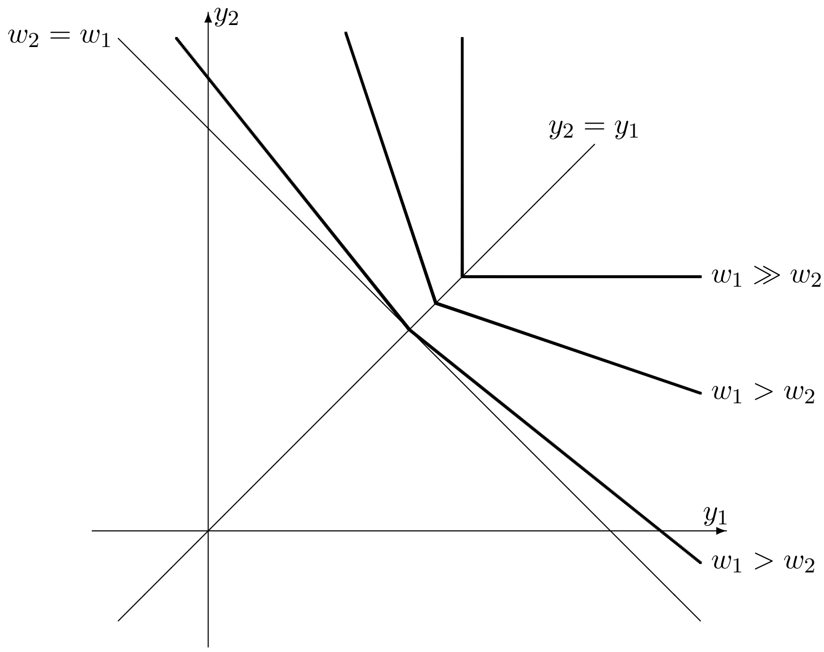

Fig. 3. Isoline contours for various equitable OWA aggregations

Actually, formulas (21) and (17) allow us to formulate any monotonic (not necessarily strictly) OWA problem (22) as the following LP extension of the original multiple criteria problem:

$$\max \sum_{k=1}^{m} w_k \eta_k \tag{23}$$

$$\text{subject to} \quad \mathbf{x} \in Q$$

$$\eta_k = k t_k - \sum_{i=1}^{m} d_{ik} \quad \text{for } k = 1, \ldots, m \tag{24}$$

$$t_k - d_{ik} \leqslant f_i(\mathbf{x}), \quad d_{ik} \geqslant 0 \quad \text{for } i, k = 1, \ldots, m \tag{25}$$

where $w_m = v_m$ and $w_k = v_k - v_{k+1}$ for $k = 1, \ldots, m-1$.

When differences among weights tend to infinity, the OWA aggregation approximates the lexicographic ranking of the ordered outcome vectors [13]. That means, as the limiting case of the OWA problem (22), we get the lexicographic problem:

$$\operatorname{lexmax} \{\Theta(\mathbf{f}(\mathbf{x})) \;:\; \mathbf{x} \in Q\} \tag{26}$$

which represents the lexicographic maximin ordering approach to the original problem (1). Problem (26) is a regularization of the standard maximin optimization (5), but in the former, in addition to the worst outcome, we maximize also the second worst outcome (provided that the smallest one remains as large as possible), maximize the third worst (provided that the two smallest remain as large as possible), and so on. Due to (14), the MMF problem (26) is equivalent to the problem:

$$\operatorname{lexmax} \{\bar{\Theta}(\mathbf{f}(\mathbf{x})) \;:\; \mathbf{x} \in Q\}$$

which leads us to a standard lexicographic optimization with predefined linear criteria defined according to (17).

Moreover, in the case of LP models, every fair solution can be identified as an optimal solution to some OWA problem with appropriate monotonic weights [6] but such a search process is usually difficult to control. Better controllability and the complete parameterization of nondominated solutions even for non-convex, discrete problems can be achieved with the direct use of the reference point methodology introduced by Wierzbicki [20] and later extended leading to efficient implementations of the so-called aspiration/reservation based decision support (ARBDS) approach with many successful applications [8]. The ARBDS approach is an interactive technique allowing the DM to specify the requirements in terms of aspiration and reservation levels, i.e., by introducing acceptable and required values for several criteria. Depending on the specified aspiration and reservation levels, a special scalarizing achievement function is built which may be directly interpreted as expressing utility to be maximized.

Maximization of the scalarizing achievement function generates an efficient solution to the multiple criteria problem. The solution is accepted by the DM or some modifications of the aspiration and reservation levels are introduced to continue the search for a better solution. The ARBDS approach provides a complete parameterization of the efficient set to multi-criteria optimization. Hence, when applying the ARBDS methodology to the ordered cumulated criteria in (15), one may generate all (fairly) equitably efficient solutions of the original problem (1).

While building the scalarizing achievement function the following properties of the preference model are assumed. First of all, for any individual outcome η_k more is preferred to less (maximization). To meet this requirement the function must be strictly increasing with respect to each outcome. Second, a solution with all individual outcomes η_k satisfying the corresponding reservation levels is preferred to any solution with at least one individual outcome worse (smaller) than its reservation level. Next, provided that all the reservation levels are satisfied, a solution with all individual outcomes η_k equal to the corresponding aspiration levels is preferred to any solution with at least one individual outcome worse (smaller) than its aspiration level. That means, the scalarizing achievement function maximization must enforce reaching the reservation levels prior to further improving of criteria. In other words, the reservation levels represent some soft lower bounds on the maximized criteria. When all these lower bounds are satisfied, then the optimization process attempts to reach the aspiration levels.

The generic scalarizing achievement function takes the following form [20]:

$$\sigma(\eta) = \min_{k=1,\ldots,m} \{\sigma_k(\eta_k)\} + \varepsilon \sum_{k=1}^{m} \sigma_k(\eta_k) \tag{27}$$

where ε is an arbitrary small positive number and σ_k, for $k = 1, \ldots, m$, are the partial achievement functions measuring actual achievement of the individual outcome η_k with respect to the corresponding aspiration and reservation levels (η_k^a and η_k^r, respectively). Thus the scalarizing achievement function is, essentially, defined by the worst partial (individual) achievement but additionally regularized with the sum of all partial achievements. The regularization term is introduced only to guarantee the solution efficiency in the case when the maximization of the main term (the worst partial achievement) results in a non-unique optimal solution.

The partial achievement function σ_k can be interpreted as a measure of the DM's satisfaction with the current value (outcome) of the k-th criterion. It is a strictly increasing function of outcome η_k with value $\sigma_k = 1$ if $\eta_k = \eta_k^a$, and $\sigma_k = 0$ for $\eta_k = \eta_k^r$. Thus the partial achievement functions map the outcomes values onto a normalized scale of the DM's satisfaction. Various functions can be built meeting those requirements [21]. We use the piece wise linear partial achievement function introduced in [12]. It is given by:

$$\sigma_k(\eta_k) = \begin{cases} \gamma(\eta_k - \eta_k^r)/(\eta_k^a - \eta_k^r), & \text{for } \eta_k \leqslant \eta_k^r \\ (\eta_k - \eta_k^r)/(\eta_k^a - \eta_k^r), & \text{for } \eta_k^r < \eta_k < \eta_k^a \\ \beta(\eta_k - \eta_k^a)/(\eta_k^a - \eta_k^r) + 1, & \text{for } \eta_k \geqslant \eta_k^a \end{cases} \quad (28)$$

where β and γ are arbitrarily defined parameters satisfying $0 < \beta < 1 < \gamma$. This partial achievement function is strictly increasing and concave which guarantees its LP computability with respect to outcomes η_k.

Recall that in our model outcomes η_k represent cumulative ordered outcomes, i.e. $\eta_k = \sum_{i=1}^{k} \theta_i(\mathbf{y})$. Hence, the reference vectors (aspiration and reservation) represent, in fact, some reference distributions of outcomes. Moreover, due to the cumulation of outcomes, while considering equal outcomes ϕ as the reference (aspiration or reservation) distribution, one needs to set the corresponding levels as $\eta_k = k\phi$. Certainly, one may specify any desired reference distribution in terms of the ordered values of the outcomes (quantiles in the probability language) $\phi_1 \leqslant \phi_2 \leqslant \ldots \leqslant \phi_m$ and cumulating them automatically get the reference values for the outcomes η_k representing the cumulated ordered values. However, such rich modeling technique may be too complicated to control effectively the search for a compromise solution.

Although defined with simple linear constraints the auxiliary conditions (17) introduces m^2 additional variables and inequalities into the original model. This may cause a serious computational burden for real-life problems containing numerous outcomes. In order to reduce the problem size one may attempt the restrict the number of criteria in the problem (15).

Let us consider a sequence of indices $K = \{k_1, k_2, \ldots, k_q\}$, where $1 = k_1 < k_2 < \ldots < k_{q-1} < k_q = m$, and the corresponding restricted form of the multiple criteria model (15):

$$\max \{(\eta_{k_1}, \eta_{k_2}, \ldots, \eta_{k_q}) \; : \; \eta_k = \bar{\theta}_k(\mathbf{f}(\mathbf{x})) \quad \text{for } k \in K, \quad \mathbf{x} \in Q\} \quad (29)$$

with only $q < m$ criteria. Following Proposition 2, multiple criteria model (15) allows us to generate any fairly efficient solution of problem (1). Reducing the number of criteria we restrict these opportunities. Nevertheless, one may still generate reasonable compromise solutions. First of all the following assertion is valid.

Theorem 1 If $\mathbf{x}^o$ is an efficient solution of the restricted problem (29), then it is an efficient (Pareto-optimal) solution of the multiple criteria problem (1) and it can be fairly dominated only by another efficient solution $\mathbf{x}'$ of (29) with exactly the same values of criteria: $\bar{\theta}_k(\mathbf{f}(\mathbf{x}')) = \bar{\theta}_k(\mathbf{f}(\mathbf{x}^o))$ for all $k \in K$.

Proof Suppose, there exists $\mathbf{x}' \in Q$ which dominates $\mathbf{x}^o$. This means, $y_i' = f_i(\mathbf{x}') \geqslant y_i^o = f_i(\mathbf{x}^0)$ for all $i \in I$ with at least one inequality strict. Hence, $\bar{\theta}_k(\mathbf{y}') \geqslant \bar{\theta}_k(\mathbf{y}^o)$ for all $k \in K$ and $\bar{\theta}_{k_q}(\mathbf{y}') > \bar{\theta}_{k_q}(\mathbf{y}^o)$ which contradicts efficiency of $\mathbf{x}^o$ within the restricted problem (29).

Suppose now that $\mathbf{x}' \in Q$ fairly dominates $\mathbf{x}^o$. Due to Proposition 2, this means that $\bar{\theta}_i(\mathbf{y}') \geqslant \bar{\theta}_i(\mathbf{y}^o)$ for all $i \in I$ with at least one inequality strict. Hence, $\bar{\theta}_k(\mathbf{y}') \geqslant \bar{\theta}_k(\mathbf{y}^o)$ for all $k \in K$ and any strict inequality would contradict efficiency of $\mathbf{y}^o$ within the restricted problem (29). Thus, $\bar{\theta}_k(\mathbf{y}') = \bar{\theta}_k(\mathbf{y}^o)$ for all $k \in K$ which completes the proof. ∎

It follows from Theorem 1 that while restricting the number of criteria in the multiple criteria model (15) we can essentially still expect reasonably fair efficient solution and only *unfairness* may be related to the distribution of flows within classes of skipped criteria. In other words, we have guaranteed some rough fairness while it can be possibly improved by redistribution of flows within the intervals $(\theta_{k_j}(\mathbf{y}), \theta_{k_{j+1}}(\mathbf{y})]$ for $j = 1, 2, \ldots, q-1$. Since the fairness preferences are usually very sensitive for the smallest flows, one may introduce a grid of criteria $1 = k_1 < k_2 < \ldots < k_{q-1} < k_q = m$ which is dense for smaller indices while sparser for lager indices and expect solution offering some reasonable compromise between fairness and throughput maximization.

CONCLUSIONS

Due to additional requirements on the utility functions the fairly efficient solutions represent a specific subset of all the Pareto-optimal solutions. However, they can be expressed as Pareto-optimal solutions to the problem with modified (ordered and cumulated) criteria. Hence, the simplest way to model a large gamut of fairly efficient decisions may depend on the use some combinations of the ordered criteria, i.e. the so-called Ordered Weighted Averaging (OWA) aggreagtions. If the weights are strictly decreasing each optimal solution corresponding to the OWA maximization is a fair (fairly efficient) solution. Moreover, in the case of LP models every fairly efficient solution can be identified as an OWA optimal solution with appropriate strictly monotonic weights. Several decreasing sequences of weights provide us with various aggregations. Better controllability and the complete parameterization of nondominated solutions even for non-convex, discrete problems can be achieved with the use of the reference point methodology.

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Jaroslav Ramík

DUALITY IN FUZZY MULTIPLE OBJECTIVE LINEAR PROGRAMMING WITH POSSIBILITY AND NECESSITY RELATIONS

INTRODUCTION

The problem of duality has been investigated since the early stage of fuzzy linear programming (FLP), see [3; 9; 11; 12]. In this paper we first introduce a broad class of fuzzy multiple objective linear programming problems (FMOLP problems) and define the concepts of β-feasible and (α, β)-maximal and minimal solutions of FMOLP problems. The class of classical MOLP problems can be embedded into the class of FMOLP ones, moreover, for FMOLP problems we define the concept of duality and prove the weak and strong duality theorems – generalizations of the classical ones. The results are compared to the existing literature [6; 7; 13; 8]. To illustrate the introduced concepts and results we present and discuss a simple numerical example.

1. PRELIMINARIES

Let X be a nonempty topological space. By $\mathcal{F}(X)$ we denote the set of all fuzzy subsets A of X, where every fuzzy subset A of X is uniquely determined by the membership function $\mu_A : X \to [0, 1]$, and $[0, 1] \subset \mathbf{R}$ is a unit interval, $\mathbf{R}$ is the Euclidean space of real numbers. We say that the fuzzy subset A is *crisp* if μ_A is a characteristic function of A, i.e. $\mu_A : X \to \{0, 1\}$. It is clear that the set of all subsets of X, $\mathcal{P}(X)$, can be isomorphically embedded into $\mathcal{F}(X)$.

Let

$$\begin{aligned} [A]_\alpha = & \quad \{x \in X | \mu_A(x) \geqslant \alpha\} \text{ for } \alpha \in (0, 1], \\ [A]_0 = & \quad cl\{x \in X | \mu_A(x) > 0\}\ . \end{aligned}$$

where clB means a topological closure of B, $B \subset X$. For $\alpha \in [0, 1]$, $[A]_\alpha$ are called *α-cuts*. $[A]_0$ is usually called a *support of* A. A fuzzy subset A of X is *closed, bounded, compact* or *convex*, if $[A]_\alpha$ are closed, bounded, compact or convex subsets of X for every $\alpha \in [0, 1]$, respectively. By the *strict α-cut* we denote $(A)_\alpha = \{x \in X | \mu_A(x) > \alpha\}$. Moreover, A is said to be *normal* if $[A]_1$is nonempty. It is a well known fact that a fuzzy subset A of X is convex if and only if its membership function μ_A is quasiconcave on X, see e.g. [6] and also

Definition 4 below.

In the set theory, a *binary relation* P *on* X is a subset of the Cartesian product $X \times X$, that is, $P \subset X \times X$. Here, a *valued relation* P *on* X is a fuzzy subset of $X \times X$. Evidently, any binary relation P *on* X can be isomorphically embedded into the class of valued relations on X by its characteristic function (i.e. membership function) μ_P. In this sense, any binary relation is valued.

Definition 1 A fuzzy subset $\tilde{P}$ of $\mathcal{F}(X) \times \mathcal{F}(X)$ is called a fuzzy relation on X, i.e. $\tilde{P} \in \mathcal{F}(\mathcal{F}(X) \times \mathcal{F}(X))$.

Definition 2 Let P be a valued relation on X. A fuzzy relation $\tilde{Q}$ on X is called a fuzzy extension of relation P, if for each $x, y \in X$, it holds

$$\mu_{\tilde{Q}}(x, y) = \mu_P(x, y) \,. \tag{1}$$

A fuzzy relations on X will be denoted by the tilde, e.g. $\tilde{P}$.

From now on, throughout this paper we shall consider $X = \mathbf{R}^n$, where $\mathbf{R}^n$ is the n-dimensional Euclidean space, particularly $X = \mathbf{R}^1 = \mathbf{R}$.

In the following definition we first present possibility and necessity indices introduced originally in [2] and then define a suitable class of fuzzy numbers called here fuzzy quantities. Then, we shall derive some basic properties of this class.

Definition 3 Let A, B be fuzzy sets with the membership functions $\mu_A : \mathbf{R} \to [0,1]$, $\mu_B : \mathbf{R} \to [0,1]$, respectively. Let

$$Pos(A \preceq B) = \sup\{\min(\mu_A(x), \mu_B(y)) | x \leqslant y, x, y \in \mathbf{R}\}, \tag{2}$$

$$Nec(A \prec B) = \inf\{\max(1 - \mu_A(x), 1 - \mu_B(y)) | x > y, x, y \in \mathbf{R}\}. \tag{3}$$

Here (2) is called the *possibility index*, (3) is called the necessity index.

The possibility and necessity index has been originally introduced in [2], where also mathematical analysis and interpretation of the one has been discussed. The indices (2), (3) can be understood as special fuzzy relations on $\mathbf{R}$ introduced by Definition 1. We write alternatively

$$Pos(A \preceq B) = \mu_{Pos}(A, B) = (A \preceq^{Pos} B), \tag{4}$$

$$Nec(A \prec B) = \mu_{Nec}(A, B) = (A \prec^{Nec} B), \tag{5}$$

where $\mu_\Omega : \mathcal{F}(\mathbf{R}) \times \mathcal{F}(\mathbf{R}) \to [0,1], \Omega \in \{Pos, Nec\}$ are the membership functions of the fuzzy relations on $\mathbf{R}$. By $A \succeq^{Pos} B$ or $A \succ^{Nec} B$ we mean $B \preceq^{Pos} A$ or $B \prec^{Nec} A$, respectively.

It can be easily verified that all possibility and necessity indices are fuzzy extensions of the classical binary relation $\leqslant$ according to Definition 2.

2. FUZZY QUANTITIES

To define a suitable class of fuzzy parameters of FMOLP problems we start with definitions of particular membership functions.

Definition 4 Let A be a normal and compact fuzzy subset of $\mathbf{R}$ with the membership function μ_A. A is called the *fuzzy quantity* if there exist $a, b, c, d \in \mathbf{R}$, $-\infty < a \leqslant b \leqslant c \leqslant d < +\infty$, such that

$$\begin{array}{cc} \mu_A(t) = 0 & \text{if } t < a \text{ or } t > d, \\ \mu_A(t) & \text{is strictly increasing if } a \leqslant t \leqslant b, \\ \mu_A(t) = 1 & \text{if } b \leqslant t \leqslant c, \\ \mu_A(t) & \text{is strictly decreasing if } c \leqslant t \leqslant d. \end{array}$$

The set of all fuzzy quantities is denoted by $\mathcal{F}_0(\mathbf{R})$.

By the definition, $\mathcal{F}_0(\mathbf{R})$ contains well known classes of fuzzy numbers: crisp (real) numbers, crisp intervals, triangular fuzzy numbers, trapezoidal and bell-shaped fuzzy numbers etc. However, $\mathcal{F}_0(\mathbf{R})$ does not contain fuzzy sets with "step-like" membership functions. The proof of the following proposition is easy and can be found also in [5], or [4].

Proposition 5 Let $A, B \in \mathcal{F}(\mathbf{R})$, $\alpha \in [0,1]$. Then
(i) $\mu_{Pos}(A, B) \geqslant \alpha$ iff $\inf[A]_\alpha \leqslant \sup[B]_\alpha$,
(ii) $\mu_{Nec}(A, B) \geqslant \alpha$ iff $\sup(A)_{1-\alpha} \leqslant \inf(B)_{1-\alpha}$.

Proposition 6 Let $A \in \mathcal{F}_0(\mathbf{R})$ be a fuzzy quantity, $\alpha \in [0,1]$. Then

$$\inf[A]_\alpha = \inf(A)_\alpha, \sup[A]_\alpha = \sup(A)_\alpha. \tag{6}$$

Corollary 7 Let $A, B \in \mathcal{F}_0(\mathbf{R})$ be fuzzy quantities $\alpha \in (0,1)$. Then combining Propositions 5 and 6 we obtain (i),(ii) as follows

$$\mu_{Pos}(A, B) \geqslant \alpha \text{ iff } \inf[A]_\alpha \leqslant \sup[B]_\alpha, \tag{7}$$

$$\mu_{Nec}(A, B) \geqslant \alpha \text{ iff } \sup[A]_{1-\alpha} \leqslant \inf[B]_{1-\alpha}. \tag{8}$$

Corollary 8 Let $A, B \in \mathcal{F}_0(\mathbf{R})$ be fuzzy quantities $\alpha \in (0,1)$. Then from (7) and (8) we obtain the following formulae

$$\mu_{Pos}(B, A) < \alpha \text{ iff } \sup[A]_\alpha < \inf[B]_\alpha, \tag{9}$$

$$\mu_{Nec}(B, A) < \alpha \text{ iff } \inf[A]_{1-\alpha} < \sup[B]_{1-\alpha}. \tag{10}$$

Corollary 8 will be useful in deriving properties of α-efficient solutions of fuzzy linear problems we deal in the next section.

3. MULTIPLE OBJECTIVE LINEAR PROGRAMMING PROBLEM WITH FUZZY COEFFICIENTS

In this section we introduce a fuzzy multiple objective linear programming problem (FMOLP problem) where coefficients are fuzzy quantities.

Let $\mathcal{K} = \{1, 2, ..., k\}$,$\mathcal{M} = \{1, 2, ..., m\}$, $\mathcal{N} = \{1, 2, ..., n\}$, k, m, n be positive integers. The *multiple objective linear programming problem* (MOLP problem) is a problem

$$\begin{array}{ll} \text{maximize} & z_q = c_{q1}x_1 + \cdots + c_{qn}x_n \ , q \in \mathcal{K}, \\ \text{subject to} & \\ & a_{i1}x_1 + \cdots + a_{in}x_n \ \leqslant \ b_i, \, i \in \mathcal{M}, \\ & x_j \geqslant 0, \, j \in \mathcal{N}. \end{array} \tag{11}$$

In contrast to the classical MOLP problem, here, the coefficients c_{qj}, a_{ij} and b_i will be fuzzy quantities. The fuzzy quantities will be denoted by symbols with the tilde above. Let $\mu_{\tilde{c}_{qj}} : \mathbf{R} \to [0, 1]$, $\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0, 1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0, 1]$, $q \in \mathcal{K}$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, be membership functions of the fuzzy quantities $\tilde{c}_{qj}$, $\tilde{a}_{ij}$ and $\tilde{b}_i$, respectively. Applying the Extension principle we can easily prove the following property.

Proposition 9 Let $\tilde{c}_{qj}, \ \tilde{a}_{ij} \in \mathcal{F}_0(\mathbf{R}), x_j \geqslant 0, \ q \in \mathcal{K}, \ i \in \mathcal{M}, \ j \in \mathcal{N}$. Then the fuzzy sets $\tilde{c}_{q1}x_1\tilde{+}\cdots\tilde{+}\tilde{c}_{qn}x_n$, $\tilde{a}_{i1}x_1\tilde{+}\cdots\tilde{+}\tilde{a}_{in}x_n$ defined by the Extension principle are again fuzzy quantities.

Let $\tilde{P}$ be a fuzzy relation - fuzzy extension of the usual binary relation $\leqslant$ on $\mathbf{R}$.

The *fuzzy multiple objective linear programming problem* (FMOLP problem) associated with a standard MOLP problem (11) is denoted as

$$\begin{array}{l} \text{"maximize"} \ \tilde{z}_q = \ \tilde{c}_{q1}x_1\tilde{+}\cdots\tilde{+}\tilde{c}_{qn}x_n \ , q \in \mathcal{K}, \\ \text{"subject to"} \\ \qquad (\tilde{a}_{i1}x_1\tilde{+}\cdots\tilde{+}\tilde{a}_{in}x_n) \ \tilde{P} \ \tilde{b}_i, \, i \in \mathcal{M}, \\ \qquad\qquad x_j \geqslant 0, \, j \in \mathcal{N}. \end{array} \tag{12}$$

In (12) the value $\tilde{a}_{i1}x_1\tilde{+}\cdots\tilde{+}\tilde{a}_{in}x_n \in \mathcal{F}_0(\mathbf{R})$ is “compared” with a fuzzy quantity $\tilde{b}_i \in \mathcal{F}_0(\mathbf{R})$ by some fuzzy relation $\tilde{P}$. The ”maximization” of the objective functions denoted by ”maximize” $\tilde{z}_q = \tilde{c}_{q1}x_1\tilde{+}\cdots\tilde{+}\tilde{c}_{qn}x_n$ (in quotation marks) will be investigated later on. Now, we shall deal with the constraints of FMOLP problem (12), see also [5; 55; 8].

4. FEASIBLE REGION, β-FEASIBLE SOLUTION

Definition 10 Let $\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0,1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, be membership functions of fuzzy quantities $\tilde{a}_{ij}$ and $\tilde{b}_i$, respectively. Let $\tilde{P}$ be a fuzzy extension of a binary relation $\leqslant$ on $\mathbf{R}$.
A fuzzy set $\tilde{X}$, whose membership function $\mu_{\tilde{X}}$ is defined for all $x \in \mathbf{R}^n$ by

$$\mu_{\tilde{X}}(x) = \begin{cases} \min\{\mu_{\tilde{P}}(\tilde{a}_{11}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{1n}x_n, \tilde{b}_1), \cdots, \mu_{\tilde{P}}(\tilde{a}_{m1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{mn}x_n, \tilde{b}_m)\} \\ \qquad \text{if } x_j \geqslant 0 \text{ for all } j \in \mathcal{N}, \\ 0 \qquad \text{otherwise,} \end{cases} \tag{13}$$

is called the *fuzzy set of feasible region* or shortly *feasible region* of the FMOLP problem (12).
For $\beta \in (0,1]$, a vector $x \in [\tilde{X}]_\beta$ is called the *β-feasible solution* of the FMOLP problem (12).

Notice that the feasible region $\tilde{X}$ of a FMOLP problem is a fuzzy set. On the other hand, β-feasible solution is a vector belonging to the β-cut of the feasible region $\tilde{X}$. It is not difficult to show, that if all coefficients $\tilde{a}_{ij}$ and $\tilde{b}_i$ are crisp fuzzy quantities, i.e. they are isomorfic to the corresponding real numbers, then the fuzzy feasible region is isomorfic to the set of all feasible solutions of the corresponding classical LP problem, see [5], or [6].

Let $\tilde{d}$ be a fuzzy quantity, i.e. $\tilde{d} \in \mathcal{F}_0(\mathbf{R})$, $\beta \in [0,1]$. We shall use the following notation:

$$\begin{aligned} \tilde{d}^{\mathrm{L}}(\beta) &= \inf\left\{t | t \in [\tilde{d}]_\beta\right\} = \inf[\tilde{d}]_\beta, \\ \tilde{d}^{\mathrm{R}}(\beta) &= \sup\left\{t | t \in [\tilde{d}]_\beta\right\} = \sup[\tilde{d}]_\beta. \end{aligned} \tag{14}$$

Proposition 11 Let $\tilde{a}_{ij}$ and $\tilde{b}_i$ be fuzzy quantities and let $x_j \geqslant 0$ for all $i \in \mathcal{M}$, $j \in \mathcal{N}$, let $\beta \in (0,1)$. Moreover, let $\preceq^{Pos}$, $\prec^{Nec}$ be fuzzy extensions of the binary relation $\leqslant$ defined by (4) and (5). Then for $i \in \mathcal{M}$ it holds
(i) $\mu_{\tilde{\preceq}_{Pos}}(\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n, \tilde{b}_i) \geqslant \beta$ if and only if

$$\sum_{j \in \mathcal{N}} \tilde{a}_{ij}^{\mathrm{L}}(\beta) x_j \leqslant \tilde{b}_i^{\mathrm{R}}(\beta), \tag{15}$$

(ii) $\mu_{Nec}^{\prec}(\tilde{a}_{i1}x_1\tilde{+}\cdots\tilde{+}\tilde{a}_{in}x_n, \tilde{b}_i) \geqslant \beta$ if and only if

$$\sum_{j\in\mathcal{N}} \tilde{a}_{ij}^{\mathrm{R}}(1-\beta)x_j \leqslant \tilde{b}_i^{\mathrm{L}}(1-\beta). \tag{16}$$

Proof The proof follows directly from notation 14, Corollary 7, (7), and (8).

Corollary 12 (i) Let $\tilde{P} = \preceq^{Pos}$. A vector $x = (x_1, \cdots, x_n)$ is a β-feasible solution of the FMOLP problem (12) if and only if it is a nonnegative solution of the system of inequalities

$$\sum_{j\in\mathcal{N}} \tilde{a}_{ij}^{\mathrm{L}}(\beta)x_j \leqslant \tilde{b}_i^{\mathrm{R}}(\beta), i \in \mathcal{M}.$$

(ii) Let $\tilde{P} = \prec^{Nec}$. A vector $x = (x_1, \cdots, x_n)$ is a β-feasible solution of the FMOLP problem (12) if and only if it is a nonnegative solution of the system of inequalities

$$\sum_{j\in\mathcal{N}} \tilde{a}_{ij}^{\mathrm{R}}(1-\beta)x_j \leqslant \tilde{b}_i^{\mathrm{L}}(1-\beta), i \in \mathcal{M}.$$

5. MAXIMIZING OBJECTIVE FUNCTIONS

Now, we return to the problem of "maximization" of objective functions $\tilde{z}_q = \tilde{c}_{q1}x_1\tilde{+}\cdots\tilde{+}\tilde{c}_{qn}x_n$ in (12). We look for the "best" fuzzy quantities $\tilde{z}_q$ with respect to the given fuzzy constraints, or, in other words, with respect to the fuzzy set of feasible region of (12).

Definition 13 Let $\tilde{P}$ be a fuzzy relation on $\mathbf{R}$, let $\alpha \in (0, 1]$.
Let $\tilde{a}, \tilde{b}$ be fuzzy quantities, we write

$$\tilde{a}\ \tilde{P}_\alpha\ \tilde{b}, \text{ if } \mu_{\tilde{P}}(\tilde{a}, \tilde{b}) \geqslant \alpha. \tag{17}$$

and call $\tilde{P}_\alpha$ the *α-relation on $\mathbf{R}$ associated to $\tilde{P}$*. We also write

$$\tilde{a}\ \tilde{P}_\alpha^*\ \tilde{b}, \text{ if } \tilde{a}\tilde{P}_\alpha\tilde{b} \text{ and } \mu_{\tilde{P}}(\tilde{b}, \tilde{a}) < \alpha, \tag{18}$$

and call $\tilde{P}_\alpha^*$ the *strict α-relation on $\mathbf{R}$ associated to $\tilde{P}$.*

Notice that $\tilde{P}_\alpha$ and $\tilde{P}_\alpha^*$ are binary relations on the set of fuzzy quantities $\mathcal{F}_0(\mathbf{R})$ being constructed from a fuzzy relation $\tilde{P}$ on the level $\alpha \in (0, 1]$. $\tilde{P}_\alpha^*$ is a strict relation to the relation $\tilde{P}_\alpha$.

If $\tilde{a}$ and $\tilde{b}$ are crisp fuzzy numbers corresponding to real numbers a and b, respectively, and $\tilde{P}$ is a fuzzy extension of relation $\leqslant$, then $a\tilde{P}_\alpha b$ if and only if $a \leqslant b$. Then for $\alpha \in (0,1)$, $a\tilde{P}^*_\alpha\ b$ if and only if $a < b$.

The following proposition is a simple consequence of the above results applied to particular fuzzy relations $\tilde{P} = \preceq^{Pos}$ and $\tilde{P} = \prec^{Nec}$, see [5]

Proposition 14 Let $\tilde{a}$ and $\tilde{b}$ be fuzzy quantities, $\alpha \in (0,1]$.
(i) Let $\tilde{P} = \preceq^{Pos}$ be a fuzzy relation on $\mathbf{R}$ defined by (4). Then

$$\begin{aligned} \tilde{a}\ \tilde{P}_\alpha\ \tilde{b} \quad &\text{iff} \quad \tilde{a}^{\mathrm{L}}(\alpha) \leqslant \tilde{b}^{\mathrm{R}}(\alpha), \\ \tilde{a}\ \tilde{P}^*_\alpha\ \tilde{b} \quad &\text{iff} \quad \tilde{a}^{\mathrm{R}}(\alpha) < \tilde{b}^{\mathrm{L}}(\alpha). \end{aligned} \tag{19}$$

(ii) Let $\tilde{P} = \prec^{Nec}$ be a fuzzy relation on $\mathbf{R}$ defined by (5). Then

$$\begin{aligned} \tilde{a}\ \tilde{P}_\alpha\ \tilde{b} \quad &\text{iff} \quad \tilde{a}^{\mathrm{R}}(1-\alpha) \leqslant \tilde{b}^{\mathrm{L}}(1-\alpha), \\ \tilde{a}\ \tilde{P}^*_\alpha\ \tilde{b} \quad &\text{iff} \quad \tilde{a}^{\mathrm{R}}(1-\alpha) \leqslant \tilde{b}^{\mathrm{L}}(1-\alpha) \text{ and} \\ &\qquad \tilde{a}^{\mathrm{L}}(1-\alpha) < \tilde{b}^{\mathrm{R}}(1-\alpha). \end{aligned} \tag{20}$$

Proof The proof follows from Definition 13, (14), Corollary 7 and Corollary 8 applied to fuzzy relations $\tilde{P} = \preceq^{Pos}$ and $\tilde{P} = \prec^{Nec}$.

An interpretation of the α-relation and strict α-relation on R associated to $\tilde{P}$ when comparing fuzzy quantities $\tilde{a}$ and $\tilde{b}$ is as follows. For a given level of satisfaction $\alpha \in (0,1]$, a fuzzy quantity $\tilde{a}$ "is not better than" fuzzy quantity $\tilde{b}$ with respect to fuzzy relation $\preceq^{Pos}$ if the smallest value of $[\tilde{a}]_\alpha$ is less or equal to the largest value of $[\tilde{b}]_\alpha$. In a sense, it is the *optimistic approach* to the comparison of fuzzy quantities $\tilde{a}$ and $\tilde{b}$ which means that among values of $[\tilde{a}]_\alpha$ and $[\tilde{b}]_\alpha$ there exists a value a of $[\tilde{a}]_\alpha$ and value b of $[\tilde{b}]_\alpha$ such that $a \leqslant b$. Moreover, fuzzy quantity $\tilde{a}$ "is worse than" fuzzy quantity $\tilde{b}$ with respect to fuzzy relation $\preceq^{Pos}$ if the largest value of $[\tilde{a}]_\alpha$ is strictly less than the smallest value of $[\tilde{b}]_\alpha$.

On the other hand, a fuzzy quantity $\tilde{a}$ "is not better than" fuzzy quantity $\tilde{b}$ with respect to fuzzy relation $\prec^{Nec}$ if the largest value of $[\tilde{a}]_{1-\alpha}$ is less or equal to the smallest value of $[\tilde{b}]_{1-\alpha}$. This could be called the *pessimistic approach* to the comparison of fuzzy quantities. The meaning of that is as follows: among all values a of $[\tilde{a}]_{1-\alpha}$ and b of $[\tilde{b}]_{1-\alpha}$ it holds $a \leqslant b$. Moreover, fuzzy quantity $\tilde{a}$ "is worse than" fuzzy quantity $\tilde{b}$ with respect to fuzzy relation $\prec^{Nec}$ if $\tilde{a}$ "is not better than" $\tilde{b}$ and the smallest value of $[\tilde{a}]_{1-\alpha}$ is strictly less than the largest value of $[\tilde{b}]_{1-\alpha}$.

Now, modifying the well known concept of efficient solution in multicriteria optimization we define "maximization" (or "minimization") of the objective functions of FMOLP problem (12). We shall consider a fuzzy relation $\tilde{P}$ on

$\mathbf{R}$ being a fuzzy extension of the usual binary relation $\leqslant$ on $\mathbf{R}$. Here, $\tilde{P}$ is used both for the objective function, and for the constraints. However, we allow for independent, i.e. different satisfaction levels: $\alpha \neq \beta$,where α is considered for the objective functions and β for the constraints, $\alpha, \beta \in (0,1]$. For our convenience we denote the value of the objective functions of (12) alternatively as follows $\tilde{z}_q = \tilde{c}_q^T x = \tilde{c}_{q1}x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_{qn}x_n = \sum_{j\in\mathcal{N}} \tilde{c}_{qj}x_j$.

Definition 15 Let $\tilde{c}_{qj}$, $\tilde{a}_{ij}$ and $\tilde{b}_i$, $q \in \mathcal{K}$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, be fuzzy quantities on $\mathbf{R}$. Let $\tilde{P}$ be a fuzzy relation on $\mathbf{R}$, being also a fuzzy extension of the usual binary relation $\leqslant$ on $\mathbf{R}$, let $\alpha, \beta \in (0,1]$. A β-feasible solution of (12) $x \in [\tilde{X}]_\beta$ is called the (α,β)*-maximal solution of* (12) if there is no $x' \in [\tilde{X}]_\beta$, $x \neq x'$, such that $\tilde{c}_q^T x \tilde{P}_\alpha \tilde{c}_q^T x'$ for all $q \in \mathcal{K}$ and $\tilde{c}_q^T x \tilde{P}_\alpha^* \tilde{c}_q^T x'$ for at least one $q \in \mathcal{K}$. Here, $\tilde{P}_\alpha^*$ is the strict α-relation on $\mathbf{R}$ associated to $\tilde{P}$.

Notice that any (α,β)-maximal solution of the FLP problem is a β-feasible solution of the FMOLP problem with some additional property concerning the values of the objective functions. Clearly, if all coefficients of FMOLP problem (12) are crisp fuzzy quantities, then (α,β)-maximal solution of the FLP problem is isomorphic to the classical Pareto-optimal solution of the corresponding LP problem (11). Comparing to the approach of satisficing solution, see [5; 6], we do not need any exogenously given additional fuzzy goal in order to optimize the objective functions.

In the following lemmas and corollary we show some important properties of (α,β)-maximal solutions of (12) in case of special fuzzy extensions of the binary relation $\leqslant$, particularly $\preceq^{Pos}$, and $\prec^{Nec}$. The corresponding proofs are straightforward.

Lemma 16 Let $\tilde{c}_{qj}$, $q \in \mathcal{K}$, $j \in \mathcal{N}$, be fuzzy quantities on $\mathbf{R}$ and let $\alpha \in (0,1)$. Let $\tilde{P} = \preceq^{Pos}$ be a fuzzy relation on $\mathbf{R}$ defined by (4) and let $\tilde{P}_\alpha^*$ be the strict α-relation on $\mathbf{R}$ associated to $\tilde{P}$. Nonnegative vectors $x = (x_1, \cdots, x_n), x' = (x'_1, \cdots, x'_n)$ satisfy

$$\tilde{c}_q^T x \tilde{P}_\alpha^* \tilde{c}_q^T x'$$

if and only if

$$\sum_{j\in\mathcal{N}} \tilde{c}_{qj}^{\mathrm{R}}(\alpha) x_j < \sum_{j\in\mathcal{N}} \tilde{c}_{qj}^{\mathrm{L}}(\alpha) x'_j. \tag{21}$$

Corollary 17 If (21) is satisfied then

$$\sum_{j\in\mathcal{N}} \tilde{c}_{qj}^{\mathrm{L}}(\alpha) x_j < \sum_{j\in\mathcal{N}} \tilde{c}_{qj}^{\mathrm{L}}(\alpha) x'_j, \tag{22}$$

$$\sum_{j\in\mathcal{N}} \tilde{c}^{\mathrm{R}}_{qj}(\alpha)x_j < \sum_{j\in\mathcal{N}} \tilde{c}^{\mathrm{R}}_{qj}(\alpha)x'_j. \tag{23}$$

Lemma 18 Let $\tilde{c}_{qj}$, $q \in \mathcal{K}$, $j \in \mathcal{N}$, be fuzzy quantities on $\mathbf{R}$ and let $\alpha \in (0,1)$. Let $\tilde{P} = \prec^{Nec}$ be a fuzzy relation on $\mathbf{R}$ defined by (5) and let $\tilde{P}^*_\alpha$ be the strict α-relation on $\mathbf{R}$ associated to $\tilde{P}$. Nonnegative vectors $x = (x_1, \cdots, x_n), x' = (x'_1, \cdots, x'_n)$ satisfy

$$\tilde{c}^T_q x \tilde{P}^*_\alpha \tilde{c}^T_q x'$$

if and only if

$$\sum_{j\in\mathcal{N}} \tilde{c}^{\mathrm{R}}_{qj}(1-\alpha)x_j \leqslant \sum_{j\in\mathcal{N}} \tilde{c}^{\mathrm{L}}_{qj}(1-\alpha)x'_j, \tag{24}$$

$$\sum_{j\in\mathcal{N}} \tilde{c}^{\mathrm{L}}_{qj}(1-\alpha)x_j < \sum_{j\in\mathcal{N}} \tilde{c}^{\mathrm{R}}_{qj}(1-\alpha)x'_j. \tag{25}$$

The following two propositions give some sufficient conditions for $x^*$to be an (α, β)-maximal solution of FMOLP problem (12).

Proposition 19 Let $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be fuzzy quantities for all $q \in \mathcal{K}$, $i \in \mathcal{M}$ and $j \in \mathcal{N}$, let $\alpha, \beta \in (0,1)$.
Let $\tilde{X}$ be a feasible region of FMOLP problem (12) with $\tilde{P} = \preceq^{Pos}$. Let c_{qj} be such that $\tilde{c}^{\mathrm{L}}_{qj}(\alpha) \leqslant c_{qj} \leqslant \tilde{c}^{\mathrm{R}}_{qj}(\alpha)$ for all $q \in \mathcal{K}$, $j \in \mathcal{N}$. If $x^* = (x^*_1, \cdots, x^*_n)$ is a Pareto-optimal solution of the MOLP problem

$$\begin{array}{ll} \text{maximize} & z_q = \sum_{j\in\mathcal{N}} c_{qj}x_j, q \in \mathcal{K}, \\ \text{subject to} & \sum_{j\in\mathcal{N}} \tilde{a}^{\mathrm{L}}_{ij}(\beta)x_j \leqslant \tilde{b}^{\mathrm{R}}_i(\beta), i \in \mathcal{M}, \\ & x_j \geqslant 0, \quad j \in \mathcal{N}, \end{array} \tag{26}$$

then x^* is an (α, β)-maximal solution of FMOLP problem (12).

The next proposition is a modification of Proposition 19 for $\tilde{P} = \prec^{Nec}$.

Proposition 20 Let $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be fuzzy quantities for all $q \in \mathcal{K}$, $i \in \mathcal{M}$ and $j \in \mathcal{N}$, $\alpha, \beta \in (0,1)$. Let $\tilde{X}$ be a feasible region of FMOLP problem (12) with $\tilde{P} = \prec^{Nec}$. Let c_{qj} be such that $\tilde{c}^{\mathrm{L}}_{qj}(\alpha) \leqslant c_{qj} \leqslant \tilde{c}^{\mathrm{R}}_{qj}(\alpha)$ for all $q \in \mathcal{K}$, $j \in \mathcal{N}$. If $x^* = (x^*_1, \cdots, x^*_n)$ is a Pareto-optimal solution of the MOLP problem

$$\begin{array}{ll} \text{maximize} & z = \sum_{j\in\mathcal{N}} c_{qj}x_j, q \in \mathcal{K}, \\ \text{subject to} & \sum_{j\in\mathcal{N}} \tilde{a}^{\mathrm{R}}_{ij}(\beta)x_j \leqslant \tilde{b}^{\mathrm{L}}_i(\beta), i \in \mathcal{M}, \\ & x_j \geqslant 0, \quad j \in \mathcal{N}, \end{array} \tag{27}$$

then x^* is an $(1-\alpha, 1-\beta)$-maximal solution of FMOLP problem (12).

6. DUAL PROBLEM

In this section we shall investigate the well known concept of duality in LP for FMOLP problems based on possibility and necessity fuzzy relations $\preceq^{Pos}$, and $\prec^{Nec}$. Similar approach for single objective FLP problems can be found in [8], some results for different concept of "optimal" solution of FLP based on satisficing solutions, can be found in [5] and [4]. Here, we derive some innovation of weak and strong duality theorems which extend the known results for LP problems.

Consider the following FMOLP problem

$$\begin{array}{lll} & \text{"maximize"} & \tilde{z}_q = \tilde{c}_{q1}x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_{qn}x_n, q \in \mathcal{K}, \\ \text{(P)} & \text{"subject to"} & (\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n)\ \tilde{P}\ \tilde{b}_i, i \in \mathcal{M}, \\ & & x_j \geqslant 0, \quad j \in \mathcal{N}, \end{array} \tag{28}$$

where $\tilde{c}_{qj}$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ are fuzzy quantities with membership functions $\mu_{\tilde{c}_{jq}} : \mathbf{R} \to [0,1]$, $\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0,1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$, $q \in \mathcal{K}, i \in \mathcal{M}$, $j \in \mathcal{N}$.

FMOLP problem (28) will be called the *primal FMOLP problem (P)*. The feasible region of (P) is introduced by Definition 10 and (α, β)-maximal solution is defined by Definition 15.

The *dual FMOLP problem (D)* can be formulated as follows

$$\begin{array}{lll} & \text{"minimize"} & \tilde{w} = \tilde{b}_1 y_1 \tilde{+} \cdots \tilde{+} \tilde{b}_m y_m \\ \text{(D)} & \text{"subject to"} & \tilde{c}_{qj}\ \tilde{Q}(\tilde{a}_{1j}y_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{mj}y_m), q \in \mathcal{K}, j \in \mathcal{N}, \\ & & y_i \geqslant 0, \quad i \in \mathcal{M}. \end{array} \tag{29}$$

Here, either $\tilde{P} = \preceq^{Pos}$, $\tilde{Q} = \prec^{Nec}$, or $\tilde{P} = \prec^{Nec}$, $\tilde{Q} = \preceq^{Pos}$. In problem (P), "maximization" is considered with respect to fuzzy relation $\tilde{P}$, in problem (D), "minimization" is considered with respect to fuzzy relation $\tilde{Q}$. Notice that the dual problem (D) is a single criterion FLP problem. The pair of FMOLP problems (P) and (D), i.e. (28) and (29), is called the *primal - dual pair of FMOLP problems*. Now, we define a concept of feasible region of (D), that is, a modification of Definition 10.

Definition 21 Let $\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0,1]$ and $\mu_{\tilde{c}_{qj}} : \mathbf{R} \to [0,1]$, $q \in \mathcal{K}, i \in \mathcal{M}$, $j \in \mathcal{N}$, be membership functions of fuzzy quantities $\tilde{a}_{ij}$ and $\tilde{c}_{qj}$, respectively. Let $\tilde{P}$ be a fuzzy extension of a binary relation P on $\mathbf{R}$.

A fuzzy set $\tilde{Y}$, whose membership function $\mu_{\tilde{Y}}$ is defined for all $y \in \mathbf{R}^m$ by

$$\mu_{\tilde{Y}}(y) = \begin{cases} \min\{\mu_{\tilde{P}}(\tilde{c}_{11}, \tilde{a}_{11}y_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{m1}y_m), \cdots, \mu_{\tilde{P}}(\tilde{c}_{kn}, \tilde{a}_{1n}y_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{mn}y_m)\} \\ \quad \text{if } y_i \geqslant 0 \text{ for all } i \in \mathcal{M}, \\ 0 \quad \text{otherwise,} \end{cases} \tag{30}$$

is called a *fuzzy set of feasible region* or shortly *feasible region* of dual FMOLP problem (29).
For $\beta \in (0, 1]$, a vector $y \in [\tilde{Y}]_\beta$ is called the *β-feasible solution* of dual FMOLP problem (29).

Now, we define an "optimal solution" of the dual FMOLP problem (D).

Definition 22 Let $\tilde{c}_{qj}$, $\tilde{a}_{ij}$ and $\tilde{b}_i$, $q \in \mathcal{K}, i \in \mathcal{M}$, $j \in \mathcal{N}$, be fuzzy quantities on $\mathbf{R}$. Let $\tilde{Q}$ be a fuzzy relation on $\mathbf{R}$ - fuzzy extension of the usual binary relation $\leqslant$ on $\mathbf{R}$, and let $\alpha, \beta \in (0, 1]$. A β-feasible solution of (29) $y \in [\tilde{Y}]_\beta$ is called the (α, β)*-minimal solution of* (29) if there is no $y' \in [\tilde{Y}]_\beta$, $y' \neq y$, such that $\tilde{b}^T y' \; \tilde{Q}^*_\alpha \; \tilde{b}^T y$. Here, $\tilde{Q}^*_\alpha$ is the strict α-relation on $\mathbf{R}$ associated to $\tilde{Q}$.

Let P be the usual binary relation $\leqslant$ on $\mathbf{R}$. Now, we shall investigate FMOLP problems (28) and (29) with pairs of dual fuzzy relations in the constraints, particularly, either $\tilde{P} = \preceq^{Pos}$, $\tilde{Q} = \prec^{Nec}$ or $\tilde{P} = \prec^{Nec}$, $\tilde{Q} = \preceq^{Pos}$, see (4), (5). The values of objective functions $\tilde{z}_q$ and $\tilde{w}$ are "maximized " and "minimized", with respect to fuzzy relation $\tilde{P}$ and $\tilde{Q}$, respectively.

The feasible region of the primal FMOLP problem (P) is denoted by $\tilde{X}$, the feasible region of the dual FMOLP problem (D) by $\tilde{Y}$. Clearly, $\tilde{X}$ is a fuzzy subset of $\mathbf{R}^n$, $\tilde{Y}$ is a fuzzy subset of $\mathbf{R}^m$.

The following proposition is a useful modification of Proposition 11.

Proposition 23 Let $\tilde{c}_{qj}$ and $\tilde{a}_{ij}$ be fuzzy quantities and let $y_i \geqslant 0$ for all $q \in \mathcal{K}, i \in \mathcal{M}$, $j \in \mathcal{N}$, $\alpha \in (0, 1)$. Let $\preceq^{Pos}$ and $\prec^{Nec}$ be fuzzy extensions of the binary relation $\leqslant$ defined by (4), (5). Then for $q \in \mathcal{K}, j \in \mathcal{N}$, it holds

(i) $\mu_{\preceq^{Pos}}(\tilde{c}_{qj}, \tilde{a}_{1j}y_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{mj}y_m) \geqslant \beta$ iff

$$\sum_{i \in \mathcal{M}} \tilde{a}^{\mathrm{R}}_{ij}(\beta) y_i \geqslant \tilde{c}^{\mathrm{L}}_{qj}(\beta), \tag{31}$$

(ii) $\mu_{\prec^{Nec}}(\tilde{c}_{qj}, \tilde{a}_{1j}y_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{mj}y_m) \geqslant \beta$ iff

$$\sum_{i \in \mathcal{M}} \tilde{a}^{\mathrm{L}}_{ij}(1-\beta) y_i \geqslant \tilde{c}^{\mathrm{R}}_{qj}(1-\beta). \tag{32}$$

Corollary 24 (i) Let $\tilde{P} = \preceq^{Pos}$. A vector $y = (y_1, \cdots, y_m)$ is an β-feasible solution of the FLP problem (29) if and only if it is a nonnegative solution of the system of inequalities

$$\sum_{i \in \mathcal{M}} \tilde{a}_{ij}^{\mathrm{R}}(\beta) y_i \geqslant \tilde{c}_{qj}^{\mathrm{L}}(\beta), q \in \mathcal{K}, j \in \mathcal{N}.$$

(ii) Let $\tilde{P} = \prec^{Nec}$. A vector $y = (y_1, \cdots, y_m)$ is an α-feasible solution of the FLP problem (29) if and only if it is a nonnegative solution of the system of inequalities

$$\sum_{i \in \mathcal{M}} \tilde{a}_{ij}^{\mathrm{L}}(1-\beta) y_i \geqslant \tilde{c}_{qj}^{\mathrm{R}}(1-\beta), j \in \mathcal{N}.$$

Lemma 25 Let $\tilde{b}_i$, $i \in \mathcal{M}$, be fuzzy quantities on $\mathbf{R}$. Let $\tilde{P} = \preceq^{Pos}$ be fuzzy relations on $\mathbf{R}$ defined by (4) and $\alpha \in (0,1)$. Then $y = (y_1, \cdots, y_m), y' = (y'_1, \cdots, y'_m)$ are nonnegative vectors such that $\tilde{b}^T y' \ \tilde{P}^*_\alpha \ \tilde{b}^T y$, where $\tilde{P}^*_\alpha$ is the strict α-relation on $\mathbf{R}$ associated to $\tilde{P}$, if and only if

$$\sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{R}}(\alpha) y'_i < \sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{L}}(\alpha) y_i. \tag{33}$$

Corollary 26 If (**??**) is satisfied then

$$\sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{L}}(\alpha) y'_i < \sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{L}}(\alpha) y_i, \tag{34}$$

$$\sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{R}}(\alpha) y'_i < \sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{R}}(\alpha) y_i. \tag{35}$$

Lemma 27 Let $\tilde{b}_i$, $i \in \mathcal{M}$, be fuzzy quantities on $\mathbf{R}$. Let $\tilde{P} = \prec^{Nec}$ be a fuzzy relation on $\mathbf{R}$ defined by (4) and $\alpha \in (0,1)$. The vectors $y = (y_1, \cdots, y_m), y' = (y'_1, \cdots, y'_m)$ are nonnegative with $\tilde{b}^T y' \ \tilde{P}^*_\alpha \ \tilde{b}^T y$, where $\tilde{P}^*_\alpha$ is the strict α-relation on $\mathbf{R}$ associated to $\tilde{P}$, if and only if

$$\sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{R}}(1-\alpha) y'_i \leqslant \sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{L}}(1-\alpha) y_i, \tag{36}$$

$$\sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{L}}(1-\alpha) y'_i < \sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{R}}(1-\alpha) y_i. \tag{37}$$

The following propositions give sufficient conditions for y^* to be an (α, β)-minimal solution of FMOLP problem (D).

Proposition 28 Let $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be fuzzy quantities for all $q \in \mathcal{K}, i \in \mathcal{M}$ and $j \in \mathcal{N}$, $\alpha, \beta \in (0,1)$. Let $\tilde{Y}$ be a feasible region of FMOLP problem (29) with $\tilde{P} = \preceq^{Pos}$. Let b_i be such that $\tilde{b}_i^{\mathrm{L}}(\alpha) \leqslant b_i \leqslant \tilde{b}_i^{\mathrm{R}}(\alpha)$ for all $i \in \mathcal{M}$. If $y^* = (y_1^*, \cdots, y_m^*)$ is an optimal solution of the LP problem

$$\begin{array}{ll} \text{minimize} & w = \sum_{i \in \mathcal{M}} b_i y_i \\ \text{subject to} & \sum_{i \in \mathcal{M}} \tilde{a}_{ij}^{\mathrm{R}}(\beta) y_i \geqslant \tilde{c}_{qj}^{\mathrm{L}}(\beta), q \in \mathcal{K}, j \in \mathcal{N}, \\ & y_i \geqslant 0, \quad i \in \mathcal{M}, \end{array} \tag{38}$$

then y^* is a (α, β)-minimal solution of FMOLP problem (D).

The following proposition is a simple and useful modification of Proposition 28.

Proposition 29 Let $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be fuzzy quantities for all $q \in \mathcal{K}, i \in \mathcal{M}$ and $j \in \mathcal{N}$, $\alpha, \beta \in (0,1)$. Let $\tilde{X}$ be a feasible region of FMOLP problem (P) with $\tilde{P} = \prec^{Nec}$. Let b_i be such that $\tilde{b}_i^{\mathrm{L}}(\alpha) \leqslant b_i \leqslant \tilde{b}_i^{\mathrm{R}}(\alpha)$ for all $i \in \mathcal{M}$. If $y^* = (y_1^*, \cdots, y_m^*)$ is an optimal solution of the LP problem

$$\begin{array}{ll} \text{minimize} & w = \sum_{i \in \mathcal{M}} b_i y_i \\ \text{subject to} & \sum_{i \in \mathcal{M}} \tilde{a}_{ij}^{\mathrm{L}}(\beta) y_i \geqslant \tilde{c}_{qj}^{\mathrm{R}}(\beta), q \in \mathcal{K}, j \in \mathcal{N}, \\ & y_i \geqslant 0, \quad i \in \mathcal{M}, \end{array} \tag{39}$$

then y^* is an $(1-\alpha, 1-\beta)$-minimal solution of FMOLP problem (D).

7. WEAK AND STRONG DUALITY THEOREMS

Now, we focus our attention to duality theory for FMOLP problems (see also [7; 12; 8]). In the following duality theorems we present always two versions: (i) for fuzzy relation $\preceq^{Pos}$ in problem (P) and (ii) for fuzzy relation $\prec^{Nec}$in problem (P). In order to prove duality results we assume that the level of satisfaction α of the objective function is equal to the level of satisfaction β of the constraints. Otherwise, the duality theorems in our formulation do not hold.

Moreover, we assume that each objective function is associated with a weight $w_q > 0, q \in \mathcal{K}$, such that $\sum_{q \in \mathcal{K}} w_q = 1$, where w_q may be interpreted as a relative importance of the q-th objective function.

Theorem 30 *First Weak Duality Theorem.* Let $\tilde{c}_{qj}$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be fuzzy quantities, $q \in \mathcal{K}, i \in \mathcal{M}$ and $j \in \mathcal{N}$, $\alpha \in (0,1)$.
(i) Let $\tilde{X}$ be a feasible region of FMOLP problem (28) with $\tilde{P} = \preceq^{Pos}$, $\tilde{Y}$ be a feasible region of FMOLP problem (29) with $\tilde{Q} = \prec^{Nec}$.
If a vector $x = (x_1, \ldots, x_n) \geqslant 0$ belongs to $[\tilde{X}]_\alpha$ and $y = (y_1, \ldots, y_m) \geqslant 0$ belongs to $[\tilde{Y}]_{1-\alpha}$, then

$$\sum_{j \in \mathcal{N}} \tilde{c}^{\mathrm{R}}_{qj}(\alpha) x_j \leqslant \sum_{i \in \mathcal{M}} \tilde{b}^{\mathrm{R}}_i(\alpha) y_i. \tag{40}$$

(ii) Let $\tilde{X}$ be a feasible region of FMOLP problem (28) with $\tilde{P} = \prec^{Nec}$, $\tilde{Y}$ be a feasible region of FMOLP problem (29) with $\tilde{Q} = \preceq^{Pos}$.
If a vector $x = (x_1, \ldots, x_n) \geqslant 0$ belongs to $[\tilde{X}]_{1-\alpha}$ and $y = (y_1, \ldots, y_m) \geqslant 0$ belongs to $[\tilde{Y}]_\alpha$, then

$$\sum_{j \in \mathcal{N}} \tilde{c}^{\mathrm{L}}_{qj}(\alpha) x_j \leqslant \sum_{i \in \mathcal{M}} \tilde{b}^{\mathrm{L}}_i(\alpha) y_i. \tag{41}$$

Proof (i) Let $q \in \mathcal{K}, x \in [\tilde{X}]_\alpha$ and $y \in [\tilde{Y}]_{1-\alpha}$, $x_j \geqslant 0$, $y_i \geqslant 0$ for all $i \in \mathcal{M}$, $j \in \mathcal{N}$. Then by Proposition 23 (ii), multiplying both sides by nonnegative x_j and summing up for $j \in \mathcal{N}$ we obtain

$$\sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{M}} \tilde{a}^{\mathrm{L}}_{ij}(\alpha) y_i x_j \geqslant \sum_{j \in \mathcal{N}} \tilde{c}^{\mathrm{R}}_{qj}(\alpha) x_j. \tag{42}$$

In a similar way, by Proposition 11 (i) we obtain

$$\sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{M}} \tilde{a}^{\mathrm{L}}_{ij}(\alpha) x_j y_i \leqslant \sum_{i \in \mathcal{M}} \tilde{b}^{\mathrm{R}}_i(\alpha) y_i. \tag{43}$$

Combining inequalities (42) and (43), we obtain

$$\sum_{j \in \mathcal{N}} \tilde{c}^{\mathrm{R}}_{qj}(\alpha) x_j \leqslant \sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{M}} \tilde{a}^{\mathrm{L}}_{ij}(\alpha) x_j y_i \leqslant \sum_{i \in \mathcal{M}} \tilde{b}^{\mathrm{R}}_i(\alpha) y_i,$$

which is the desired result.

(ii) Let $q \in \mathcal{K}, x \in [\tilde{X}]_{1-\alpha}$ and $y \in [\tilde{Y}]_\alpha$, $x_j \geqslant 0, y_i \geqslant 0$ for all $i \in \mathcal{M}$, $j \in \mathcal{N}$. Then by Proposition 23 (i), multiplying both sides by x_j and summing up we obtain

$$\sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{M}} \tilde{a}^{\mathrm{R}}_{ij}(\alpha) y_i x_j \geqslant \sum_{j \in \mathcal{N}} \tilde{c}^{\mathrm{L}}_{qj}(\alpha) x_j.$$

In a similar way, by Proposition 11 (ii) with α instead of $1-\alpha$ we obtain

$$\sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{M}}\tilde{a}^{\mathrm{R}}_{ij}(\alpha)x_jy_i \leqslant \sum_{i\in\mathcal{M}}\tilde{b}^{\mathrm{L}}_i(\alpha)y_i.$$

Combining the last two inequalities, we obtain

$$\sum_{j\in\mathcal{N}}\tilde{c}^{\mathrm{L}}_{qj}(\alpha)x_j \leqslant \sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{M}}\tilde{a}^{\mathrm{R}}_{ij}(\alpha)x_jy_i \leqslant \sum_{i\in\mathcal{M}}\tilde{b}^{\mathrm{L}}_i(\alpha)y_i.$$

Corollary 31 Let $w_q > 0$, for all $q\in\mathcal{K}$, such that $\sum_{q\in\mathcal{K}} w_q = 1$.

(i) If a vector $x=(x_1,\dots,x_n)\geqslant 0$ belongs to $[\tilde{X}]_\alpha$ and $y=(y_1,\dots,y_m)\geqslant 0$ belongs to $[\tilde{Y}]_{1-\alpha}$, then

$$\sum_{q\in\mathcal{K}}\sum_{j\in\mathcal{N}}w_q\tilde{c}^{\mathrm{R}}_{qj}(\alpha)x_j \leqslant \sum_{i\in\mathcal{M}}\tilde{b}^{\mathrm{R}}_i(\alpha)y_i. \tag{44}$$

(ii) If a vector $x=(x_1,\dots,x_n)\geqslant 0$ belongs to $[\tilde{X}]_{!-\alpha}$ and $y=(y_1,\dots,y_m)\geqslant 0$ belongs to $[\tilde{Y}]_\alpha$, then

$$\sum_{q\in\mathcal{K}}\sum_{j\in\mathcal{N}}w_q\tilde{c}^{\mathrm{L}}_{qj}(\alpha)x_j \leqslant \sum_{i\in\mathcal{M}}\tilde{b}^{\mathrm{L}}_i(\alpha)y_i. \tag{45}$$

Theorem 32 *Second Weak Duality Theorem.* Let $\tilde{c}_{qj}$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be fuzzy quantities for all $q\in\mathcal{K}, i\in\mathcal{M}$ and $j\in\mathcal{N}$, $\alpha\in(0,1)$, moreover, $w_q>0, q\in\mathcal{K}$, such that $\sum_{q\in\mathcal{K}} w_q = 1$.

(i) Let $\tilde{X}$ be a feasible region of FMOLP problem (28) with $\tilde{P} = \preceq^{Pos}$, $\tilde{Y}$ be a feasible region of FMOLP problem (29) with $\tilde{Q} = \prec^{Nec}$.

If for some $x=(x_1,...,x_n)\geqslant 0$ belonging to $[\tilde{X}]_\alpha$ and $y=(y_1,...,y_m)\geqslant 0$ belonging to $[\tilde{Y}]_{1-\alpha}$ it holds

$$\sum_{q\in\mathcal{K}}\sum_{j\in\mathcal{N}}w_q\tilde{c}^{\mathrm{R}}_{qj}(\alpha)x_j = \sum_{i\in\mathcal{M}}\tilde{b}^{\mathrm{R}}_i(\alpha)y_i, \tag{46}$$

for some $q\in\mathcal{K}$, then x is an (α,α)-maximal solution of FMOLP problem (P), (28) and y is an $(1-\alpha,1-\alpha)$-minimal solution of FMOLP problem (D), (29).

(ii) Let $\tilde{X}$ be a feasible region of FMOLP problem (28) with $\tilde{P} = \prec^{Nec}$, $\tilde{Y}$ be a feasible region of FMOLP problem (29) with $\tilde{Q} = \preceq^{Pos}$.

If for some $x=(x_1,...,x_n)\geqslant 0$ belonging to $[\tilde{X}]_{1-\alpha}$ and $y=(y_1,...,y_m)\geqslant 0$ belonging to $[\tilde{Y}]_\alpha$ it holds

$$\sum_{q\in\mathcal{K}}\sum_{j\in\mathcal{N}}w_q\tilde{c}^{\mathrm{L}}_{qj}(\alpha)x_j = \sum_{i\in\mathcal{M}}\tilde{b}^{\mathrm{L}}_i(\alpha)y_i, \tag{47}$$

for some $q \in \mathcal{K}$, then x is an $(1-\alpha, 1-\alpha)$-maximal solution of FMOLP problem (P), (28) and y is an (α, α)-minimal solution of FMOLP problem (D), (29).

Proof (i) Let $x \in [\tilde{X}]_\alpha$ and $y \in [\tilde{Y}]_{1-\alpha}$, $x_j \geqslant 0$, $y_i \geqslant 0$ for all $i \in \mathcal{M}$, $j \in \mathcal{N}$. Then by Theorem 30 (i), inequality (40) is satisfied.

By equality (46), x is a Pareto-optimal solution of LP problem (26) with $\beta = \alpha$, $z_q = \sum\limits_{j\in\mathcal{N}} c_{qj}x_j = \sum\limits_{j\in\mathcal{N}} \tilde{c}^{\mathrm{R}}_{qj}(\alpha)x_j$ and y is an optimal solution of MOLP problem (39) with $\beta = \alpha$, $w = \sum\limits_{i\in\mathcal{M}} b_i y_i = \sum\limits_{i\in\mathcal{M}} \tilde{b}^{\mathrm{R}}_i(\alpha)y_i$. By Proposition 19, x is an (α, α)-maximal solution of FMOLP problem (P) and by Proposition 29, y is an $(1-\alpha, 1-\alpha)$-minimal solution of FMOLP problem (D).

(ii) Let $x \in [\tilde{X}]_{1-\alpha}$ and $y \in [\tilde{Y}]_\alpha$, $x_j \geqslant 0, y_i \geqslant 0$ for all $i \in \mathcal{M}$, $j \in \mathcal{N}$. Then by Theorem 30 (ii), inequality (41) is satisfied.

By equality (47), x is an optimal solution of MOLP problem (27) with $\beta = \alpha$, $z_q = \sum\limits_{j\in\mathcal{N}} c_{qj}x_j = \sum\limits_{j\in\mathcal{N}} \tilde{c}^{\mathrm{L}}_{qj}(\alpha)x_j$ and y is an optimal solution of MOLP problem (39) with $\beta = \alpha$, $w = \sum\limits_{i\in\mathcal{M}} b_i y_i = \sum\limits_{i\in\mathcal{M}} \tilde{b}^{\mathrm{L}}_i(\alpha)y_i$. By Proposition 20, x is an $(1-\alpha, 1-\alpha)$-maximal solution of FLP problem (P) and by Proposition 28, y is an (α, α)-minimal solution of FLP problem (D).

Remarks.

1. In the crisp and single-objective case, Theorems 30 and 32 are the standard LP Weak Duality Theorems.

2. Let $\alpha \geqslant 0,5$. Then $[\tilde{X}]_\alpha \subset [\tilde{X}]_{1-\alpha}$, $[\tilde{Y}]_\alpha \subset [\tilde{Y}]_{1-\alpha}$, hence in the First Weak Duality Theorem we can change the assumptions as follows: $x \in [\tilde{X}]_\alpha$ and $y \in [\tilde{Y}]_\alpha$. However, the statements of the theorem remain unchanged. The same holds for the Second Weak Duality Theorem.

Finally, let us turn to the *strong duality*. Motivated by the pairs of Propositions 19, 29 and Propositions 20, 28 in Theorem 32, we consider a pair of dual LP problems corresponding to FLP problems (28) and (29) with fuzzy relations $\tilde{P} = \preceq^{Pos}$, $\tilde{Q} = \prec^{Nec}$, assuming $\alpha = \beta$, particularly

$$
\text{(P1)}\quad \begin{array}{ll} \text{maximize} & z_q = \sum_{j\in\mathcal{N}} \tilde{c}^{\mathrm{R}}_{qj}(\alpha)x_j, q \in \mathcal{K}, \\ \text{subject to} & \sum_{j\in\mathcal{N}} \tilde{a}^{\mathrm{L}}_{ij}(\alpha)x_j \leqslant \tilde{b}^{\mathrm{R}}_i(\alpha), i \in \mathcal{M}, \\ & x_j \geqslant 0, \quad j \in \mathcal{N}, \end{array} \tag{48}
$$

$$
\text{(D1)}\quad \begin{array}{ll} \text{minimize} & w = \sum_{i\in\mathcal{M}} \tilde{b}^{\mathrm{R}}_i(\alpha)y_i \\ \text{subject to} & \sum_{i\in\mathcal{M}} \tilde{a}^{\mathrm{L}}_{ij}(\alpha)y_i \geqslant \tilde{c}^{\mathrm{R}}_{qj}(\alpha), q \in \mathcal{K}, j \in \mathcal{N}, \\ & y_i \geqslant 0, \quad i \in \mathcal{M}. \end{array} \tag{49}
$$

Moreover, we consider a pair of dual LP problems with fuzzy relations $\tilde{P} = \prec^{Nec}$, $\tilde{P}^D = \preceq^{Pos}$:

$$\text{(P2)} \quad \begin{array}{ll} \text{maximize} & z_q = \sum_{j\in\mathcal{N}} \tilde{c}^{\mathrm{L}}_{qj}(\alpha)x_j, q \in \mathcal{K}, \\ \text{subject to} & \sum_{j\in\mathcal{N}} \tilde{a}^{\mathrm{R}}_{ij}(\alpha)x_j \leqslant \tilde{b}^{\mathrm{L}}_i(\alpha), i \in \mathcal{M}, \\ & x_j \geqslant 0, \quad j \in \mathcal{N}, \end{array} \tag{50}$$

$$\text{(D2)} \quad \begin{array}{ll} \text{minimize} & w = \sum_{i\in\mathcal{M}} \tilde{b}^{\mathrm{L}}_i(\alpha)y_i \\ \text{subject to} & \sum_{i\in\mathcal{M}} \tilde{a}^{\mathrm{R}}_{ij}(\alpha)y_i \geqslant \tilde{c}^{\mathrm{L}}_{qj}(\alpha), q \in \mathcal{K}, j \in \mathcal{N}, \\ & y_i \geqslant 0, \quad i \in \mathcal{M}. \end{array} \tag{51}$$

Notice that in case of single objective problem, (P1) and (D1) are classical dual LP problems and the same holds for (P2) and (D2).

Theorem 33 *Strong Duality Theorem.* Let $\tilde{c}_{qj}$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be fuzzy quantities for all $q \in \mathcal{K}, i \in \mathcal{M}$ and $j \in \mathcal{N}$, let $w_q > 0$, for all $q \in \mathcal{K}$, such that $\sum_{q\in\mathcal{K}} w_q = 1$.
(i) Let $\tilde{X}$ be a feasible region of FMOLP problem (28) with $\tilde{P} = \preceq^{Pos}$, $\tilde{Y}$ be a feasible region of FMOLP problem (29) with $\tilde{Q} = \prec^{Nec}$. If for some $\alpha \in (0, 1)$, $[\tilde{X}]_\alpha$ and $[\tilde{Y}]_{1-\alpha}$ are nonempty, then there exists x^*- an (α, α) -maximal solution of FMOLP problem (P), and there exists y^* - an $(1-\alpha, 1-\alpha)$-minimal solution of FMOLP problem (D) such that

$$\sum_{q\in\mathcal{K}} \sum_{j\in\mathcal{N}} w_q \tilde{c}^{\mathrm{R}}_{qj}(\alpha)x^*_j = \sum_{i\in\mathcal{M}} \tilde{b}^{\mathrm{R}}_i(\alpha)y^*_i. \tag{52}$$

(ii) Let $\tilde{X}$ be a feasible region of FMOLP problem (28) with $\tilde{P} = \prec^{Nec}$, $\tilde{Y}$ be a feasible region of FMOLP problem (29) with $\tilde{Q} = \preceq^{Pos}$. If for some $\alpha \in (0, 1)$, $[\tilde{X}]_{1-\alpha}$ and $[\tilde{Y}]_\alpha$ are nonempty, then there exists x^*- an $(1-\alpha, 1-\alpha)$-maximal solution of FLP problem (P), and y^* - an (α, α)-minimal solution of FLP problem (D) such that

$$\sum_{q\in\mathcal{K}} \sum_{j\in\mathcal{N}} w_q \tilde{c}^{\mathrm{L}}_{qj}(\alpha)x^*_j = \sum_{i\in\mathcal{M}} \tilde{b}^{\mathrm{L}}_i(\alpha)y^*_i. \tag{53}$$

Proof (i) Clearly, $[\tilde{X}]_\alpha$ is the set of all $\alpha-$feasible solutions of MOLP problem (P1) and $[\tilde{Y}]_{1-\alpha}$ is the set of all $(1-\alpha)-$feasible solutions of MOLP problem (D1), we assume that they are both nonempty. As (P1) and (D1) are dual MOLP problems in the usual sense, there exists $x^* \in [\tilde{X}]_\alpha$- a Pareto-optimal solution of (P1), and $y^* \in [\tilde{Y}]_{1-\alpha}$- an optimal solution of (D1), such that (52) holds.

It remains to prove that x^* is an (α, α)-maximal solution of FMOLP problem (28), and $y^*$is an $(1-\alpha, 1-\alpha)$-minimal solution of FMOLP problem (29).

By Proposition 19, x^* is an (α, α)-maximal solution of FLP problem (P1) and by Proposition 29, y^* is an $(1-\alpha, 1-\alpha)$-minimal solution of FLP problem (D1).

Part (ii) can be proven analogically using Propositions 20 and 28.

Remarks.

1. In the crisp and single-objective case, Theorem 33 is a standard LP (Strong) Duality Theorem.

2. Let $\alpha \geqslant 0,5$. Then $[\tilde{X}]_\alpha \subset [\tilde{X}]_{1-\alpha}$, $[\tilde{Y}]_\alpha \subset [\tilde{Y}]_{1-\alpha}$, hence in the Strong Duality Theorem we can assume $x \in [\tilde{X}]_\alpha$ and $y \in [\tilde{Y}]_\alpha$. Evidently, the statement of the theorem remains unchanged.

3. Theorem 33 provides only the existence of the (α, α)-maximal solution (or $(1-\alpha, 1-\alpha)$-maximal solution) of FMOLP problem (P), and $(1-\alpha, 1-\alpha)$-minimal solution ((α, α)-minimal solution) of FMOLP problem (D) such that (52) or (53) holds. However, the proof of the theorem gives also the method for finding the solutions by solving (MO)LP problems (P1) and (D1).

4. The following questions remain open and can be investigated in the future:

(1) How the theorems could be modified for more general fuzzy extensions of $\leqslant$.

(2) Duality theorems allowing for different satisfaction levels α and β would be interesting.

8. ILLUSTRATIVE EXAMPLE

In this section we discuss a simple illustrative example to clarify the introduced concepts and results, to provide some interpretation and features of possible applications. Last but not least, to solve the multi-objective FLP problem (P) by the single-objective FLP problem (D).

Let two new products A and B be manufactured. The manufacturing process is composed of two sub-processes, Processes 1 and 2. The estimated processing resources (e.g. processing time, materials) for manufacturing a batch of Product A for each process are the following: $\tilde{a}_{11}$ units for Process 1 and $\tilde{a}_{21}$ units for Process 2. On the other hand, the processing resources for manufacturing a batch of Product B for each process are as follows: $\tilde{a}_{12}$ units for Process 1, $\tilde{a}_{22}$ units at Process 2. The working resource for Process 1 is restricted by $\tilde{b}_1$ units, for Process 2 by $\tilde{b}_2$ units. The "profit" rates (1000 CZK/batch) of Products A and B are estimated as $\tilde{c}_{11}$ and $\tilde{c}_{12}$, respectively. The "utility" rates (1000 CZK/batch) of Products A and B are estimated as $\tilde{c}_{21}$ and $\tilde{c}_{22}$, respectively. The weights of the criteria are $w_1 = 0,6$ and $w_2 = 0,4$. All mentioned parameters $\tilde{a}_{ij}$, $\tilde{b}_i$ and

$\tilde{c}_{qj}$ are subjected to uncertainty and they are expressed by fuzzy quantities. We shall investigate what quantity of Products A and B should be manufactured in order to "maximize" the total "profit" and total "utility". For this purpose we formulate the following FMOLP problem (primal problem)

$$
\text{(PE)} \quad \begin{array}{ll}
\text{"maximize"} & \tilde{z}_1 = \tilde{c}_{11}x_1 \tilde{+} \tilde{c}_{12}x_2, \\
& \tilde{z}_2 = \tilde{c}_{21}x_1 \tilde{+} \tilde{c}_{22}x_2 \\
\text{"subject to"} & (\tilde{a}_{11}x_1 \tilde{+} \tilde{a}_{12}x_2)\ \tilde{P}\ \tilde{b}_1, \\
& (\tilde{a}_{21}x_1 \tilde{+} \tilde{a}_{22}x_2)\ \tilde{P}\ \tilde{b}_2, \\
& x_1, x_2 \geqslant 0,
\end{array} \tag{54}
$$

where $\tilde{c}_{qj} = (c^L_{qj}, c_{qj}, c^R_{qj})$, $\tilde{a}_{ij} = (a^L_{ij}, a_{ij}, a^R_{ij})$ and $\tilde{b}_i = (b^L_i, b_i, b^R_i)$ are triangular fuzzy quantities (with triangular piecewise linear membership functions) given by the triples, as usual. Here, we shall consider the following triangular fuzzy quantities

$$
\begin{array}{ll}
\tilde{c}_{11} = (3,4,5), & \tilde{c}_{12} = (2,4,6), \\
\tilde{c}_{21} = (2,3,4), & \tilde{c}_{21} = (3,4,5), \\
\tilde{a}_{11} = (1,3,5), & \tilde{a}_{12} = (1,1,1), \\
\tilde{a}_{21} = (1,3,5), & \tilde{a}_{22} = (3,3,3), \\
\tilde{b}_1 = (8,11,14), & \tilde{b}_2 = (11,12,15).
\end{array} \tag{55}
$$

Notice that $\tilde{a}_{12}$ and $\tilde{a}_{22}$ are crisp fuzzy numbers.

The dual FLP problem to (PE) is formulated as follows

$$
\text{(DE)} \quad \begin{array}{ll}
\text{"minimize"} & \tilde{w} = \tilde{b}_1 y_1 \tilde{+} \tilde{b}_2 y_2 \\
\text{"subject to"} & \tilde{c}_{11} \tilde{Q} (\tilde{a}_{11}y_1 \tilde{+} \tilde{a}_{21}y_2), \\
& \tilde{c}_{12}\ \tilde{Q} (\tilde{a}_{12}y_1 \tilde{+} \tilde{a}_{22}y_2), \\
& \tilde{c}_{21} \tilde{Q} (\tilde{a}_{11}y_1 \tilde{+} \tilde{a}_{21}y_2), \\
& \tilde{c}_{22} \tilde{Q} (\tilde{a}_{12}y_1 \tilde{+} \tilde{a}_{22}y_2), \\
& y_1, y_2 \geqslant 0,
\end{array} \tag{56}
$$

Here, $\tilde{P}$ and $\tilde{Q}$ is a pair of dual fuzzy relations, particularly $\tilde{P} = \preceq^{Pos}$ and $\tilde{Q} = \prec^{Nec}$, see (4), (5).

Given $\alpha, \beta \in (0,1)$, $\alpha = \beta$, by (48) and (49) we obtain the following couple of dual problems

$$
\begin{array}{ll}
\text{maximize} & z_1 = \tilde{c}^{\mathrm{R}}_{11}(\alpha)x_1 \tilde{+} \tilde{c}^{\mathrm{R}}_{12}(\alpha)x_2, \\
& z_2 = \tilde{c}^{\mathrm{R}}_{21}(\alpha)x_1 \tilde{+} \tilde{c}^{\mathrm{R}}_{22}(\alpha)x_2, \\
\text{subject to} & \tilde{a}^{\mathrm{L}}_{11}(\alpha)x_1 \tilde{+} \tilde{a}^{\mathrm{L}}_{12}(\alpha)x_2 \leqslant \tilde{b}^{\mathrm{R}}_1(\alpha), \\
& \tilde{a}^{\mathrm{L}}_{21}(\alpha)x_1 \tilde{+} \tilde{a}^{\mathrm{L}}_{22}(\alpha)x_2 \leqslant \tilde{b}^{\mathrm{R}}_2(\alpha), \\
& x_1, x_2 \geqslant 0,
\end{array} \tag{57}
$$

$$\begin{array}{ll} \text{minimize} & w = \tilde{b}_1^{\mathrm{R}}(\alpha)y_1 \tilde{+} \tilde{b}_2^{\mathrm{R}}(\alpha)y_2, \\ \text{subject to} & \tilde{a}_{11}^{\mathrm{L}}(\alpha)y_1 \tilde{+} \tilde{a}_{21}^{\mathrm{L}}(\alpha)y_2 \geqslant \tilde{c}_{11}^{\mathrm{R}}(\alpha), \\ & \tilde{a}_{12}^{\mathrm{L}}(\alpha)y_1 \tilde{+} \tilde{a}_{22}^{\mathrm{L}}(\alpha)y_2 \geqslant \tilde{c}_{12}^{\mathrm{R}}(\alpha), \\ & \tilde{a}_{11}^{\mathrm{L}}(\alpha)y_1 \tilde{+} \tilde{a}_{21}^{\mathrm{L}}(\alpha)y_2 \geqslant \tilde{c}_{21}^{\mathrm{R}}(\alpha), \\ & \tilde{a}_{12}^{\mathrm{L}}(\alpha)y_1 \tilde{+} \tilde{a}_{22}^{\mathrm{L}}(\alpha)y_2 \geqslant \tilde{c}_{22}^{\mathrm{R}}(\alpha), \\ & y_1, y_2 \geqslant 0, \end{array} \tag{58}$$

As $\preceq^{Pos}$ is an "optimistic" fuzzy relation and $\prec^{Nec}$ is a "pessimistic" one, this couple can be called "optimistic-pessimistic" dual couple. Particularly, substituting (55) into (57), (58), we obtain

$$\begin{array}{ll} \text{maximize} & z_1 = (5-\alpha)x_1 + (6-2\alpha)x_2, \\ & z_2 = (4-\alpha)x_1 + (5-\alpha)x_2, \\ \text{subject to} & (1+2\alpha)x_1 + x_2 \leqslant 14 - 3\alpha, \\ & (1+\alpha)x_1 + 3x_2 \leqslant 15 - 3\alpha, \\ & x_1, x_2 \geqslant 0, \end{array} \tag{59}$$

$$\begin{array}{ll} \text{minimize} & w = (14-3\alpha)y_1 + (15-3\alpha)y_2, \\ \text{subject to} & (1+2\alpha)y_1 + (1+\alpha)y_2 \geqslant 5 - \alpha, \\ & y_1 + \quad 3y_2 \geqslant 6 - 2\alpha, \\ & (1+2\alpha)y_1 + (1+\alpha)y_2 \geqslant 4 - \alpha, \\ & y_1 + \quad 3y_2 \geqslant 5 - \alpha, \\ & y_1, y_2 \geqslant 0. \end{array} \tag{60}$$

On the other hand, let $\tilde{P} = \prec^{Nec}$ and $\tilde{Q} = \preceq^{Pos}$. Then by (50) and (51), with $\alpha = \beta$ we obtain the following couple of dual problems

$$\begin{array}{ll} \text{maximize} & z_1 = \tilde{c}_{11}^{\mathrm{L}}(\alpha)x_1 \tilde{+} \tilde{c}_{12}^{\mathrm{L}}(\alpha)x_2, \\ & z_2 = \tilde{c}_{21}^{\mathrm{L}}(\alpha)x_1 \tilde{+} \tilde{c}_{22}^{\mathrm{L}}(\alpha)x_2, \\ \text{subject to} & \tilde{a}_{11}^{\mathrm{R}}(\alpha)x_1 \tilde{+} \tilde{a}_{12}^{\mathrm{R}}(\alpha)x_2 \leqslant \tilde{b}_1^{\mathrm{L}}(\alpha), \\ & \tilde{a}_{21}^{\mathrm{RL}}(\alpha)x_1 \tilde{+} \tilde{a}_{22}^{\mathrm{R}}(\alpha)x_2 \leqslant \tilde{b}_2^{\mathrm{L}}(\alpha), \\ & x_1, x_2 \geqslant 0, \end{array} \tag{61}$$

$$\begin{array}{ll} \text{minimize} & w = \tilde{b}_1^{\mathrm{L}}(\alpha)y_1 \tilde{+} \tilde{b}_2^{\mathrm{L}}(\alpha)y_2, \\ \text{subject to} & \tilde{a}_{11}^{\mathrm{R}}(\alpha)y_1 \tilde{+} \tilde{a}_{21}^{\mathrm{R}}(\alpha)y_2 \geqslant \tilde{c}_{11}^{\mathrm{L}}(\alpha), \\ & \tilde{a}_{12}^{\mathrm{R}}(\alpha)y_1 \tilde{+} \tilde{a}_{22}^{\mathrm{R}}(\alpha)y_2 \geqslant \tilde{c}_{12}^{\mathrm{L}}(\alpha), \\ & \tilde{a}_{11}^{\mathrm{R}}(\alpha)y_1 \tilde{+} \tilde{a}_{21}^{\mathrm{R}}(\alpha)y_2 \geqslant \tilde{c}_{21}^{\mathrm{L}}(\alpha), \\ & \tilde{a}_{12}^{\mathrm{R}}(\alpha)y_1 \tilde{+} \tilde{a}_{22}^{\mathrm{R}}(\alpha)y_2 \geqslant \tilde{c}_{22}^{\mathrm{L}}(\alpha), \\ & y_1, y_2 \geqslant 0, \end{array} \tag{62}$$

This couple can be called "pessimistic-optimistic" dual couple. Again, substituting (55) into (62), we obtain

$$\begin{array}{ll} \text{maximize} & z_1 = (3+\alpha)x_1 + (2+2\alpha)x_2, \\ & z_2 = (2+\alpha)x_1 + (3+\alpha)x_2, \\ \text{subject to} & (5-2\alpha)x_1 + x_2 \leqslant 8+3\alpha, \\ & (3-\alpha)x_1 + 3x_2 \leqslant 11+\alpha, \\ & x_1, x_2 \geqslant 0, \end{array} \tag{63}$$

$$\begin{array}{ll} \text{minimize} & w = (8+3\alpha)y_1 + (11+\alpha)y_2, \\ \text{subject to} & (5-2\alpha)y_1 + (3-\alpha)y_2 \geqslant 3+\alpha, \\ & y_1 + \quad 3y_2 \geqslant 2+2\alpha, \\ & (5-2\alpha)y_1 + (3-\alpha)y_2 \geqslant 2+\alpha, \\ & y_1 + \quad 3y_2 \geqslant 3+\alpha, \\ & y_1, y_2 \geqslant 0. \end{array} \tag{64}$$

Let $\alpha = \beta = 0,7$ be an appropriate level of satisfaction (degree of satisfaction or, necessity degree) for the objective function and for the constraints. By Simplex method we obtain the following numerical results. The optimal solutions of problems (59), (60), i.e. "optimistic-pessimistic" dual couple are displayed in Table 1.

Table 1

$\tilde{P}$ =	$\preceq^{Pos}$		$\tilde{Q}$ =	$\prec^{Nec}$
$x_1^* =$	4,15		$y_1^* =$	0,74
$x_2^* =$	1,95		$y_2^* =$	1,25
$z^* =$	24,91		$w^* =$	24,91

The optimal solutions of problems (63), (64), i.e. the "pessimistic - optimistic" dual couple are displayed in Table 2.

Table 2

$\tilde{P}$ =	$\prec^{Nec}$		$\tilde{Q}$ =	$\preceq^{Pos}$
$x_1^{**} =$	2,19		$y_1^{**} =$	0,39
$x_2^{**} =$	2,22		$y_2^{**} =$	1,00
$z^{**} =$	15,65		$w^{**} =$	15,65

As is evident from Table 1, the value $z^* =$ 24,91 of the optimal solution of the "optimistic" primal problem is greater than the value $z^{**} =$ 15,65 of the optimal solution of the "pessimistic" primal one. This result is in a correspondence with our expectation. By Strong Duality Theorem $x^* = (4,15; 1,95)$ is a

(0,7; 0,7)-maximal solution of FLP problem (PE), and y^* =(0,74; 1,25) is a (0,3; 0,3)-minimal solution of FLP problem (DE) such that (52) holds, i.e. $z^* = w^*$. Moreover, y^* =(0,74; 1,25) is a vector of dual (shadow) prices of the resources $\tilde{b}_i$ at disposition. The vector y^* is a $(1-\alpha, 1-\alpha)$-minimal solution of the "pessimistic" dual problem with the meaning that the smallest value of $\tilde{a}_{1j}y_1 \tilde{+} \tilde{a}_{2j}y_2$ with the degree of satisfaction at least $1-\alpha$, is less or equal to the largest value of $\tilde{c}_j$ with the degree of satisfaction at least $1-\alpha = 0{,}3$.

Analogical explanation could be formulated for the other dual couple (PE) and (DE) with $\tilde{P} = \prec^{Nec}$ and $\tilde{Q} = \preceq^{Pos}$, i.e. for "pessimistic - optimistic" dual couple. Again by Strong Duality Theorem $x^{**} = (2{,}19;\ 2{,}22)$ is a $(0{,}3;\ 0{,}3)$-maximal solution of FLP problem (PE), and $y^{**} = (0{,}39;\ 1{,}00)$ is a (0,7; 0,7)-minimal solution of FLP problem (DE) such that (52) holds, i.e. $z^{**} = w^{**} =$ 15,65. Here, y^{**} =(0,39; 1,00) is a vector of dual (shadow) prices of the resources $\tilde{b}_i$ at disposition. The vector y^* is a (0,7; 0,7)-minimal solution of the "optimistic" dual problem with the meaning that the largest value of $\tilde{a}_{1j}y_1 \tilde{+} \tilde{a}_{2j}y_2$ with the degree of satisfaction at least 0,7 is at most equal to the smallest value of $\tilde{c}_j$ with the degree of satisfaction at least 0,7.

CONCLUSION

In this paper we introduced a class of FMOLP problems and defined the concepts of β-feasible and (α, β)-maximal and minimal solutions. Our approach here is different to the approaches used in [5] and [6]. Particularly, in [5] and [6] we investigated different concept of "optimal" solution of FLP problem, namely, the concept of satisficing solution, for the comparison of these approaches (see [8]).

In [8], we used a similar concept of α-efficient solutions, however, it was applied in a different way to the single objective FLP problem. Here, we present a more detailed analysis of the MFLP problems focused on duality theory, moreover, an illustrative example is discussed.

In [11] a problem of LP with coefficients belonging to given usual sets have been investigated and duality results have been derived (see also [1]).

Recently, in [13], duality in FLP is investigated for a special relation used for comparing fuzzy numbers, based on other two possibility and necessity indices, namely (2) and (3). A fuzzy relation investigated in [13] is a fuzzy extension of the usual binary relation $\leqslant$, in the sense of Definition 2, however, it is different to fuzzy relations $\preceq^{Pos}$or $\prec^{Nec}$ investigated here.

It is possible to investigate duality in FLP problems even in more general settings. There exist several ways of generalization. For instance, it is possible to extend the duality results to some other classes of fuzzy relations, or, to find some necessary conditions that fuzzy relations for comparing fuzzy numbers should satisfy in order to provide a duality result, or, eventually a duality gap. Moreover, in [6], the concept of dual couples of t-norms and t-conorms has been formulated and dual fuzzy relations have been defined. The role of dual relations in the couple of dual FLP problems should be also clarified and a more general duality theory could be derived. The other way of generalization is based on introducing interactive fuzzy coefficients, or oblique fuzzy vectors (see e.g. [6]).

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Jaideep Roy

Honorata Sosnowska

IMPOSSIBILITY OF STRATEGY-PROOFNESS WITH COALITION FORMATION UNDER TRANSFERABLE UTILITY[*]

INTRODUCTION

A society consists of a set of individuals $I = \{1,\ldots, N\}$ and a set alternatives $A = \{a_1 ,\ldots, a_M\}$. Each individual $i \in I$ has rational preference relation on A. A central issue here is that of aggregating individual preferences into a social preference ordering. Arrow showed that if the admissible domain of preferences is unrestricted and the number of alternatives is at least three, then the aggregate preference is Paretian and Independent of Irrelevant Alternatives if and only if there is an Arrovian dictator[1]. An important aspect of such aggregation programs is that the aggregator (or the planner) is not always aware of the true individual preferences and hence individuals may misrepresent their preferences to affect the social outcome. Such social choice problems may be interpreted as multi-criterial decision making where individual preferences are the criteria and the social planner is a decision maker. Gibbart [6] and Satterthwaite [12] (GS henceforth) showed that any mechanism that wishes to implement truth-telling (that is strategy-proofness) on part of the individuals and is unanimous must also be dictatorial in the sense of Arrow. Several attempts have been made to circumvent this dictatorial misfortune of such aggregation program[2]. In tandem, probabilistic social choice rules have also received significant attention, partly motivated by the desire to escape the dictatorial results generated in the deterministic framework[3]. Under such non-deterministic programs a new and significantly weaker version of dictator-

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[*] We thank Arunava Sen for introducing us probabilistic social choice shemes and random dictatorship and for a discussion on coalitional strategy-proofness. The usual disclaimer applies.

[1] An Arrovian dictator is an individual $h \in I$ such that for every individual preference profile of the society his individual preference is identical with the social preference.

[2] Among several possibility results are ones which allow the social preference to be less than fully rational or where individual preferences are single peaked.

[3] See for example [10; 7; 5; 1].

ship, called random dictatorship, is proposed (for example, see [10]). A probabilistic social choice rule is random-dictatorial if independent of the announcement of profile of preferences, there is a pre-determined probability for each individual with which that individuals most preferred alternative is ranked most preferred by the society as well. In a recent work by Dutta, Peters and Sen (DSP) [4] it is shown that if the planner wishes to implement a strategy-proof probabilistic social decision scheme (which is cardinal) and the number of alternatives is at least three, then the mechanism is unanimous if and only if is a random dictatorship.

In this note we test the GS and the DPS results in face of coalitional mis--reporting. A coalition is any non-empty subset of *I* who agree to jointly report their preference profile. Coalitional strategy-proofnesss has been studied in [3] and [8; 9]. All these papers defined coalitional strategy proofness as situation where for every preference profile of the society, for every non-empty coalition of individuals and for every joint deviation profile for such coalitions, there always exists some member of that coalition for whom the initial profile is not worse. With this approach, Dasgupta et. Al show that the social choice function is monotonic, the domain is rich and the preference domain has a product structure, then the social choice functions coalitionally strategy proof. Allowing for infinitely many individual, Mihara [8] show that any coalitionally strategy proof social choice function must depend only on the top most elements of each individual's preferences. In [9] a concrete example is provided for a coalitionally strategy proof non-dictatorial social choice in case of countably infinite societies. Such an existence was also shown in [11] but in a non-constructive manner.

The definition of coalitional strategy-proofness used in the papers cited above deals with non-transferable utility problems since it does not specify any notion of imputations or redistributions of worth's of coalitions amongst their numbers. In this note we instead concentrate on transferable utility scenarios. We consider two cases, one with a deterministic social choice function where GS is readily applicable and its probabilistic counterpart where DPS is the most natural extension of GS. We show that neither Arrovian nor random dictatorship are sufficient to guarantee coalitional strategy-proofness and particular that it one cannot guarantee to achieve truth-telling in environments where coalitions may be formed and utility is transferable. In the next section we define some terms and in section 3 we prove our main results.

1. DEFINITIONS

Let $I = \{1,...,N\}$ and $A = \{a_1,...,a_M\}$ be the set of individuals and alternatives as defined above. Each individual $i \in I$ has a reference relation on A represented by a utility function $u_i : A \to R$. Let U denote the domain of all such utility functions normalized such that $\max_A \{u_i\} = 1$ and $\min_A \{u_i\} = 0$ for every $i \in I$.

A society is *cooperative* if there exists at least to individuals $i, j \in I$ such that they reveal truthfully their individual utilities to each other and agree to announce jointly a utility pair $(u_i, u_j) \in U^2$.

A society is *minimally cooperative* if there are at most two individuals for whom the society becomes cooperative.

A society is *fully cooperative* if for any nonempty coalition $S \subseteq I$, each member $i \in S$ reveals truthfully his utility to every $j \in S/\{i\}$ and the coalition S agrees to jointly announce a utility profile $u_S = (u_i)_{i \in S} \in U^{|S|}$.

A *cardinal social choice scheme* is a mapping $\varphi : U^N \to O$ where O is the set of outcomes. φ is *deterministic* if $O = A$ while it is *probabilistic* if O is the set $L(A)$ of all possible lotteries over A[4].

Let V_{u_i} be *a value operator* under the utility function u_i. That is $V_{u_i} : O \to R$. We consider only two cases for V_{u_i} :(i) if $O = A$ then $V_{u_i}(a_t) = u_i(a_t), a_t \in A,$ while (ii) if $O = L(A)$ then V_{u_i} is the mathematical expectation under u_i given a lottery λ in $L(A)$. That is:

$$V_{u_i}(\lambda) = \sum_{t=1}^{M} \lambda_t u_i(a_t)$$

where λ_t is the probability of alternative $a_t \in A$ under the lottery λ in $L(A)$.

Given a utility profile $u \in U^N$ and any coalition $S \subseteq I$, the profile $w = (u'_S, u_{-S})$ is:

$$w_j = \begin{cases} u_j & \text{if } j \notin S \\ u'_j & \text{if } j \in S. \end{cases}$$

[4] There are other versions of deterministic schemes, for example, set-valued outcomes. We restrict attention to singleton-set outcomes here.

A social choice scheme φ is Individually *Strategy-proof* if for all $i \in I$, for all $u \in U^N$ and for all $u_i^{'} \in U$ we have $V_{u_i}(\varphi(u)) \geq V_{u_i}(\varphi(u_i^{'}, u_{-i}))$.

We now come to our transferable-utility notion of coalitional strategy proofness.

A social choice scheme φ is *coalitionally manipulable* by some non--empty coalition $S \subseteq I$ at the utility profile $u \in U^N$ via a joint announcement $u_S^{'} \in U^{|S|}$ if $\exists (x_i)_{i \in S}, x_i \in R$ such that:

(i) $\sum_{i \in S} x_i \leq \sum_{i \in S} V_{u_i}(\varphi(u_S^{'}, u_{-S}))$,

(ii) $x_i \geq V_{u_i}(\varphi(u))$ for every $i \in S$ with strict inequality for at least some $i \in S$.

Consequently, a social choice scheme φ is *coalitionally strategy proof* if it is not coalitionally manipulable at any utility profile by any coalition.

Remark 1

If φ is Coalitionally Strategy-proof then must it be Individually Strategy-proof.

We now state clearly the GS and the DPS theorems in our setting.

GS Theorem

Let $\varphi : U \to A$ be a deterministic social choice scheme with $|A| \geq 3$. Then φ is Individually Strategy-proof if and only if it is dictatorial a la Arrow.

Consider any probabilistic cardinal social choice scheme $\varphi : U \to L(A)$. Then φ is a *random dictatorship* if independent of the profile $u \in U^N$, there exists non-negative real numbers δ_i with $\sum_{i \in I} \delta_i = 1$ such that:

$$\varphi^t(u) = \sum_{i \in \{j \in I | u_j(a_t) = 1\}} \delta_i$$

where $\varphi^t(u)$ is the probability that φ attaches to the alternative $a_t \in A$ under the profile $u \in U^N$.

DPS Theorem

Let $\varphi: U \to L(A)$ be a probabilistic social choice scheme with $|A| \geq 3$. Then φ is Unanimous and Individually Strategy-proof if and only if it is a random dictatorship.

2. MAIN RESULT

Theorem 1

Let $\varphi: U \to \mathrm{O}$ be a cardinal social choice scheme with $|A| \geq 3$. Then φ cannot be Coalitionally Strategy-proof even in a Minimally Cooperative Society.

Proof

By the GS and the DPS theorems, it is sufficient to consider dictatorial schemes. The proof is by example. Since U is an unrestricted domain, consider the following preference profile:

$$\begin{bmatrix} 1 & 2 & \ldots & N \\ a & b & . & . \\ b & . & . & . \\ & & & \\ . & a & . & . \end{bmatrix}$$

Here for each individual 1, 2,…. and so on, the top element above is the best etc. Without any loss of generality, asseme that individual 1 receives a utility equal to α from alternative b. Now, consider any dictatorial scheme such that $\delta_i \in [0,1]$ and $\sum_{i \in N} \delta_i = 1$. Note, that this include Arrovian and Random dictatorships. Suppose the coalition {1,2} is formed.. Truth telling yields a total payoff of:

$$\delta_1 + \delta_2(1+\alpha) + B$$

where $B = \sum_{i=3}^{N} \delta_i [u_1(\tau(\succ_i)) + u_2(\tau(\succ_i))]$ and $\tau(\succ_i)$ is the top element of individual i's preference $\succ_i$. Consider the following mis-representation by the coalition $\{1,2\}$:

$$\begin{bmatrix} 1 & 2 & \dots & N \\ b & b & . & . \\ . & . & . & . \\ & & & \\ . & a & . & . \end{bmatrix}$$

Then, total payoff for the coalition $\{1,2\}$ from this misrepresentation, given the previous δ_i is:

$$\delta_1(\alpha+1)+\delta_2(\alpha+1)+B$$

Clearly, for any $\alpha > 0$ we have:

$$\delta_1(\alpha+1)+\delta_2(\alpha+1)+B > \delta_1+\delta_2(1+\alpha)+B$$

which completes the proof.

The theorem shows that whenever utility is expressed in terms of money, strategy-proofness is an impossible virtue.

CONCLUSIONS

The transferable utility may be interpreted as utility measured in terms of money, and money is transferable. Hence any society where there is money, one cannot guarantee truth-telling in case of coalition formation for problems where utility is equivalent to money. However, there are problems where utility is not transferable, for example, happiness, fear etc. In such cases, our results does not apply.

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Edita Šarkienė
Vaidotas Šarka
Leonas Ustinovichius

A MODEL FOR EVALUATING THE INVESTMENT IN THE CONSTRUCTION OF DWELLING HOUSES BASED ON MULTIPLE CRITERIA DECISION SYNTHESIS METHODS

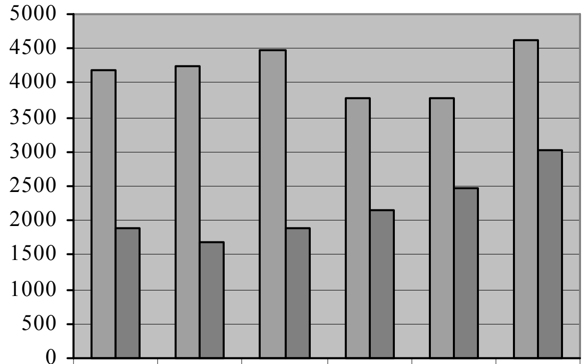
INTRODUCTION

The aim of the present investigation is to answer the question whether it is more advisable to invest in individual dwelling-houses construction in Vilnius or in the construction of apartment houses.

The analysis of the market of individual dwelling-houses in Lithuania has revealed that the construction of many houses is not completed.

In the period from 1998 to 2003 there was a boom in the construction of new dwelling-housing in Lithuania (Figure 1) [12].

In order to achieve the specified objective, an analysis of works of foreign and Lithuanian authors was performed and a decision model based on financial and economical indicators for evaluating dwelling-house construction in Vilnius and profitability of investment in this area compared to that of apartment houses was developed.



	1998	1999	2000	2001	2002	2003
Total number of flats, units	4176	4254	4463	3785	3785	4628
1 - 2 flat dwelling-houses, units	1890	1674	1904	2150	2485	3012

Fig. 1. Statistical data on the construction of new dwelling-houses in Lithuania [12]

1. A SURVEY OF DECISION SUPPORT SYSTEMS AND EVALUATION REQUIREMENTS

The authors carried out an analysis of the decision support systems developed by Lithuanian and foreign researchers, and suggested a decision making model for the analysis of the planning stage of individual house construction. This problem is rather complicated; therefore, at this stage, the authors suggest to analyze the problems of site selection and financial and investment evaluation.

The significant problems of housing construction relating to design, construction, finance and maintenance were considered by many authors [2; 15; 16; 20; 21]. The tools proposed by Park [10] and Piramuthu [11] use information technology to support decision making as well as evaluation methods and data managing in construction.

According to the expert system developed by Christian, preliminary building cost and construction time may be determined at the pre-design stage. Using the data from earlier projects (project price) and referring to experts' knowledge, the system may predict the cost of forthcoming developments [3].

Mohan described the expert systems created in various countries and intended for calendar planning – PLATFORM, site preparation – SI-TEPLAN, analysis of industrial safety – HOWSAFE (Stanford University, USA), as well as a system for determining preliminary cost and time of construction, for making construction plans – PLANEX, analysis of building site – DSCAS (Colorado State University, USA), strategic planning of construction projects – ELSIE (University of Salford, UK), and analysis of construction process and risk analysis (University of Texas, USA and Georgia Institute of Technology, USA) [7].

Ozdoganm and Birgonul (Middle East Technical University, Turkey) developed a model on the basis of which multiple decisions in BOT (Build--operate-transfer) type projects were made in municipal projects with private capital investment. The model was developed to evaluate the selection of contractors and the risks involved. The model was intended to help select a municipality where a private company is allowed to finance, build and operate a project for some time before it is transferred to municipality [9]. This model could be used for municipality housing program development.

Multicriteria analysis of project selection problems was also performed by Nowak [8].

A new multiple criteria decision methodology developed by Zavadskas and Kaklauskas was proposed for solving various multi-objective problems [4; 16; 17; 19].

Realization of a model for complex analysis of a single-family house life cycle was proposed by Kvederyte [5]. Complex analysis model of rational Lithuanian housing was developed by Banaitis [1].

When solving technological and economic problems in construction to find optimal solutions, it is advisable to use decision support systems based on game theory and the normalisation methods proposed and described by Migilinskas and Ustinovichius [6].

2. A MODEL OF ECONOMIC EVALUATION OF DWELLING-HOUSES CONSTRUCTION BASED ON MULTIPLE CRITERIA DECISION SYNTHESIS METHODS

A two-stage model for evaluating the profitability of investment in the construction of dwelling-houses is selected.

At the first stage, the land plot for individual dwelling-houses is evaluated, while at the second stage, a comparative analysis of the possible construction variants of individual dwelling-houses and apartment houses from economical perspective is made.

For decision-making, the authors selected a modern method of multiple criteria project synthesis DSS1 [13; 14]. The method consists in the synthesis of a number of interrelated engineering solutions by selecting the best decisions at any stage of analysis.

Synthesis is a solution incorporating various problems (stage solutions) into a general project based on the relationship matrices [13].

The underlying principle of the two main structural elements of decision support systems is based on the data (DB and their control system) and the major elements of economic evaluation of the construction model of dwelling-houses (the structure of the expert methods, integrated solutions and multiple criteria assessment methods) which are shown in Figure 2.

The first stage of decision-making is intended for stating the problem and selecting a database structure (DBS). At this stage, the data are being collected for a database (i.e. the criteria to be used, the data on the variants, the relationship matrices needed etc.). Sets of closely interrelated criteria to be used at both stages are defined. The criteria referring to the variants under consideration are subdivided into qualitative and quantitative. For calculating the criteria values, descriptions or particular formulas will be used in the model.

For evaluating the qualitative criteria, the descriptions for expert assessment in decision making or integrated pre-assessed qualitative criteria are provided.

Based on the approach described, a decision tree related to the economic evaluation model to be used in the construction of dwelling-houses has been created (Figure 3).

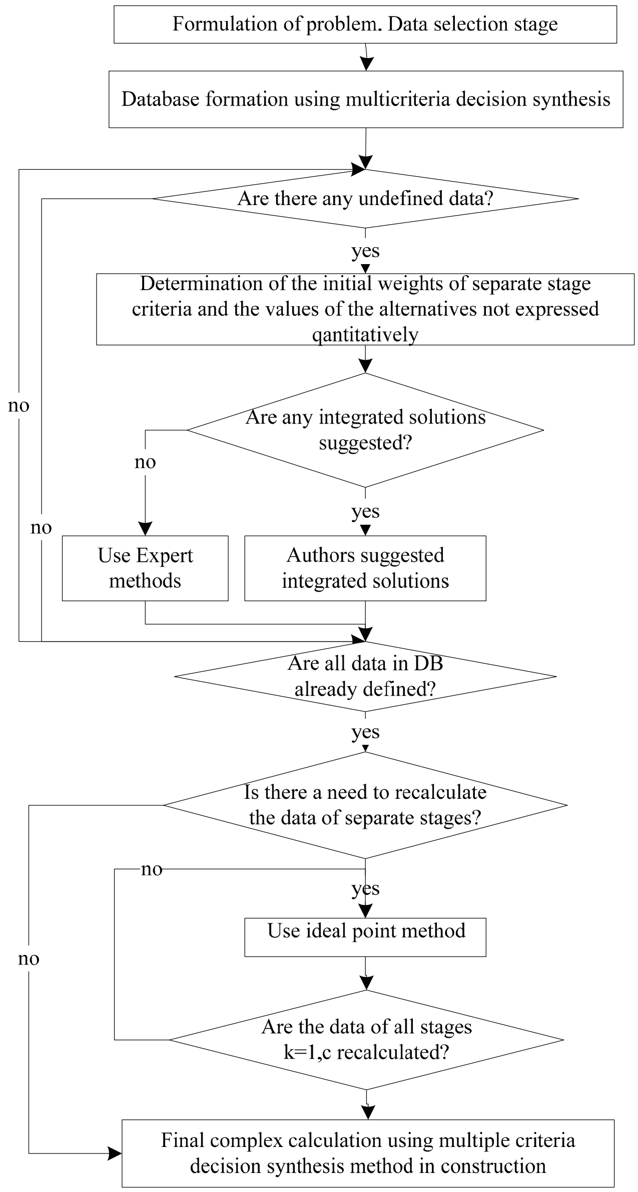


Fig. 2. Economic evaluation of the construction of dwelling-houses based on multiple criteria decision synthesis methods

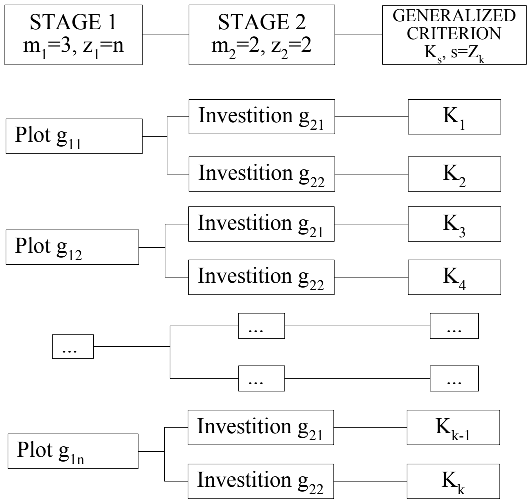


Fig. 3. Fragment of a decision tree (DT) structure created by multiple criteria decision synthesis methods

A set of criteria to be used in selecting a rational investment alternative in the construction of dwelling houses was defined (Figure 4).

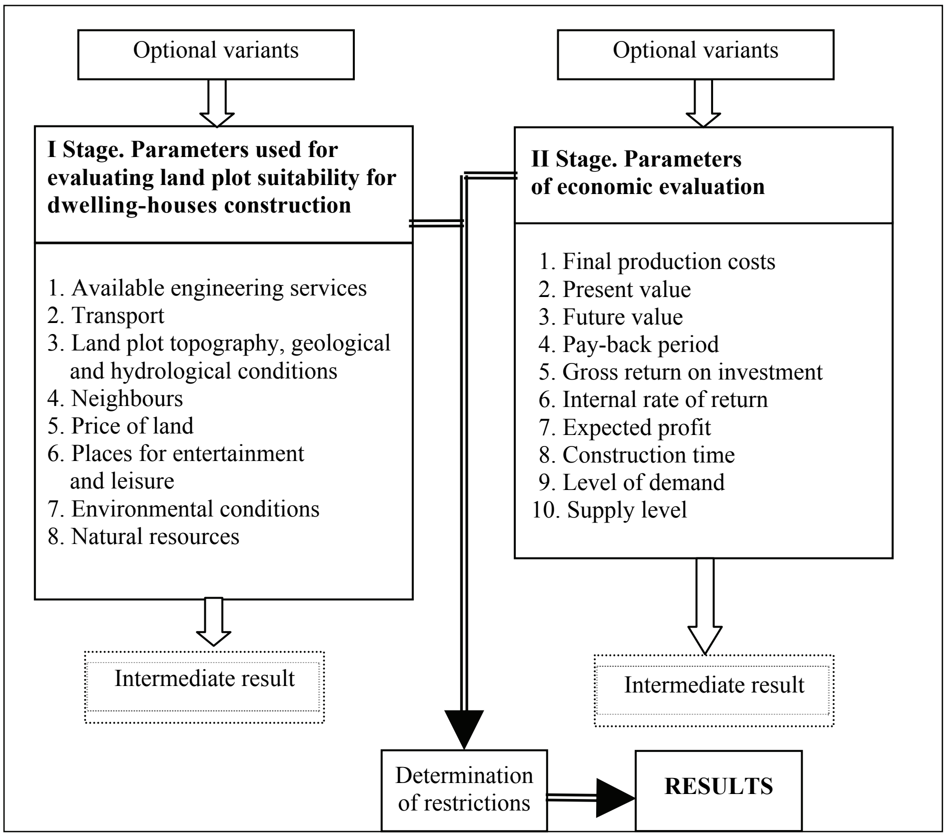


Fig. 4. Major criteria used for evaluating the construction of dwelling-houses

2.1. The criteria describing the land plot

For siting (plot) evaluation the authors have developed a set of integrated criteria. To increase the evaluation reliability and to decrease the influence of subjective factors on the expected result, internal parameters were determined for each criterion. According to the determined internal parameters, possible variants of the criteria were developed and evaluated by the experts. A description of the criteria is presented together with the particular values. If in the process of evaluation the selection is made from the alternatives to which the integrated criterion values cannot be applied, it is recommended to apply the interpolation method and to recalculate the value for a new variant of the newly created criteria.

The criteria describing the land plot are as follows:

Engineering services. According to the existing project, it is necessary to join the water-supply system, sewage, heating and electricity networks. Each of these connections may be assigned points from 1 to 5, depending on the distance to the connection points, because the highest costs are associated with the need to cutting, uplifting or going around the existing systems.

The connections are evaluated in the following way:

– Connection to the existing systems at 50 m distance – 5 points.
– New filiations to existing systems (in suburbs), when there are no more networks requiring uplifting or going around – 4 points.
– Connection to existing systems at 50-100 m distance – 3 points.
– Connection to existing systems over 100 m distance – 2 points.
– Connection to existing systems requiring performing the renovation of the main line or meeting other special requirements – 1 point.

Experts helped to determine the economic effect of the available engineering services and their connections on the project.

Table 1

	P1	P2	...	Pn
E1	xp11	xp21	...	xpn1
E2	xp12	xp22	...	xpn2
...	...	...	...	...
Em	xp1m	xp2m	...	xpnm

Experts with experience of more than 10 years in construction and engineering were selected. The experts were asked to assess the economic effect of each connection by ranking the alternatives (the most significant is No 1, the less significant is No X). By collecting the estimates of the experts, the significance of each connection was determined, with the maximum significance equal to 5 points (Table 1).

Here, E is an expert, P – a connection and x − an estimate.

The expert's estimate of each connection P is obtained from the sum:

$$A_n = \sum_{n=1}^{m} xp_{nm} \tag{1}$$

The average estimate value of each expert is calculated as X:

$$Xpn_{vid} = A_n / m \tag{2}$$

It is assumed that the least value Xpn_{vid} is the best (in other words, this will be the factor with the highest influence on the project) – 5 points (i.e. 100%), thus every other Xp_{vid} is calculated as the deviation from the best result, because it also indicates that in the process of assessment the experts expressed their opinions that the costs of the works of the smallest and the largest assessed Pmn will differ proportionally. Therefore, the significance may be determined for each P:

$$Kpn = 5 / Xpn_{vid} \tag{3}$$

Transport. Laying of roads requires additional costs and is associated with the restrictions of traffic. Therefore, roads and streets and their condition influence the evaluation of land plot. This, however, depends on land usage and its intensity.

As the road network in Vilnius is sufficiently developed, the main factor for transport evaluation is the time needed to get from the land plot to shops and institutions most often visited by people. Each value is assigned points from 1 to 0.1 (Table 2). The maximum estimate is 1 point if the land plot is 5 min. away from the object, 0.1 points if 90 min. or more are needed to reach it. The maximum allowable value of the criteria could be 5 points.

Table 2

	<5 min.	<15 min.	<30 min.	<45 min.	<60 min.	><90 min.
City centre	1	0.8	0.6	0.4	0.2	0.1
School	1	0.8	0.6	0.4	0.2	0.1
Shopping centre	1	0.8	0.6	0.4	02	0.1
Hospital	1	0.8	0.6	0.4	0.2	0.1
Eating house	1	0.8	0.6	0.4	0.2	0.1

The total estimate of each Transport value T is obtained from the sum:

$$T_n = \sum_{n=1}^{m} xp_{nm} \tag{4}$$

Land plot topography, geological and hydrological conditions may restrict decisions on construction and the total price of the building. Soil properties are the decisive factors in design and laying of foundations. The following criteria values are recommended to be used for the evaluation:

- 3 points – good conditions (natural cohesive soil, no remains of demolished buildings, low groundwater level),
- 2 points – average conditions (cohesive soil, remains of demolished buildings, high level of groundwater),
- 1 point – satisfactory conditions (unstable, bulk soil, remains of former demolished buildings, high level of groundwater).

Neighbouring areas. The evaluation is made according to the district and the purposes of nearby buildings. The smaller the built-up area or the farther the neighbouring areas from the project area, the smaller the required investments. The estimates offered are as follows:

– The areas close to the land plot are not built-up (no restrictions for constructions) – 4 points.
– Neighbouring land plots are industrial areas (minimum restrictions) – 3 points.
– Neighbouring land plots are commercial or cultural heritage areas, partially restricted access (many minor restrictions) – 2 points.
– Neighbouring land plots are residential or secured areas (many particular and strictly defined restrictions) – 1 point.

Price of land. Price is one of the most important criteria in siting evaluation. It includes not only the technical and qualitative data of the land plot, but also the prestige of location, traditions, as well as possibilities for expansion and growth. It is recommended to use the average district price for the area.

Places for leisure and entertainment. When evaluating the place with respect to this criterion it is necessary to take into account the location of the recreational centres, i.e. the accessibility (preferably on foot) of cafes, bars, night clubs, shops, museums and theatres. The estimate shows the concentration of these centres in a particular neighbourhood within 1 km^2. The following evaluation criteria for Vilnius were proposed (Table 3):

Table 3

Vilnius district	Value
Old town	8.4
New city	22.4
Zverynas	2.8
Zujūnai	2.8
Valakampiai	3.3
Naujininkai	5.1
Riešė	2.0
Antakalnis	3.9
Antaviliai	0.5

Environmental conditions. Ecologically clean accommodation environment should be one of the most important criteria in evaluation of the land plot in order to ensure the most favourable conditions for the inhabitants of dwelling-houses. The criterion is affected by many factors: traffic flow, overall air and soil pollution in the area, development of manufacture, noise and radiation level.

- Good environmental conditions (not including any aforementioned factors) – 3 points.
- Satisfactory (1-2 factors) – 2 points.
- Poor (3-4 factors) – 1 point.

Natural resources. For each of the described natural resources close to the potential land plot 0.20 points are added to the initial 0.20 points: park/forest, natural water body, ornamental water (basins, ponds), spectacular landscape.

2.2. Economic evaluation

Economic evaluation is made at all stages including activity preparation, the process itself and selling of the product in the past, present and future (formation and usage of labour, material resources, economic and financial indicators of activity).

Economic and financial evaluation of the investment projects is a rational way to make a decision.

A brief description of the economic criteria is presented below.

Final production costs include all the costs necessary for the item production, including land plot.

Economic evaluation of an engineering project is based on the planned cash flow. The flow shows the future investments, incomes and expenses. One of the most significant and complicated issues in evaluation is the forecasting of the future cash flow for the period of 5 years. In this way, the worth of the future money at present is determined.

Pay-back period represents the amount of time needed for the project to recover the investments made.

Gross return on investment is the total profit, received from the project during its lifetime, divided by the amount of investments and expressed in percentage.

Internal rate of return is the rate of return of the project when the present value of all cash flows received from the project is equal to zero. Thus, this rate is a certain profitability measure, while the external economic factors, capable to affect the planned investment flows, are not taken into account.

In the evaluation process, the project with the highest positive internal rate of return is selected.

Expected profit is the calculated weighted average profit based on probabilities. It describes the average or the main trend in probabilistic profit distribution.

The expected profit is the mathematical average of the different possible profit rates. In mathematical statistics the expected magnitude is referred to as the first moment of stochastic distribution. The second moment of stochastic distribution (around the mean) is called the average square deviation.

Construction time. The duration of construction is calculated from the beginning of the pre-design stage to the planned date of the project commissioning.

Level of demand. The year 2003 was one of the most successful for the developers of all types of real estate. Particularly high demand for all types of real estate determined the increase in sales price by 10-40%, depending on the type of real estate. Although quite a high demand was for the rent of premises as well, there was no price increase in this sector of real estate market – prices for rent (especially those for old flats, luxury apartments and houses) are constantly decreasing as the supply of new apartments and houses is increasing.

Under these market conditions, when sales prices are increasing and rent prices are decreasing or remain constant, the profitability of investment in real estate purchasing and the subsequent renting fell to 6-12% in the sector of residential premises. Depending on the market sector, the profit of real estate developers is about 15-30%. However, the increasing building costs and price of land plots as well as growing competition are likely to reduce the profit of real estate valuators in the nearest future.

Determination of supply level. The supply of construction products is evaluated by experts according to the following factors [2]:

- Costs.
- Technologies. With the advances in technology, the price of the products decreases and the quality increases.
- Prices of complementary goods. Manufactured products may complement or substitute each other. For example, cheaper alternative windows decrease the cost of building renovation.
- Other factors: political crises, wars, strikes, interest rate variation etc.

Table 4

Criterion	Arbitrary notation of the criterion	Integrated value of the criterion
Demand level increases	A11	10.0
Demand level is constant	A12	8.0
Demand level decreases	A13	4.0

Table 5

Criterion	Arbitrary notation of the criterion	Integrated value of the criterion
Supply level is lower than demand level (shortage)	A21	10.0
Supply is equal to demand	A22	6.7
Supply level is higher than demand (market surplus)	A23	3

3. EVALUATION OF THE CONSTRUCTION OF DWELLING-HOUSES IN VILNIUS

3.1. Selection of location

Following the description presented in section 2, possible location alternatives for dwelling-houses were selected. For each variant the initial data were generated (presented in Table 6). When the model for evaluating the construction of dwelling-houses based on multiple criteria decision synthesis methods is used, the integrated solutions are available, and the decision maker does not need to ask an expert to evaluate the situation (except for the case when there is a recommended alternative). This saves time of the decision maker. At the first stage of research based on the application of multiple criteria synthesis method (DSS1), the intermediate result was obtained.

As shown in the table, the most suitable land plot for housing construction from economical perspective is in Riese. The land plot suitable for the construction of dwelling-houses in Zujunai was the second, and the third was a plot in New City.

Table 6

Criteria	Expressed or not expressed in monetary units	Units of measure	Min or Max	Weight	New City	Riese	Zujunai
Engineering services	-	points	+	0.17	5	2.5	1.5
Transport	-	points	+	0.14	5	3.6	3.6
Topography of land plot	-	points	+	0.09	2	3	2
Neighbours	-	points	+	0.09	3	4	4
Price of land	+	1000 Lt.	-	0.18	100.00	10.00	15.00
Places for entertainment and leisure	-	%	+	0.07	22.4	2	2.8
Environmental conditions	-	points	+	0.17	2	3	3
Natural resources	-	points	+	0.9	0.20	0.80	0.60

3.2. Economic evaluation

In order to determine the more economical engineering project at the second stage, as a variant of housing construction, the possibility to build an apartment house was selected out of all the criteria described in the methodological part of the paper. The indicators whose values are not the same were used for calculations. For example, only the gross return on investment was selected from the profitability indices because the values of other criteria are identical. When selecting the criteria, it is important to group together related criteria, e.g. costs, pay-back period, profitability indices and so on.

Table 7

Criteria	Units of measurement	Min or Max	Initial significance	Dwelling house (2 units*160m²)	Apartment (8flat*100m²)
Costs:					
Final production costs	Thousand Lt	-	0.10	1009.00	2005.00
Money time value:					
Present value	Lt	-	0.10	0.95	0.95
Future value	Lt	+	0.10	1.34	1.39
Pay-back period	Years	-	0.11	2.5	3.5
Profitability indices:					
Gross return on investment	Percent	+	0.10	111.28	119.70
Internal rate of return	Percent	+	0.10	11.33	9.12
Risk:					
Expected profit	Percent of 1 Lt	+	0.11	0.11	0.14
Construction time	months	-	0.90	10.00	12.00
Level of demand	points	+	0.50	8.00	10.00
Supply level	points	-	0.50	10.00	8.00

The significance of the criteria is determined by expert methods, i.e. by interviewing the experts [14]. The initial data are presented in Tables 6 and 7.

By performing the calculations, the restrictions for dwelling-house siting were determined. The restrictions for the location of an apartment house were not established.

Table 8

Priority of alternatives	Alterna-tive №	K_{bita} value	Variant composition		Value of K_{bit} alternatives
1	3	1.00			
			6	Riese	1.000
			3	Dwelling-house	1.000
2	4	0.99			
			6	Riese	1.000
			4	Apartment	0.989
3	5	0.76			
			10	Zujunai	0.758
			3	Dwelling-house	1.000
4	6	0.75			
			10	Zujunai	0.758
			4	Apartment	0.989

The calculations performed by the SPS_DS system applying multiple criteria synthesis methods yielded a final result. According to it, the construction of a dwelling-house in Riese or Zujunai is most promising from economic perspective.

In general, project alternatives (options) have been made using the initial data, expert methods and multi-stage multiple criteria decision synthesis DSS1 approach. Next, the most efficient siting alternative for dwelling house construction and economic evaluation were calculated by DSS1 method (Table 8). The results were:

- Riese.
- Dwelling-house.

It should be noted that the above system does not provide precise data. A model is based on generalized criteria giving a possibility to choose among a number of the available decisions.

The authors believe that the model suggested may help harmonize the needs of various interested parties at minimum expenses at the initial stage of investment, thereby allowing for contracting and further development and implementation of other steps in project execution.

CONCLUSIONS AND RECOMMENDATIONS

1. A model for evaluating the construction of dwelling-houses has been developed. The model is designed for complex evaluation of their location and for a comparative analysis of profitability of investment in the construction of dwelling-houses and apartment buildings.
2. The model is designed for the construction in Vilnius. In order to apply the model in other localities, some of the criteria should be revised.
3. The model was applied to the real problem solution. The result obtained shows that the construction of dwelling-houses is more economically promising in 2004.
4. The system of integrated indicators presented in the model allows for the reduction of the amount of expert evaluation work as well as for saving his time.

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Ralph E. Steuer
Yue Qi
Markus Hirschberger

DEVELOPMENTS IN MULTI-ATTRIBUTE PORTFOLIO SELECTION

INTRODUCTION

It is a rare person in finance that sees portfolio selection as a multiple criteria problem. And it is a rare person in multiple criteria optimization that sees portfolio selection as a single criterion problem.

Also, when finance people think of portfolio selection, they typically think of it in terms of the "mean-variance" formulation:

$$\begin{aligned} &\min\{\, \mathbf{x}^T \boldsymbol{\Sigma}\, \mathbf{x}\, \} \\ &\text{s.t.} \quad \boldsymbol{\mu}^T \mathbf{x} \geqslant \rho \\ &\mathbf{x} \in S \end{aligned} \tag{1}$$

in which ρ is to be parameterized over a wide enough range to compute the "efficient frontier" (but what we will call the "nondominated frontier"). In practice, most people in finance settle for the repetitive solving of (1) for different values of ρ to obtain a dotted or piecewise linear characterization of the nondominated frontier. Ten to twenty such optimizations would not be uncommon (see Figure 1). However, when (1) is viewed by a multiple criteria optimization person, it is recognized as an ε-constraint program. An ε-constraint program is a multiple objective program that has been reformulated for solution as a single objective problem in which all but one of the objectives have been converted to constraints (see for example [18], Chap. 8). Thinking of the problem behind the ε-constraint program, a multiple criteria optimization person sees the "mean-variance" problem of (1) more aptly expressed as:

$$\begin{aligned} &\min\{\, \mathbf{x}^T \boldsymbol{\Sigma}\, \mathbf{x}\, \} \\ &\max\{\, \boldsymbol{\mu}^T \mathbf{x}\} \\ &\text{s.t.} \quad \mathbf{x} \in S \end{aligned} \tag{2}$$

in which the endeavor is to compute all points in S that are *efficient* to define, by taking their images, the *nondominated set* (or in this case because there are only two objectives, the “nondominated frontier”). Either way, with the same intended solution sets, (1) and (2) are simply two different ways of expressing exactly the same thing.

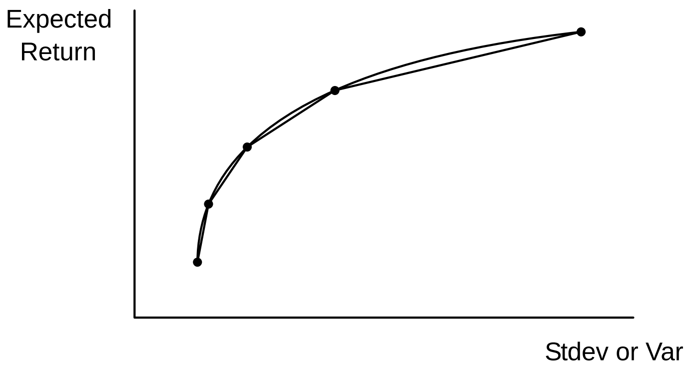

Fig. 1. Nondominated frontier (curved line) along with, using only five points, dotted and piecewise linear characterizations. Note that oftentimes nondominated frontiers are portrayed with standard deviation rather than variance on the horizontal axis. The vertical axis is always expected return

To make clear the notation employed above along with the basic problem of portfolio selection attributable to Markowitz [13; 14], let:

(a) there be a beginning of a holding period,
(b) there be an end of the holding period,
(c) there be an initial sum to be invested,
(d) n be the number of securities in the pool from which a portfolio is to be formed,
(e) $\mathbf{x} = (x_1, \ldots, x_n)$ denote a *portfolio* where x_i specifies the proportion of the sum invested in security i,
(f) $S \subset \mathbb{R}^n$ be the set of all feasible portfolios which often is expressed, as assumed in this paper, as simply as $S = \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1,\ 0 \leqslant x_i \leqslant 1\}$.

With expected value $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ and $n \times n$ covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & & \\ \vdots & & & \vdots \\ \sigma_{n1} & & \cdots & \sigma_{nn} \end{bmatrix},$$

let $\mathbf{r} = (r_1, \ldots, r_n)$ be the random vector that specifies the returns of the n securities to be realized over the course of the holding period. While the realized values of the r_i are not known until the end of the holding period, basic theory assumes that $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, also known as "Markowitz inputs", are known with certainty at the beginning of the holding period.

The rest of the paper is organized as follows. In Section 1, we show how both finance and multiple criteria optimization can each be correct in viewing portfolio selection with one and two "objectives", respectively. In Section 2 we discuss the introduction of multiple objectives into the theory of portfolio selection, and in Section 3 we demonstrate the platelet-wise hyperboloidic nature of the nondominated set in multiple objective portfolio selection programming. Section 4 comments on a Java code under development and Section 5 ends the paper with concluding remarks.

1. TWO LEVELS OF ASSUMPTIONS

We note that neither (1) nor (2) is a starting point. Rather, the formulations, which represent two ways of writing the same problem, are a consequence of two levels of assumptions.

In conventional theory, the assumptions at the highest level are that the self-interest model of economics applies and that markets are efficient. This means that investors need only concern themselves with "making money" (the more the better). The belief here is that there is no need to take into account factors such as dividends, quality of corporate governance, social responsibility, and so forth, as all such effects are assumed to be already in the prices.

Portraying the situation of conventional portfolio selection, we have Figure 2. At the top is the investor's *overall focus*, to make money. Commencing the operationalization of the overall focus then results in the formulation:

$$\begin{aligned} &\max\{\, \mathbf{r}^T\mathbf{x}\} \\ &\text{s.t.} \quad \mathbf{x} \in S \end{aligned} \tag{3}$$

in which the objective of *portfolio return*, given by $\mathbf{r}^T\mathbf{x}$, is observed to be a random variable. This is because $\mathbf{r}^T\mathbf{x}$ involves the random vector $\mathbf{r}$. Consequently (3), which might look like a linear program, is not a linear program. It is a *stochastic program* as a result of its objective being stochastic. Thus we see the difficulty. While the $\mathbf{r}$ are not known until the end of the holding period, $\mathbf{x} \in \mathbb{R}^n$ must nevertheless be selected at the beginning of the holding period.

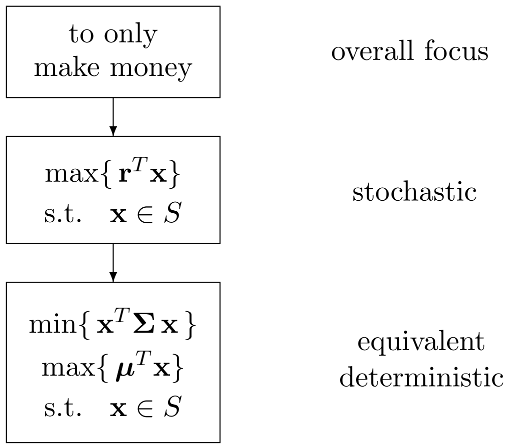

Fig. 2. Hierarchical structure of the overall focus, stochastic, and equivalent deterministic stages in conventional portfolio selection

Since a stochastic program cannot be solved in its present form, the assumptions at the second level involve how to replace (3) with an *equivalent deterministic program*. Caballero, R. and Cerdá, E. and Muñoz, M. M. and Rey, L. and Stancu-Minasian [3], enumerate several possibilities, the third of which is mean-variance. Thus, and bear in mind that this was before either stochastic programming or multiple criteria optimization came on the scene, the contributions of Markowitz's were (a) his choice of the problem of mean-variance (regardless of whether it is in the form of (1) or (2)) as the equivalent deterministic problem and (b) his protocol of computing, and then searching for the most preferred point on, the nondominated frontier.

We are now able to reconcile the two viewpoints and the two formulations. Looking at portfolio selection from an overall focus perspective, finance only sees the single stochastic objective of portfolio return. Note that writing the equivalent deterministic problem in the form of (1) only tends to reinforce the single criterion viewpoint. On the other hand, multiple criteria optimization, taking the equivalent deterministic problem in the form of (2) at face value, sees portfolio selection as possessing the two deterministic objectives of expected return and variance. How many objectives ones sees depends upon whether one is looking at the equivalent deterministic program or not.

2. MULTIPLE CRITERIA PORTFOLIO THEORY

With the overall focus, stochastic, and equivalent deterministic stages of portfolio selection as developed in the previous section, we now show where and how multiple objectives can enter the picture. Multiple criteria portfolio selection is for at least three groups of investors. One group could consist of investors who believe in the self-interest model and efficient markets, but do not believe in the 100% certainty assumption about the Markowitz inputs at the beginning of the holding period. Such investors might well wish to monitor their portfolios with regard to other measures such as dividends, growth in sales, amount invested in R&D, and so forth, in order to hedge against errors that might be made by selecting portfolios based upon expected return and variance alone. Another group could consist of people who do not believe, or only partially believe, in efficient markets. For them, prices do not always reflect all known information as they see it and they might wish to identify value and desirability using additional or other measures. A third group could consist of investors also interested in "portfolio-as-whole" criteria. To indicate this class of criteria, consider the two groups of possible objectives:

$\max\{ z_1 = \text{portfolio return}\}$
$\max\{ z_2 = \text{dividends}\}$
$\max\{ z_3 = \text{amount invested in R\&D}\}$
$\max\{ z_4 = \text{social responsibility}\}$
$\max\{ z_5 = \text{liquidity}\}$

$\min\{ z_6 = \text{deviations from target asset allocation percentages}\}$
$\min\{ z_7 = \text{number of securities in portfolio}\}$
$\min\{ z_8 = \text{turnover (i.e. costs of adjustment)}\}$
$\min\{ z_9 = \text{maximum investment proportion weight}\}$
$\min\{ z_{10} = \text{amount of short selling}\}$

In the first group, the criteria are derived from random variable attributes of the individual securities, and are thus stochastic. In the second group, the criteria are derived from the properties of the portfolio as a whole, that is, they can be ascertained with certainty by examining the components of the $\mathbf{x}$-vector, and are thus deterministic.

The overall focus, stochastic, and equivalent deterministic stages of portfolio selection with multiple criteria are shown in Figure 3. To illustrate, assume that an investor has in his or her overall focus portfolio return and dividends as objectives. Let portfolio return be split into mean and variance as usual in accordance with the third choice enumerated by Caballero et al. in [3]. However, since the variability of dividends is of much less importance than the variability of portfolio return, let us assume that dividends can be adequately represented by its mean vector in the equivalent deterministic program in accordance with the first choice enumerated by Caballero et al. in [3]. In this multiple objective illustration, the resulting equivalent deterministic program would then be of a 1-quadratic-2-linear variety.

Two more comments before leaving Figure 3. The sets of three vertical dots signify that there can be more than one stochastic objective that is to be split in to mean and variance, and that there can be more than one stochastic objective that need only be represented by its respective mean vector, in the equivalent deterministic program. As for deterministic objectives such as from the second group in the list above, they appear as objectives in the equivalent deterministic program unchanged from as they appear in the stochastic program.

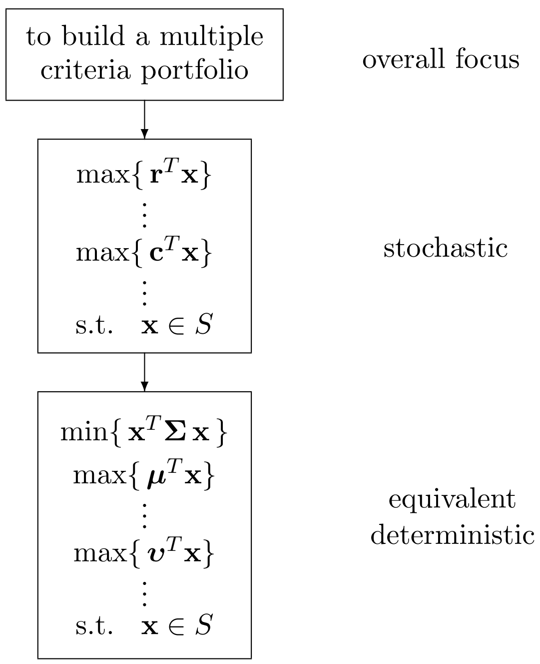

Fig. 3. Hierarchical structure of the overall focus, stochastic, and equivalent deterministic stages in multiple criteria portfolio selection where the expected value vector of $\mathbf{r}$ is $\boldsymbol{\mu}$ and the expected value vector of $\mathbf{c}$ is $\boldsymbol{\upsilon}$

3. PIECEWISE HYPERBOLIC AND HYPERBOLOIDIC

Before commenting on methods under development for solving equivalent deterministic programs with one quadratic and two or more linear objective functions, let us say a few things about the 1-quadratic-1-linear equivalent deterministic problems of conventional portfolio selection. As depicted in Figure 4, the efficient set is piecewise linear in **x**-space as on the left, and the nondominated set is piecewise *hyperbolic* in (standard deviation, expected return) space[1] as on the right .

As for equivalent deterministic problems with one quadratic and two or more linear objective functions that can easily arise in multiple criteria portfolio selection, the efficient set is a connected union of polyhedral sets in **x**-space and the nondominated set is platelet-wise *hyperboloidic* in (standard deviation, expected return) space[2]. This is as in Figure 5 with the efficient set portrayed as on the left and the platelet-wise (like on the back of a turtle) nondominated set portrayed as on the right.

Whereas methods for solving for the nondominated frontiers of mean-variance problems have been well studied, methods for solving for the nondominated surfaces of equivalent deterministic problems with additional linear objective functions are only under development (as being worked on for instance by Fliege [5], Kliber [10] and Hirschberger, Qi and Steuer [7; 8]).

4. JAVA CODE

One of the items under development in [7; 8] is a Java code for portfolio selection. The purposes of the code are that it

(a) be fast and easy to use,
(b) be able to address large-scale conventional and multiple objective portfolio selection problems,
(c) be able to compute all hyperbolic segments or hyperboloidic platelets of the nondominated set,
(d) be able to handle covariance matrices that are up to 100% dense in nonzero elements,
(e) be equipped with a built-in random portfolio selection problem generator.

[1] Or piecewise *parabolic* in (variance, expected return space).
[2] Or platelet-wise *paraboloidic* in (variance, expected return space).

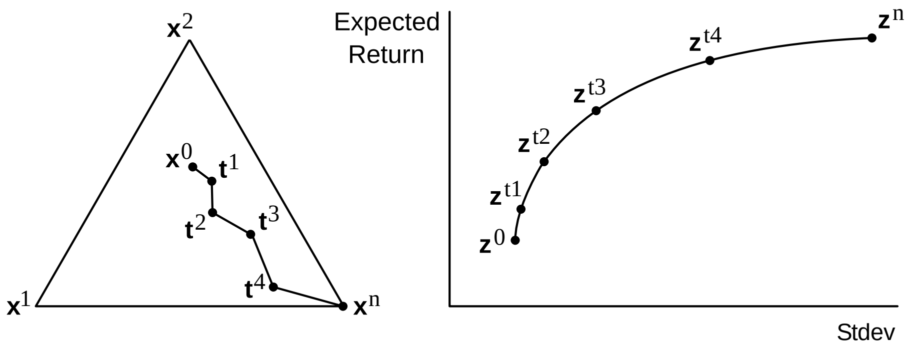

Fig. 4. Portrayal of the efficient and nondominated sets of a conventional mean-variance portfolio optimization problem

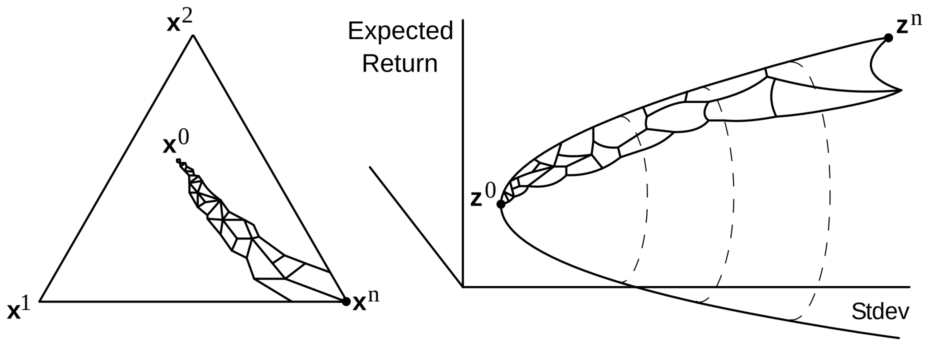

Fig. 5. Portrayal of the efficient and nondominated sets of an equivalent deterministic problem with one quadratic and two linear objective functions

This means that with this code it is no longer necessary in large problems to diagonalize the covariance structure, and endure the resulting inevitable loss of information, to achieve computational feasibility as the code can handle 100% dense covariance matrices directly. Moreover, because the code provides for the exact computation of the nondominated set, it is no longer necessary to utilize ε-constraint methods to obtain approximations of the nondominated set as are our only choices when using software such as Matlab [15], Mathematica [21], Cplex [4], LINGO [17], or SAS.

Although the code is not yet available for distribution, some preliminary computational results can be reported. For instance, in a normal (i.e.

1-quadratic-1-linear) mean-variance portfolio optimization problem

$$\begin{gathered}\min\{\,\mathbf{x}^T\boldsymbol{\Sigma}\,\mathbf{x}\,\}\\ \max\{\,\boldsymbol{\mu}^T\mathbf{x}\}\\ \text{s.t.}\quad \mathbf{x}\in S\end{gathered}$$

with $n = 500$, $S = \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1,\ 0 \leqslant x_i \leqslant 1\}$, and a 100% dense $\boldsymbol{\Sigma}$, one would expect about 200 nondominated hyperbolic segments. But in a 1-quadratic-2-linear problem

$$\begin{gathered}\min\{\,\mathbf{x}^T\boldsymbol{\Sigma}\,\mathbf{x}\,\}\\ \max\{\,\boldsymbol{\mu}^T\mathbf{x}\}\\ \max\{\,\boldsymbol{\upsilon}^T\mathbf{x}\}\\ \text{s.t.}\quad \mathbf{x}\in S\end{gathered}$$

with the same n, S and $\boldsymbol{\Sigma}$, one would expect about 5,000 nondominated hyperboloidic platelets.

With regard to the last item in the above list, the Java code under development contains a built-in random problem generator. Using the generator, portfolio selection problems with one quadratic and one or more linear objective functions can be randomly generated for any number of securities up to at least 3,000. What is non-trivial about the random problem generation task is how to generate the $\boldsymbol{\Sigma}$ covariance matrices. Unfortunately, it is nearly impossible to create covariance matrices larger than about 20×20 by simply assigning random numbers. The reason is that for a matrix to be a covariance matrix, it must be positive semidefinite. To save a user from having to resort to different universes of historical data to obtain a valid $\boldsymbol{\Sigma}$s, the built-in random problem generator employs a method for randomly generating realistic covariance matrices that have pre-chosen distributional characteristics. The method employed is taken from [9]. With this capability, the code should be highly useful for computational and benchmark testing purposes in the portfolio optimization area.

5. FUTURE DIRECTIONS

As exemplified by the works of Aouni, Ben Abdelaziz and El-Fayedh [1], Bana e Costa and Soares [2], Hallerbach and Spronk [6], Lo, Petrov and Wierzbicki [12], Ogryczak [16], Steuer, Qi and Hirschberger [19], Xu and Li [22], and others, the general area of multi-attribute portfolio selection has begun to attract increased attention. With regard to the content of this paper, we see two particularly fertile areas for future research. One is on further work to compute the nondominated sets of multiple objective portfolio selection problems, and the other is on how to effectively search a portfolio-selection nondominated set for a most preferred point contained therein.

With regard to computing or characterizing nondominated sets, what we have done in this paper is talk about the easy case, when there is one quadratic and all other objective functions are linear. More difficult cases would involve equivalent deterministic problems (a) in which there are several quadratic and several linear objective functions, or (b) in which one or more of the non-quadratic objective functions are discrete or non-smooth. In the first case, a weighted-sums objective function could probably be formed so as to facilitate the repetitive sampling of the nondominated set using the Java code. Admittedly, this could involve many optimizations and consume considerable CPU time. In the second case, evolutionary algorithms such as employed in [20] would presumably be necessary to obtain a discretized representation of the nondominated set.

Either way, the nondominated sets of most multiple criteria portfolio optimization problems are likely to be known only via a number of given points (ideally, a very very large number of given points). Then the task becomes how to search among perhaps tens or hundreds of thousands of points to find a most preferred. Among several methods that might be considered is the projected line search method proposed in [11]. As one can see from this paper, with multiple criteria portfolio selection only now emerging, much work yet remains to be done.

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Tadeusz Trzaskalik
Sebastian Sitarz

TRIANGULAR NORMS IN DISCRETE DYNAMIC PROGRAMMING

INTRODUCTION

Decision problems with conflicting objectives and multiple stages can be considered as multi-objective dynamic programming problems. A survey has been presented by Li and Haimes [4], more recently by Trzaskalik [9]. Another way of generalization single-criterion dynamic programming models is to consider outcomes in partially ordered criteria set. Mitten [6] described a method for solving a variety of multistage decisions in which the real value objective function is replaced by preference relation. Sobel [7] extended Mitten's result to infinitive horizon for deterministic and stochastic problems. Preference order dynamic programming was described by Steinberg and Parks [8]. Henig [3] defined a general sequential model with returns in partially ordered set. It is shown that Bellman's principle of optimality [1] is valid with respect to maximal returns and leads to an algorithm to approximate these returns. Application of fuzzy logic to control started with the work written by Bellman and Zadeh [2]. Many contemporary approaches in this field are presented in Kacprzyk [5].

The present paper is devoted to investigate how triangular norms can be applied in the discrete dynamic programming and is the continuation of our previous papers. Basic backward procedure was formulated in Trzaskalik and Sitarz [11] and the forward procedure was worked out in Trzaskalik and Sitarz [12]. In the next paper [13] we considered dynamic programming with outcomes in fuzzy ordered structures. Fuzzy numbers and triangular norms were considered there. Some other examples of ordered structures and products of ordered structures were given in Trzaskalik and Sitarz [10; 14].

The paper consists of 4 sections. In Section 1 basic notation, backward and forward procedure are reminded. In Chapter 2 a wide extension of the idea of applying triangular norms to create ordered structures and 15 exemplary ordered structures based on triangular norms are given. Numerical analysis for these structures is performed in Section 3. Some concluding remarks are given in Section 4.

1. DYNAMIC PROGRAMMING IN PARTIALLY ORDERED STRUCTURE

Discrete dynamic process P which consists of T periods is considered. Let us assume that for t = 1,…,T:

Y_t is the set of all feasible state variables at the beginning of period t,
Y_{T+1} is the set of all states at the end of the process,
$X_t(y_t)$ is the set of all feasible decision variables for period t and state $y_t \in Y_t$.

We assume that all above sets are finite. Now let us define:

$D_t = \{d_t = (y_t , x_t): y_t \in Y_t, x_t \in X_t(y_t)\}$ – the set of all period realizations in period t,

$\Omega_t: D_t \rightarrow Y_{t+1}$ – transformations.

Process P is given if sets $Y_1, \ldots, Y_{T+1}, X_1(y_1), \ldots X_T(y_T)$ and transformations $\Omega_1, \ldots, \Omega_T$ are identified.

Let us denote:

$D = \{d = (d_1,\ldots,d_T): \forall_{t\in\{1,\ldots,T\}} \; y_{t+1} = \Omega_t(y_t, x_t) \;\; and \;\; x_T \in X_T(y_T)\}$ – the set of all process realizations

$D_t(y_t) = \{(y_t, x_t): x_t \in X_t(y_t)\}$ – the set of all realizations in period t which begin at y_t.

$d(y_t) = (y_t\, x_t, \ldots, y_T, x_T)$ – the backward partial realization which begins at y_t.

$D(y_t) = \{d(y_t): d \in D\}$ – the set of all backward partial realizations, which begin at y_t.

$D(Y_t) = \{D(y_t): y_t \in Y_t\}$ – the set of all backward partial realizations for Y_t.

$d'(y_t) = (y_1, x_1, \ldots, y_{t-1}, x_{t-1})$ – the forward partial realization which ends at $y_t = \Omega_t(y_{t-1}, x_{t-1})$.

$D'(y_t) = \{d'(y_t): d \in D\}$ – the set of all forward partial realizations, which end at y_t.

$D'(Y_t) = \{D'(y_t): y_t \in Y_t\}$ – the set of all forward partial realizations for Y_t.

$(W, \preceq, \circ)$ – ordered structure with binary relation ≤ and binary operator ∘ fulfilling following conditions:

$$\forall_{a \in W} \ a \preceq a \tag{1}$$

$$\forall_{a,b \in W} \ a \preceq b \wedge b \preceq a \Rightarrow a = b \tag{2}$$

$$\forall_{a,b,c \in W} \ a \preceq b \wedge b \preceq c \Rightarrow a \preceq c \tag{3}$$

$$\forall_{a,b,c \in W} \ a \circ (b \circ c) = (a \circ b) \circ c \tag{4}$$

$$\forall_{a,b,c \in W} \ a \preceq b \ \Rightarrow \ a \circ c \preceq b \circ c \wedge \ c \circ a \preceq c \circ b \tag{5}$$

Relation < is defined as follows:

$$a < b \Leftrightarrow \ a \preceq b \wedge a \neq b \tag{6}$$

Applying relation $\preceq$ we define for each finite subset $A \subset W$ the set of maximal elements:

$$max(A) \ = \{a^* \in A: \sim \exists_{a \in A} \ \ a^* < a\} \tag{7}$$

For $t = 1,\ldots,T$ let $(W, \preceq, \circ_t)$ be a sequence of ordered structures and $f_t{:}D_t \to W$ – a sequence of period criteria functions. Applying period criteria functions f_t, we define functions $F_t{:}D(Y_t) \to W$ in the following way:

$$F_T = f_T \tag{8}$$

$$F_t = f_t \ \circ_t \ F_{t+1} \qquad t = T\text{-}1,\ldots,1 \tag{9}$$

Functions $G_t{:}D'(Y_{t+1}) \to W$ are built as follows:

$$G_1 = f_1 \tag{10}$$

$$G_t = G_{t-1} \ \circ_{t-1} f_t \qquad t = 2,\ldots,T \tag{11}$$

According to (11) we obtain:

$$F_1 = G_T \tag{12}$$

Let $F{:}\ D \to W$ be the function defined in one of the following ways:

$$F = F_1 \tag{13}$$

$$F = G_T \tag{14}$$

F is called the multiperiod criteria function. Discrete dynamic decision process (P, F) is given if the discrete dynamic process P and the multiperiod criteria function F are defined.

Realization $d^* \in D$ is efficient, iff:

$$F(d^*) \in max\ F(D) \qquad (15)$$

Our problem is to find the set of all maximal values of the process, i.e. set max F(D).

Theorem 1

Decision dynamic process *(P, F)* is given. For all $t=T-1,\ldots,1$ and all $y_t \in Y_t$ holds:

$$max\ \{F_t(D(y_t)\} = max\ \{ f_t(d_t) \circ_t max(F_{t+1}(d(\Omega_t(d_t))): d_t \in D_t(y_t)\} \qquad (16)$$

$$max\{F(D)\} = max\ \{maxF_1(d(y_1)): y_1 \in Y_1\} \qquad (17)$$

Proof. Trzaskalik and Sitarz (11).

Theorem 1 yields backward iterative computational method.

Backward Procedure

Step B$_1$. Compute $max\{F_T(D(y_T)\}$ for all states $y_T \in Y_T$.

Step B$_t$ (for t = T-1, …, 1). Compute $max\{F_t(D(y_t)\}$ for all states $y_t \in Y_t$ applying (16).

Step B$_{T+1}$. Compute $max\{F(D)\}$ applying (17).

Theorem 2

Decision dynamic process *(P, F)* is given. For all $t=2,\ldots,T$ and all $y_t \in Y_t$ holds:

$$max\ \{G_t(D'(y_{t+1})\} = max\ \{ max\ G_{t-1}(d'(y_t)) \circ_t f_t(y_t, x_t):\ \Omega_t(y_t, x_t) = y_{t+1}\} \qquad (18)$$

$$max\{F(D)\} = max\ \{max\ G_T(d'(y_{T+1})): y_{T+1} \in Y_{T+1}\} \qquad (19)$$

Proof. Trzaskalik and Sitarz [11].

Theorem 2 yields forward iterative computational method.

Forward Procedure

Step F$_1$. Compute $max\{G(D(y_2)\}$ for all states $y_2 \in Y_2$.

Step F$_t$. (for t = 3, …, T). Compute $max\ \{G_t(D'(y_{t+1})\}$ for all states $y_t \in Y_t$ applying (18).

Step F$_{T+1}$. Compute $max\{F(D)\}$ applying (19).

2. FUZZY ORDERED STRUCTURES

As examples of fuzzy ordered structures we will consider triangular norms and products of triangular norms.

2.1. Triangular norms

Function T : [0, 1] × [0, 1] → [0, 1] is called t-norm iff:

$$T(a, b) = T(b, a) \tag{20}$$

$$T(a, T(b, c)) = T(T(a, b), c) \tag{21}$$

$$a \le a' \wedge b \le b' \Rightarrow T(a, b) \le T(a', b') \tag{22}$$

$$T(a, 1) = a \tag{23}$$

We denote:

$$[0,1]^n = [0, 1] \times \ldots \times [0, 1]$$

Each t-norm T may be extended to the function of n-arguments $a^1,\ldots,a^n$:

$$T^n : [0, 1]^n \to [0,1]$$

as follows:

$$T^2(a^1, a^2) = T(a^1, a^2) \tag{24}$$

$$T^i(a^1, a^2, \ldots, a^i) = T(T^{i-1}(a^1, a^2, \ldots, a^{i-1}), a^i) \text{ for } i = 2,\ldots,n \tag{25}$$

In the further considerations we will omit index n.

Let us consider the structure $([0,1]^n, \le, T)$. It is easy to show, that conditions (4) and (5) are fulfilled. Applying the definition of t-norms (20) – (23) we see, that:

$$\forall_{a,b,c \in [0,1]} \; T(a,T(b,c)) = T(T(a,b),c) \tag{26}$$

$$\forall_{a,b,c \in [0,1]} \; a \le b \Rightarrow T(a,c) \le T(b,c) \wedge T(c,a) \le T(c,b) \tag{27}$$

It means, that such a triple constitute ordered structure.

2.2. Product of triangular norms

We denote:

$$a = [a_1, \ldots, a_m],\; b = [a_1, \ldots, a_m]$$

and define relation $\le^m$ as a product of standard relations $\le$, i.e:

$$a \le^m b \Leftrightarrow \forall_{i=1,\ldots,m} \; a_i \le b_i \tag{28}$$

Let T be the set of triangular norms. Function $T = T_1 \times ... \times T_m$ is defined as the product of triangular norms $T_1,...,T_m \in T$, iff:

$$\forall_{a,b \in [0,1]^m} \; T(a,b) = [T_1(a_1,b_1),...,T_m(a_m, b_m)] \tag{29}$$

Applying formulas (24) and (25), each product of triangular norms T may be extended to the function of n-arguments $a^1 = [a_1^1,...,a_m^1]$, ..., $a^n = [a_1^n,...,a_m^n]$:

$$T^n : [0, 1]^{m \times n} \to [0,1]^m$$

Again in the further considerations we will omit index n.

Let us consider the structure $([0,1]^{m \times n}, \leq^m, T)$. It is easy to show, that conditions (4) and (5) are fulfilled. It means, that such a triple constitutes ordered structure.

2.3. Examples of ordered structures applying triangular norms

We will consider the following t-norms:

$$T_1(a, b) = \max\{a+b-1, 0\}, \quad (\text{Łukasiewicz}) \tag{30}$$

$$T_2(a,b) = \begin{cases} \min\{a,b\}, & \text{if } \max\{a,b\} = 1 \\ 0 & \text{otherwise} \end{cases}, \quad (weak) \tag{31}$$

$$T_3(a, b) = a \cdot b, \quad (probablistic) \tag{32}$$

$$T_4(a, b) = \min\{a, b\}, \quad (minimum) \tag{33}$$

and define following ordered structures:

Structure S_1: $([0, 1], \leq, T_1)$

Structure S_2: $([0, 1], \leq, T_2)$

Structure S_3: $([0, 1], \leq, T_3)$

Structure S_4: $([0, 1], \leq, T_4)$

Structure S_{12}: $([0, 1], \leq^2, T_{12})$ $\quad T_{12} = T_1 \times T_2$

Structure S_{13}: $([0, 1], \leq^2, T_{13})$ $\quad T_{13} = T_1 \times T_3$

Structure S_{14}: $([0, 1], \leq^2, T_{14})$ $\quad T_{14} = T_1 \times T_4$

Structure S_{23}: $([0, 1], \leq^2, T_{23})$ $\quad T_{23} = T_2 \times T_3$

Structure S_{24}: $([0, 1], \leq^2, T_{24})$ $\quad T_{24} = T_2 \times T_4$

Structure S_{34}: $([0, 1], \leq^2, T_{34})$ $T_{34} = T_3 \times T_4$

Structure S_{123}: $([0, 1], \leq^3, T_{123})$ $T_{123} = T_1 \times T_2 \times T_3$

Structure S_{124}: $([0, 1], \leq^3, T_{124})$ $T_{124} = T_1 \times T_2 \times T_4$

Structure S_{134}: $([0, 1], \leq^3, T_{134})$ $T_{134} = T_1 \times T_3 \times T_4$

Structure S_{234}: $([0, 1], \leq^3, T_{234})$ $T_{234} = T_2 \times T_3 \times T_4$

Structure S_{1234}: $([0, 1], \leq^4, T_{1234})$ $T_{1234} = T_1 \times T_2 \times T_3 \times T_4$

These structures will be used in numerical analysis, performed below.

3. NUMERICAL ILLUSTRATIONS

We consider a dynamic process which consists of 3 periods [$T=3$]. We have:

$$Y_t = \{0,1\}, \quad for \quad t\in\{1, 2, 3, 4\}$$

$$X_t(y_t) = Y_{t+1} \quad for \quad t\in\{1, 2, 3\} \quad and \quad y_t \in Y_t$$

$$\Omega_t(y_t, x_t) = x_t \quad for \quad y_t\in Y_t \quad and \quad x_t \in X_t(y_t)$$

The sets of period realizations for t = 1,2,3 are as follows:

$$D_t = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

The values of period realizations are listed below:

$f_1(0, 0) = 0{,}35$	$f_2(0, 0) = 0{,}5$	$f_3(0, 0) = 0{,}5$
$f_1(0, 1) = 0{,}93$	$f_2(0, 1) = 1$	$f_3(0, 1) = 0{,}65$
$f_1(1, 0) = 0{,}3$	$f_2(1, 0) = 0{,}8$	$f_3(1, 0) = 0{,}6$
$f_1(1, 1) = 0{,}85$	$f_2(1, 1) = 0{,}82$	$f_3(1, 1) = 0{,}61$

The structure of the process and the values of period criteria are shown in Figure 1.

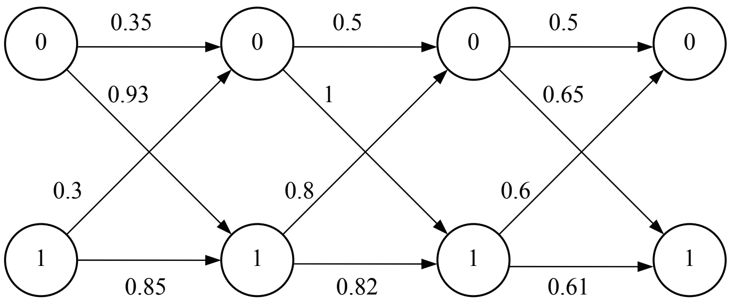

Fig. 1. The graph of the process

The sets of all realizations in period t which begin at y_t are as follows:

$$D_t(0) = \{(0, 0), (0, 1)\}$$

$$D_t(1) = \{(1, 0), (1, 1)\}$$

The sets of backward partial realizations are as follows:

$$D(y_3 = 0) = \{(0,0), (0,1)\}$$

$$D(y_3 = 1) = \{(1,0), (1,1)\}$$

$$D(y_2 = 0) = \{((0,0), (0,0)); \quad ((0,0), (0,1)); \quad ((0,1), (1,0)); \quad ((0,1), (1,1))\}$$

$$D(y_2 = 1) = \{((1,0), (0,0)); \quad ((1,0), (0,1)); \quad ((1,1), (1,0)); \quad ((1,1), (1,1))\}$$

$$\begin{aligned} D(y_1 = 0) = \ & \{((0,0), (0,0), (0,0)); && ((0,0), (0,0), (0,1)); \\ & ((0,0), (0,1), (1,0)); && ((0,0), (0,1), (1,1); \\ & ((0,1), (1,0), (0,0)); && ((0,1), (1,0), (0,1); \\ & ((0,1), (1,1), (1,0)); && ((0,1), (1,1),(1,1))\} \end{aligned}$$

$$\begin{aligned} D(y_1 = 1) = \ & \{((1,0), (0,0), (0,0)); && ((1,0), (0,0), (0,1)); \\ & ((1,0), (0,1), (1,0)); && ((1,0), (0,1), (1,1); \\ & ((1,1), (1,0), (0,0)); && ((1,1), (1,0), (0,1); \\ & ((1,1), (1,1), (1,0)); && ((1,1), (1,1),(1,1))\} \end{aligned}$$

The sets of all backward partial realizations are as follows:

$$D(Y_3) = D(y_3 = 0) \cup D(y_3 = 1)$$

$$D(Y_2) = D(y_2 = 0) \cup D(y_2 = 1)$$

$$D(Y_1) = D(y_1 = 0) \cup D(y_1 = 1)$$

The set of the realizations of the process can be presented as:

$$D = D(Y_1)$$

The sets of forward partial realizations are as follows:

$$D'(y_2 = 0) = \{(0,0); (1,0)\}$$
$$D'(y_2 = 1) = \{(0,1); (1,1)\}$$

$$D'(y_3 = 0) = \{((0,0), (0,0)); \quad ((0,1), (1,0)); \quad ((1,0), (0,0)); \quad ((1,1), (1,0))\}$$
$$D'(y_3 = 1) = \{((0,0), (0,1)); \quad ((0,1), (1,1)); \quad ((1,0), (0,1)); \quad ((1,1), (1,1))\}$$

$$\begin{aligned} D'(y_4 = 0) = \ & \{((0,0), (0,0), (0,0)); \quad ((0,0), (0,1), (1,0)); \\ & ((0,1), (1,0), (0,0)); \quad ((0,1), (1,1), (1,0)); \\ & ((1,0), (0,0), (0,0)); \quad ((1,0), (0,1), (1,0); \\ & ((1,1), (1,0), (0,0)); \quad ((1,1), (1,1),(1,0))\} \end{aligned}$$

$$\begin{aligned} D'(y_4 = 1) = \ & \{((0,0), (0,0), (0,1)); \quad ((0,0), (0,1), (1,1)); \\ & ((0,1), (1,0), (0,1)); \quad ((0,1), (1,1), (1,1)); \\ & ((1,0), (0,0), (0,1)); \quad ((1,0), (0,1), (1,1); \\ & ((1,1), (1,0), (0,1)); \quad ((1,1), (1,1),(1,1))\} \end{aligned}$$

Let us consider process realization $d = (d_1, d_2, d_3)$. According to formulas (8) and (9) we have:

$$F_3(d_3) = f_3(d_3)$$
$$F_2(d_2, d_3) = T_i(f_2(d_2), F_3(d_3))$$
$$F_1(d_1, d_2, d_3) = T_i[f_1(d_1), (F_2(d_2, d_3)]$$
$$F(d) = F_1(d_1, d_2, d_3)$$

The same result we will obtain applying formulas (10) and (11):

$$G_1(d_1) = f_1(d_1)$$
$$G_2(d_1, d_2) = T_i[F_1(d_1), f_2(d_2)]$$
$$G_3(d_1, d_2, d_3) = T_i[F_2(d_1, d_2), f_3(d_3)]$$
$$F(d) = G_3(d_1, d_2, d_3)$$

At the beginning we will consider ordered structures $S_1 - S_4$. The process is a single criterion maximization problem now and we will apply norms $T_1 - T_4$ as the operators.

Let us look at the numerical computations applying formulas (8) and (9) for subsequent norms $T_1 – T_4$ and the process realization *d = [(0, 0), (0, 0), (0, 0)]*.

For the norm T_1 we obtain:

$F_3(d_3) = F_3(0, 0) = f_3(0, 0) = 0,5$

$F_2(d_2, d_3) = T_1(f_2(0, 0), F_3(0, 0)) = T_1(0,5, 0,5) = max \{0,5 + 0,5 – 1, 0\} = 0$

$F_1(d_1, d_2, d_3) = T_1(f_1(d_1), F_2(d_2, d_3)) = T_1(0,35, 0) = max (0,35 + 0 – 1, 0\} = 0$

For the norm T_2 we obtain:

$F_3(d_3) = F_3(0, 0) = f_3(0, 0) = 0,5$

$F_2(d_2, d_3) = T_2(f_2(0, 0), F_3(0, 0)) = T_2(0,5, 0,5) = 0$

$F_1(d_1, d_2, d_3) = T_2(f_1(d_1), F_2(d_2, d_3)) = T_2(0,35, 0) = 0$

For the norm T_3 we obtain:

$F_3(d_3) = F_3(0, 0) = f_3(0, 0) = 0,35$

$F_2(d_2, d_3) = T_3(f_2(0, 0), F_3(0, 0)) = T_3(0,5, 0,5) = 0,5 \cdot 0,5 = 0,25$

$F_1(d_1, d_2, d_3) = T_3[f_1(d_1), F_2(d_2, d_3)] = T_1(0,35, 0,25) = 0,875$

For the norm T_4 we obtain:

$F_3(d_3) = F_3(0, 0) = f_3(0, 0) = 0,35$

$F_2(d_2, d_3) = T_4[f_2(0, 0), F_3(0, 0)] = T_4(0,5, 0,5) = min \{0,5, 0,5\} = 0,5$

$F_1(d_1, d_2, d_3) = T_4(f_1(d_1), F_2(d_2, d_3)) = T_4(035, 0,5) = min \{0,35, 0,5\} = 0,35$

We can continue computations for the next realization of the process. The results are gathered in Table 1.

Table 1

Values of multiperiod criterion function

d	F(d)			
	T_1	T_2	T_3	T_4
(0,0, 0,0, 0,0)	0	0	0,09	0,35
(0,0, 0,0, 0,1)	0	0	0,11	0,35

d	F(d)			
	T_1	T_2	T_3	T_4
(0,0, 0,1, 1,0)	0	0,35*	0,21	0,35
(0,0, 0,1, 1,1)	0	0,35*	0,21	0,35
(0,1, 1,0, 0,0)	0,23	0	0,37	0,50
(0,1, 1,0, 0,1)	0,33	0	0,45	0,60
(0,1, 1,1, 1,0)	0,35	0	0,46	0,60
(0,1, 1,1, 1,1)	0,36*	0	0,47*	0,61
(1,0, 0,0, 0,0)	0	0	0,08	0,30
(1,0, 0,0, 0,1)	0	0	0,10	0,30
(1,0, 0,1, 1,0)	0	0,3	0,18	0,30
(1,0, 0,1, 1,1)	0	0,30	0,18	0,30
(1,1, 1,0, 0,0)	0,15	0	0,34	0,50
(1,1, 1,0, 0,1)	0,30	0	0,44	0,65*
(1,1, 1,1, 1,0)	0,27	0	0,42	0,60
(1,1, 1,1, 1,1)	0,28	0	0,43	0,61

The best realizations in the considered ordered structures are marked in the Table 1.

Instead of inspection process, the methods described in Section 2 ensure to obtain optimal solutions. We will apply the backward method.

Step B_1

$\max \{F_3(D(y_3 = 0)\} = \max \{F_3(0,0), F_3(0,1)\} = \max \{0,5, 0,65\} = 0,65$

$\max \{F_3(D(y_3 = 1)\} = \max \{F_3(1,0), F_3(1,1)\} = \max \{0,6, 0,61\} = 0,61$

Step B_2

$\max F_2(D_2(0)) = \max T_1(f_2(0,x_2), \max F_3(D(y_3 = x_2) =$

$= \max \{T_1(f_2(0,0), \max F_3(D(0)), T_1(f_2(0,1), \max F_3(D(1))\}$

$= \max \{T_1(0,5, 0,65), T_1(1, 0,61) = \max \{0,15, 0,61\} = 0,61$

$\max F_2D_2(1) = \max T_1(f_2(1,x_2), \max F_3(D(y_3 = x_2) =$

$= \max \{T_1(f_2(1,0), \max F_3(D(0)), T_1(f_2(1,1), \max F_3(D(1))\}$

$= \max \{T_1(0,8, 0,65), T_1(0,82, 0,61) = \max \{0,45, 0,43\} = 0,45$

Step B_3

max $F_1(D(0))$ = max $T_1(f_1(0,x_2)$, max $F_2(D(y_2 = x_1)$ =

= max $\{T_1(f_1(0,0)$, max $F_2(D(0))$, $T_1(f_1(0,1)$, max $F_2(D(1))\}$

= max $\{T_1(0{,}35, 0{,}61)$, $T_1(0{,}93, 0{,}45)$ = max $\{0, 0{,}38\}$ = 0,38

max $F_2D(1)$ = max $T_1(f_2(1,x_2)$, max $F_3(D(y_3 = x_2)$ =

= max $\{T_1(f_1(1,0)$, max $F_2(D(0))$, $T_1(f_1(1,1)$, max $F_2(D(1))\}$

= max $\{T_1(0{,}3, 0{,}65)$, $T_1(0{,}85, 0{,}45)$ = max $\{0, 0{,}30\}$ = 0,30

Step B_4

max $\{F_1(D(0), F_1(D(1)\}$ = max $\{0{,}38, 0{,}30\}$ = 0.38

Now we will consider multicriteria processes based on the next ordered structures, described in Section 2.3. We will assume that we have two, three or four criteria and the value for all the period criteria are the same for a given realization. For instance, in the four criteria process and the ordered structure S_{1234} for t = 1,2,3 we have:

$$f_t(y_t, x_t) = [f_t^1(y_t, x_t), f_t^2(y_t, x_t), f_t^3(y_t, x_t), f_t^4(y_t,x_t)]$$

and for a given period realization (y_t, x_t) it holds:

$$f_t^1(y_t,x_t) = f_t^2(y_t,x_t) = f_t^3(y_t,x_t) = f_t^4(y_t,x_t)$$

We will apply forward and backward procedure for the considered process in the structure S_{1234}. The consecutive stages of computations in the forward procedure are given in Table 2, and for the backward procedure – in Table 3.

Table 2

The forward method in the structure S_{1234}

Step	*max $G_t(d(0))$*	*max $G_t(d(1))$*
F_1	[0.35, 0.35, 0.35, 0.35]	[0.93, 0.93, 0.93, 0.93]
F_2	[0.73, 0, 0.74, 0.8]	[0.75, 0, 0.76, 0.82] [0.35, ,35, 0.35, 0.35]
F_3	[0, 0.35, 0.21, 0.35] [0.35, , 0, 0.46, 0.60]	[0, 0.35, 0.21, 0.35] [0.3, 0, 0.44, 0.65] [0.36, 0, 0.47, 0.61]

Step	*max* $G_t(d(0))$	*max* $G_t(d(1))$
F_4	max F(D) [0, 0.35, 0.21, 0.35] [0.3, 0, 0.44, 0.65] [0.36, 0, 0.47, 0.61]	

Table 3

The backward method in the structure S_{1234}

Step	max $F_t(d(0))$	max $F_t(d(1))$
B_1	[0.65, 0.65, 0.65, 0.65]	[0.61, 0.61, 0.61, 0.61]
B_2	[0.61, 0,61, 0.61, 0.61]	[0.45, 0, 0.52, 0.65] [0.43, 0, 0.50, 0.61]
B_3	[0, 0.35, 0.21, 0.35] [0.36, 0, 0.47, 0.61]	[0.3, 0, 0.44, 0.65]
B_4	Max F(D) [0, 0.35, 0.21, 0.35] [0.3, 0, 0.44, 0.65] [0.36, 0, 0.47, 0.61]	

We can consider this problem as mulitcriteria one. Every realization is desribed by 4 numbers (for each T-norm). The efficient realizations obtained by using one of forward or backward method are presented in Table 2.

Table 4

The efficient realizations (x)

	Ordered structures										
	S_{12}	S_{13}	S_{14}	S_{23}	S_{24}	S_{34}	S_{123}	S_{124}	S_{134}	S_{234}	S_{1234}
(0,0, 0,0, 0,0)											
(0,0, 0,0, 0,1)											
(0,0, 0,1, 1,0)	x				x			x			
(0,0, 0,1, 1,1)	x			x	x		x	x		x	x
(0,1, 1,0, 0,0)											
(0,1, 1,0, 0,1)											

	Ordered structures										
	S_{12}	S_{13}	S_{14}	S_{23}	S_{24}	S_{34}	S_{123}	S_{124}	S_{134}	S_{234}	S_{1234}
(0,1, 1,1, 1,0)											
(0,1, 1,1, 1,1)	x	x	x	x		x	x	x	x	x	x
(1,0, 0,0, 0,0)											
(1,0, 0,0, 0,1)											
(1,0, 0,1, 1,0)											
(1,0, 0,1, 1,1)											
(1,1, 1,0, 0,0)											
(1,1, 1,0, 0,1)			x		x	x		x	x	x	x
(1,1, 1,1, 1,0)											
(1,1, 1,1, 1,1)											

CONCLUDING REMARKS

The considerations in the paper show that applying t-norms to discrete programming models seems to be easy and natural. In the numerical examples the number of efficient realizations was not large. The next step will be to consider the possibility to apply the proposed methodology to model decision makers' preferences.

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Małgorzata Trzaskalik-Wyrwa

Maciej Nowak

Tadeusz Trzaskalik

APPLICATION OF MULTICRITERIA ANALYSIS TO RESTORATION OF HISTORICAL PORTABLE ORGAN

INTRODUCTION

As time goes by, every historic object dilapidates and wears out. As the result, the values that it used to represent become obliterated and its effect on the public is weakened. Conservation and restoration of art works aim at preserving the extant matter and, if possible, at bringing the antiques to their former glory; the more so that the historical value of the objects increases with time.

Conservators' work, independently of their special fields of interest, should be preceded by research whose goal is the determination of the guidelines for the conservation efforts and the selection of the best methods of action. Inventory, documentation and research efforts are completed by a value analysis whose purpose is to precisely define several values of the object so as to emphasise and reveal the most important of them. A thorough analysis determines several possible methods of action, emphasising various groups of values. The basic value groups of historic objects and monuments have been formulated by Walter Frodl. These groups, expanded by musical issues, are used in this paper.

The possibility of a variant-based approach to the issue of the value analysis of historic items suggests that the methodology of the multicriteria decision support can be used for the selection of the best variant of conservation method of the individual item or monument. The possibility of shaping the selected values after the reconstruction of the object allows regarding the values as decision criteria. Possible ways of the instrument reconstruction constitute here decision variants.

In the 17th- and 18th-century Poland the portable organ, called the positive organ, was a very popular instrument[1]; almost every parish was equipped with one. It was not only a church instrument, since the portable organ was used also to accompany dancers in ballrooms. Its popularity was due above all to the ease of handling and the possibility of easy transportation. Unfortunately, only 18 copies of this once so common instrument are nowadays extant in Poland (according to current research). One of the extant instruments from this group, found only recently, comes from Sokoły near Łapy in the Podlasie region of Poland. For many years the instrument had been stored disassembled, undergoing atmospheric and biological damage. Its condition made it impossible to use it either as a visual historic item ("piece of furniture") or as a musical instrument. Such condition is called in Polish conservation science terminology a "destrukt".

The value analysis of historic items is not only a theoretical consideration, but aims at determining the guidelines of conservation efforts and, in connection with experience and conservation science, allows for the selection of the best conservation methods for individual works of art. The precise estimation of value of the extant elements of the instrument became thus a research problem; on this basis the determination of several (10 to 20) variants of conservation programmes will be made. The purpose of this paper is the joint application of the analysis evaluating an historic organ and the Electre I method in the selection of the guidelines for conservation efforts in the case of the recently discovered organ.

This paper consists of six chapters. In Chapter 1 selected problems related to the analysis for the evaluation of an historic organ are described. Specified groups of values have been used for the construction of decision criteria. The history and original condition of the instrument in question have been described in Chapter 2. Possible methods of restoration of this instrument, treated as decision variants in multivariate analysis, have been presented in Chapter 3. The Electre I method is described in Chapter 4. Chapter 5 presents the application of the Electre I method proposed for the analysis of the problem in question, as well as conclusions following from it. The development of the restoration of the organ and the condition of the instrument after the restoration have been described in Chapter 6. In Conclusions the directions of further works are given.

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[1] More on this topic see [6].

1. VALUES OF AN HISTORIC ORGAN AS DECISION CRITERIA

The first person to recognise and define the value of a separate group of historic objects – historic musical instruments – was the German scholar and musician Albert Schweitzer[2]. Thanks to his authority the cause of preservation of historic organs gained many advocates among musicians as well as conservators and researchers.

The value analysis of historic objects, used nowadays in conservation science with respect to all kinds of historic objects and monuments, has been defined by Walter Frodl in the middle of the 20th century[3], and was subsequently expanded and completed [2]; in the Polish legislation it resulted in an act concerning the preservation and protection of historic monuments [8]. According to this document an "historic monument or object" is "a building or an object, its element or subsystem, man-made or related to human activities which is an evidence of an epoch or an event from the past, whose preservation is of social value due to its historic, artistic or scientific value".

Taking into account the synthetic character of the group of objects dealt with in this paper – historic organs – one should add to the values listed above musical and technical values of historic instruments[4]; a precise definition of such values will help improve value analysis.

In the following discussion we suggest a division of the values of historical organs into four groups: historic, artistic, musical and utilitarian values. We will now describe the values constituting each of the four groups.

Historic values determine the character of the object as a document and its influence on the development of historical knowledge. Among the values of this group are *scientific values*, due to the fact that an organ is an historic object, requiring a scholarly description. Also in this group are *technical values*, determining the ingenuity of the construction, the quality of the workmanship and the scientific value of its current condition. Also *historic emotional values*, perceived not only by scientists and scholars, but also

[2] During the Third International Congress of the International Musical Society, which took place in Vienna on May 25-29, 1909, he was the first to direct attention to the necessity of the preservation of old instruments due to their numerous values.
[3] The author used the Polish translation of Frodl's work [1].
[4] More on this topic in [7].

by the public at large, belong here. The *ownership values*, i.e., values stemming from the ownership of the original item (without hypothetical additions) are connected with honest approach of the conservators to the historic object, in which that what is preserved should be emphasised above all, as opposed to that what we think might have been there. The group of *artistic values* is related to the perception of historic organs as works of art, and this is connected with the instrument's case. To this group belong *historic-artistic values*, determining whether the solutions chosen by the builders are typical or atypical as well as the importance of the original, its copy or its hypothetical reconstruction. *Artistic qualities* affect the public independently of the current fashion or style. The *artistic effect* of the case of historic organ should match musical impressions received by the audience from the musical compositions heard by it. *Musical values* become apparent during a musical performance. We deal here with the issue of style (*historical musical value*) and of sound (*musical quality*). All of them taken together may reinforce the *musical influence* on the amateur listener. It can happen that the regaining of musical value and the preservation of the original technical solutions are conflicting goals. In such case we face the problem of *utilitarian values* of the historic instrument. The notions of live organ and dead organ are related to this group of values. A *musically dead organ* is an instrument that nowadays cannot fulfil its function of a musical instrument. A *live instrument* is an instrument capable of being used in musical performance, affecting the audience in various ways. Like any historic object, an organ as a piece of furniture can be also visually dead – not suitable for being exhibited, or else visually alive (independently of its musical "vitality") – beautiful, but unplayable.

A comparison of the values described above is shown in Figure 1.

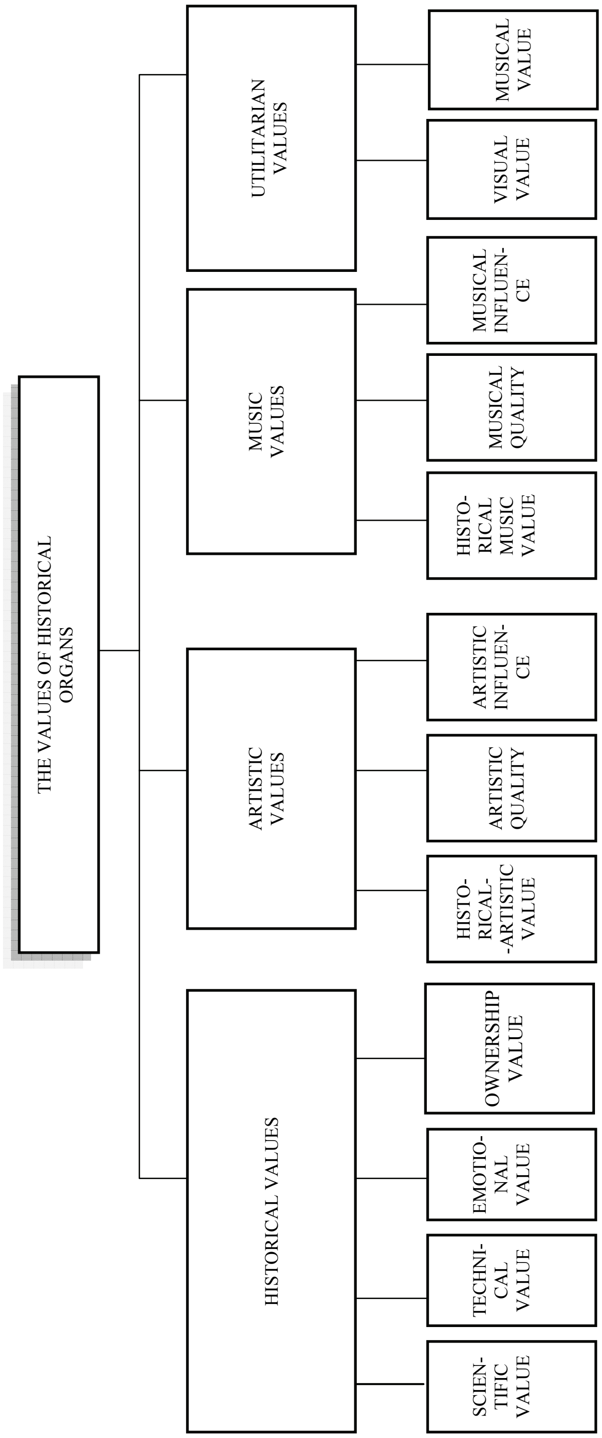

Fig. 1. The values of historical organs

The values listed above will be used as decision criteria in the problem of the selection of the best conservation variant[5], discussed below. The decision criteria admit the values from 0 (lowest grade) to 5 (highest grade). The decision variants defined later in the paper will be evaluated by an expert, utilising during the evaluation his or her expert knowledge of the topic.

2. THE DESCRIPTION OF THE INSTRUMENT UNDERGOING THE RESTORATION

The basic feature distinguishing a portable organ from a stationary one are its small dimensions and a design allowing for placing of all elements characteristic for the organ-like instruments (pipes, wind chest, action, bellows) in a small, easy to handle case. The "compression" of the instrument's mechanism is achieved by making the dimensions of the wind chest as small as possible, restricting the action to the direct transfer of the movement from the key to the pallet and mitering (often repeated) of the pipes or the use of common side walls of wooden pipes. Bellows of small dimensions always require a certain space for proper functioning, and that is why they are located on the instruments, underneath, or next to the case wall opposite the keyboard. The placing of the keyboard is also related to the localisation of the bellows, which follows from the construction of the wind chest.

The positive organ from Sokoły is an instrument of the two-chamber type, in which the lower one (larger) contains two wedge bellows, while the upper one (smaller), wind chest, pipes and keyboard. It is characteristic for this instrument that the lower chamber cover can be taken out and, after the bellows have been blocked, the upper chamber can be inserted into the lower one. Once this "package" is closed, the instrument is secured and can be transported conveniently. After the arrival at its destination, a two-part positive organ, when taken apart, is independent and does not require any auxiliary furniture.

The positive organ from Sokoły is preserved as a non-functional and visually unattractive object ("destrukt") – each element was stored separately and individual parts were damaged. About 70% of the case, 90% of the mechanism and 10% of the sound system have been preserved. In this condition the value of the positive organ is recognisable by a narrow group of researchers who are able to visualise how to combine the individual parts.

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[5] In the following analysis we disregard the financial criterion, which should not influence the choice of the best (from the point of view of reconstruction) decision criterion.

3. POSSIBLE METHODS OF INSTRUMENT RESTORATION AS DECISION VARIANTS

On the basis of research and evaluation of the condition of the individual parts of the instrument (or their lack) 12 renovation treatments of the re-discovered instrument have been suggested. They are decision variants in the multivariate analysis conducted later. A grade has been attached to each variant, depending of the values that the instrument would gain after the reconstruction according to the given decision variant.

Variants discussed later are described below. The set of expert evaluations is shown in Table 1.

Situation I

Preservation of the instrument as a non-functional, visually unattractive object ("destrukt") and its exhibition in the form of a group of museum exhibits.

Action: Securing of the individual parts by means of the so-called conservative treatment; reinforcement of the historic substance.

Result: Preservation of the 100% of the historic substance, but the organ is dead, visually interesting only for a small group of researchers (a collection of parts not resembling a musical instrument).

Situation II

Integration of the elements of the instrument using racks necessary to place the individual elements in proper places.

Action: Reinforcement of parts by means of conservative method, installation of racks (as little visible as possible).

Result: Minimal loss of the historic substance (ca. 1%) in order to assemble the racks. The organ remains dead, visually interesting only for a small group of researchers, but the ordered collection of parts begins to resemble a musical instrument.

Situation III

Integration of the parts of the instrument with full completion of the construction elements of the case (without covering the "windows" with reconstructed wood carved ornaments) according to their former shape as concluded from the preserved elements; completion of the missing parts of the mechanism. The pipes remain secured, but do not play.

Action: Cutting down the historic elements to join them with the added completions.

Result: Loss of a certain part of the historic substance (ca. 5%). The historical object becomes alive visually and is understandable for about half of the audience, but the organ as a musical instrument remains dead.

Situation IV

Integration of the parts of the instrument with full completion of the construction elements of the case according to their former shape, as concluded from the preserved elements; completion of the missing parts of the mechanism. Reconstruction of the polychrome and covering of the "windows" by a neutral filling (canvas, wooden grill). The pipes remain secured, but do not play.

Action: Cutting down the historic elements to join them with the added completions; covering of the valuable for researches carpentry joints by polychrome.

Result: Loss of a certain part of the historic substance (ca. 5%); visually, the object becomes definitely more attractive. The historic object becomes visually alive and pleasing to the audience; it is not uninteresting for a large part of audience, but it is still a dead instrument, without the functionality of a musical instrument.

Situation V

Integration of the parts of the instrument with full completion of the construction elements of the case according to their former shape, as concluded from the preserved elements; completion of the missing parts of the mechanism. Reconstruction of the polychrome. Hypothetical reconstruction of the wood carved ornaments filling out the "windows" (on the basis of comparative analysis – it is impossible to achieve the historical truth). The pipes remain secured, but do not play.

Action: Cutting down the historical elements to join them with the added completions; covering of the valuable for researches carpentry joints by polychrome.

Result: Loss of a certain part of the historic substance (ca. 5%); visually, the object becomes maximally attractive, but it still lacks the functionality of a musical instrument.

Situation VI

Integration of the parts of the instrument with full completion of the construction elements of the case (without covering the “windows” by reconstructed wood carved ornaments) according to their former shape, as concluded from the preserved elements; completion of the missing parts of the mechanism. Bringing the extant pipes to working condition and reconstruction of the missing pipes, so as to match the sound capabilities of the extant pipes.

Action: Cutting down the historical elements to join them with the added completions; aggressive conservation of the extant pipes.

Result: Loss of a certain part of the historic substance of the mechanism and case (ca. 5%), significant intervention into the condition of the historic pipes and loss of about 50% of their original condition to raise their technical value.

Result: Visually, the object is moderately attractive; utilitarian musical value appears, especially for people appreciating the original, historical sound.

Situation VII

Integration of the parts of the instrument with full completion of the construction elements of the case (without covering the “windows” by reconstructed wood carved ornaments) according to their former shape, as concluded from the preserved elements; completion of the missing parts of the mechanism. Exhibition of the extant historic pipes in a display case without giving them their former technical functionality. Reconstruction of the entire sound system according to preserved models.

Action: Cutting down the historical elements to join them with the added completions; complete reconstruction of the pipes.

Result: Loss of a certain part of the historic substance of the mechanism and case (ca. 5%), no aggressive intervention into historic pipes, achieving a hypothetical, reconstructed sound. The historic object becomes alive visually and understandable for about half of the audience. The instrument is alive, but its sound is entirely reconstructed.

Situation VIII

Integration of the parts of the instrument with full completion of the construction elements of the case according to their former shape, as concluded from the preserved elements; completion of the missing parts of the mechanism. Reconstruction of the polychrome and covering the "windows" by a neutral filling (canvas, wooden grill). Bringing the pipes to a working condition and reconstruction of the missing pipes, so as to match the sound capabilities of the extant pipes.

Action: Cutting down the historical elements to join them with the added completions; covering of the valuable for researches carpentry joints by polychrome; aggressive conservation of the extant pipes.

Result: Loss of a certain part of the historic substance of the mechanism and case (ca. 5%); visually the instrument becomes definitely more attractive; significant intervention into historic pipes and loss of about 50% of their original condition in order to raise their technical value.

Situation IX

Integration of the parts of the instrument with full completion of the construction elements of the case according to their former shape, as concluded from the preserved elements; completion of the missing parts of the mechanism. Reconstruction of the polychrome and covering the "windows" by a neutral filling (canvas, wooden grill). Exposition of the extant historical pipes in a display case without bringing them to a working condition. Reconstruction of the whole sound system according to preserved models.

Action: Cutting down the historical elements to join them with the added completions; covering of the valuable for researches carpentry joints by polychrome; full reconstruction of the pipes.

Result: Loss of a certain part of the historic substance of the mechanism and case (ca. 5%); visually the instrument becomes definitely more attractive. No aggressive intervention into historic pipes; achieving of a hypothetical, reconstructed sound.

Situation X

Integration of the parts of the instrument with full completion of the construction elements of the case according to their former shape, as concluded from the preserved elements and completion of the missing parts of the mechanism. Reconstruction of the polychrome. Hypothetical reconstruction of the wood carved ornaments filling out the "windows" (on the basis of comparative analysis – it is impossible to achieve historical truth). Bringing the pipes to a working condition and reconstruction of the missing pipes so as to match the sound of the sound capabilities of the preserved pipes.

Action: Cutting down the historical elements to join them with the added completions; covering of the valuable for researches carpentry joints by polychrome; aggressive restoration of the preserved pipes.

Result: Loss of a certain part of the historic substance (ca. 5%); visually the instrument becomes maximally attractive. Significant aggressive intervention into historic pipes and loss of about 50% of their original state to raise their technical value.

Situation XI

Integration of the parts of the instrument with full completion of the construction elements of the case according to their former shape, as concluded from the preserved elements and completion of the missing parts of the mechanism. Reconstruction of the polychrome. Hypothetical reconstruction of the wood carved ornaments filling out the "windows" (on the basis of comparative analysis – it is impossible to achieve historical truth). Exhibition of the preserved historic pipes in a display case without bringing them to a working condition. Reconstruction of the whole sound system according to preserved models.

Action: Cutting down the historical elements to join them with the added completions; covering of the valuable for researches carpentry joints by polychrome; complete reconstruction of the pipes.

Result: Loss of a certain part of the historic substance (ca. 5%); visually the instrument becomes maximally attractive; no aggressive intervention into historic pipes; achieving a hypothetical, reconstructed sound.

Situation XII

Preservation of the instrument in its non-functional, visually unattractive condition (as a "destrukt"). Making of an accurate copy. The evaluation focuses on the values of the copy, which is presented to the public.

We do not deal here with an historic object anymore, but with a new, functional musical instrument.

Table 1

Comparison of the value criteria

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
Historical-scientific value	10	8	6	6	6	6	6	6	6	6	6	0
Historical-technical value	4	6	10	8	8	6	10	6	10	6	10	0
Emotional value	10	10	10	8	6	10	6	6	4	4	2	0
Ownership value	10	10	10	9	5	8	5	9	4	5	0	0
Historical-artistic value	0	2	4	6	6	4	4	6	6	8	8	0
Artistic quality	0	0	2	4	8	2	2	4	4	8	8	8
Artistic influence	2	2	6	8	10	6	6	8	8	10	10	10
Historical-musical value	0	0	0	0	0	10	4	10	4	10	4	4
Musical quality	0	0	0	0	0	8	10	8	10	8	10	10
Musical influence	0	0	0	0	0	8	10	8	10	8	10	10
Visual-utilitarian value	2	4	6	8	10	6	6	8	8	10	10	10
Musical-utilitarian value	0	0	0	0	0	8	10	8	10	8	10	10

4. THE ELECTRE I METHOD

Let A be the set of decision alternatives, and F be the set of criteria:

$$A = \{ a_1, a_2, \ldots, a_m \}$$

$$F = \{ f_1, f_2, \ldots, f_n \}$$

Let's assume that criteria are defined in such a way that larger values are preferred to smaller ones. A weighting coefficient w_k is assigned to each criterion. It reflects the importance of the criterion for the decision maker.

For each pair (a_i, a_j) the concordance index is calculated as follows:

$$c(a_i, a_j) = \frac{\sum_{k=1}^{n} w_k \varphi_k (a_i, a_j)}{\sum_{k=1}^{n} w_k}$$

where:

$$\varphi_k(a_i, a_j) = \begin{cases} 1, & \text{if } f_k(a_i) \geq f_k(a_j) \\ 0 & \text{otherwise} \end{cases}$$

We assume that the global outranking relation between a_i and a_j takes place if two conditions hold: concordance condition and non-discordance condition. The former can be formulated as follows:

$$c(a_i, a_j) \geq s \quad \wedge \quad s \in [0,5;\, 1]$$

where s is the concordance threshold, defined by the decision maker.

Non-discordance condition is defined as follows:

$$f_k(a_i) + v_k[f_k(a_i)] \geq f_k(a_j) \qquad \text{for } k = 1, \ldots, n$$

where $v_k[f_k(a_i)]$ is the veto threshold for criterion f_k, defined by the decision maker.

Non-discordance condition means that the hypothesis "a_i outranks a_j" should be rejected if for at least one criterion the difference between criterion values for alternatives a_j i a_i is greater than the value of the veto threshold $v_k[f_k(a_i)]$.

The ELECTRE I procedure operates as follows:

1. Construction of the set of concordances C_s:

$$\mathrm{C}_s = \{(a_i, a_j) \in A \times A : c(a_i, a_j) \geq s \wedge s \in [0{,}5;1]\}$$

2. Construction of the set of discordances:

$$\mathrm{D}_v = \{(a_i, a_j) \in A \times A : \exists_k \; f_k(a_j) > f_k(a_i) + v_k[f_k(a_i)]\}$$

3. Composition of the outranking relation is defined as follows:

$$\mathrm{S}(s, v) = \mathrm{C}_s \cap \overline{\mathrm{D}}_v$$
$$\text{where: } \overline{\mathrm{D}}_v = (A \times A) \setminus \mathrm{D}_v$$

4. Construction of the graph reflecting relations between the alternatives.

5. VALUATION OF INSTRUMENT RECONSTRUCTION VARIANTS

On the basis of experts' opinion, the following weights were assigned to the individual criteria:

$k_1 = 50$; $k_2 = 50$; $k_3 = 30$; $k_4 = 130$;
$k_5 = 50$; $k_6 = 30$; $k_7 = 30$; $k_8 = 80$;
$k_9 = 50$; $k_{10} = 30$; $k_{11} = 15$; $k_{12} = 30$

It was also assumed that the use of veto thresholds was not necessary.

The application of the Electre I method requires the definition of the concordance set and, on this basis, of the outranking relation (since we do not use the veto thresholds, the discordance set plays no role here). Next, graphs of dependence between alternatives have been constructed, with the thresholds being gradually lowered until a sufficiently rich outranking relation is achieved.

The calculated values of the concordance indices are shown in Table 2.

Table 2

Concordance matrix

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
a_1	1.00	0.80	0.70	0.70	0.70	0.37	0.37	0.37	0.37	0.37	0.37	0.54
a_2	0.91	1.00	0.70	0.70	0.70	0.45	0.37	0.45	0.37	0.45	0.37	0.54
a_3	0.91	0.91	1.00	0.78	0.78	0.67	0.67	0.45	0.45	0.45	0.45	0.54
a_4	0.63	0.63	0.63	1.00	0.87	0.62	0.58	0.67	0.58	0.45	0.37	0.54
a_5	0.63	0.63	0.63	0.72	1.00	0.39	0.58	0.44	0.58	0.58	0.50	0.67
a_6	0.69	0.69	0.69	0.47	0.70	1.00	0.72	0.56	0.50	0.78	0.50	0.68
a_7	0.63	0.63	0.72	0.50	0.78	0.58	1.00	0.42	0.78	0.64	0.78	0.87
a_8	0.63	0.63	0.63	0.86	0.78	0.95	0.72	1.00	0.72	0.78	0.50	0.68
a_9	0.63	0.63	0.72	0.72	0.59	0.58	0.72	0.58	1.00	0.42	0.78	0.87
a_{10}	0.63	0.63	0.63	0.63	0.86	0.72	0.67	0.72	0.72	1.00	0.72	0.81
a_{11}	0.63	0.63	0.72	0.72	0.72	0.58	0.72	0.58	0.72	0.58	1.00	1.00
a_{12}	0.55	0.46	0.46	0.46	0.46	0.32	0.46	0.32	0.46	0.32	0.69	1.00

Graph of the outranking relation for s = 0.95 is presented in Figure 2.

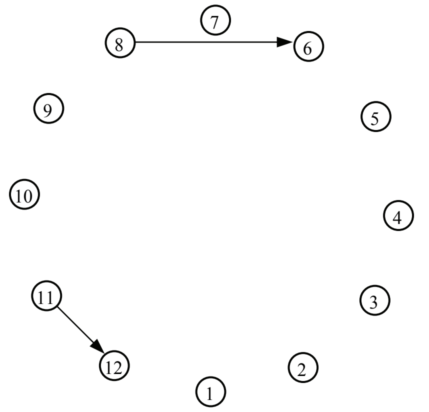

Fig. 2. The graph for s = 0,95

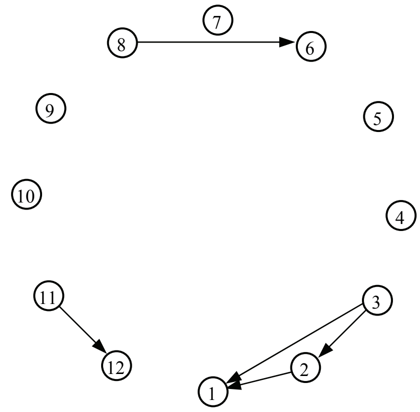

Fig. 3. The graph for s = 0,91

We obtain the following ranking:

1. a_1, a_2, a_3, a_4, a_5, a_7, a_8, a_9, a_{10}, a_{11}
2. a_6, a_{12}

The obtained ranking is not sufficient to determine the best decision alternative. Therefore, we lower the threshold. Figure 3 shows the outranking relation graph for s = 0.91.

We obtain the following ranking:

1. a_3, a_4, a_5, a_7, a_8, a_9, a_{10}, a_{11}
2. a_2, a_6, a_{12}
3. a_1

Again, the obtained ranking does not allow us to choose unambiguously the best alternative either. We continue to lower the threshold. Figure 4 shows the outranking relation graph for s = 0.86.

We obtain the ranking:

1. a_3, a_7, a_8, a_9, a_{10}, a_{11}
2. a_2, a_4, a_6, a_{12}
3. a_1

Six decision variants, a_3, a_7, a_8, a_9, a_{10} and a_{11}, get the best evaluation. We continue to lower the threshold. Figure 5 shows the outranking relation graph for s = 0.78.

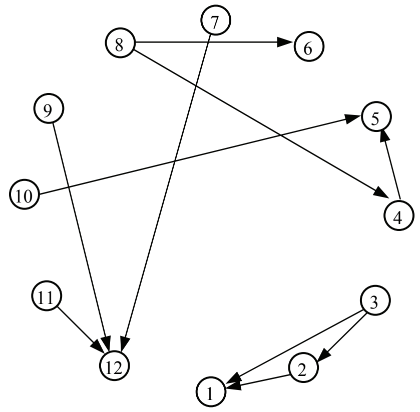

Fig. 4. The graph for s= 0,86

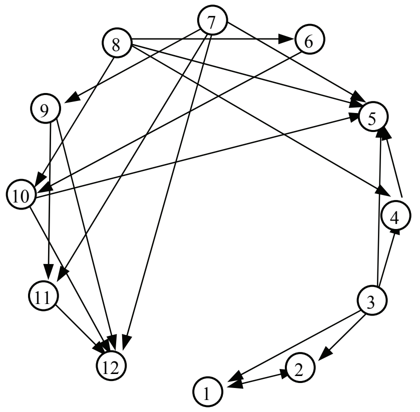

Fig. 5. The graph for s= 0,78

We obtain ranking:

1. a_3, a_7, a_8
2. a_1, a_2, a_4, a_6, a_9
3. a_{10}, a_{11}
4. a_5, a_{12}

Three decision alternatives, a_3, a_7 and a_8, get the best evaluation. It turns out that further lowering of the threshold will enable us to distinguish between them. Figure 6 shows the outranking relation graph for s = 0.72.

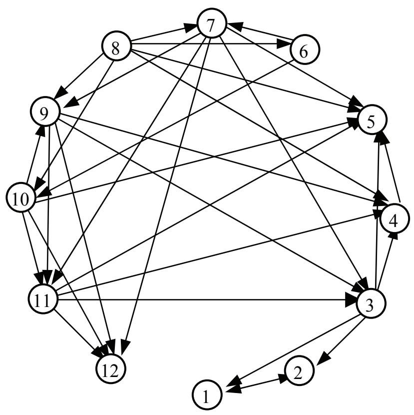

Fig. 6. The graph for s= 0,72

We obtain ranking:

1. a_8, a_{10}
2. a_6, a_9, a_{11}
3. a_7
4. a_3, a_{12}
5. a_1, a_2, a_4, a_5

Thus, two alternatives, a_8 and a_{10} get the best evaluation. As continuing lowering the value of the concordance threshold does not result in generating more detailed ranking, so we decide to stop the procedure and propose the decision maker to choose between alternative a_8 and a_{10}. Alternative a_8 is recommended, as it has got the best evaluation in all rankings that have been constructed.

6. THE PROCESS OF RECONSTRUCTION OF THE INSTRUMENT

In accordance with the assumptions of the variant a_{11}, recommended by the Electre I method, the conservation of the positive organ from Sokoły aims at emphasising the musical values of the instrument; at the same time we will try to preserve as much of the historic substance as possible and to introduce as few hypothetically reconstructed elements as possible. This was done by integration of the instrument parts with full completion of the construction elements of the case according to their original form as concluded from the preserved elements and by completion of the missing parts of the mechanism. The polychrome has been reconstructed, while the windows have been covered with a neutral filling (canvas). Missing parts of the sound system have been reconstructed and adapted to the 21 preserved pipes.

The reconstruction of the instrument was finished in 2004. Figure 7 shows the instrument before the reconstruction and Figure 8 – after the reconstruction.

Fig. 7. The instrument before the reconstruction

Fig. 8. The instrument after the reconstruction

CONCLUSIONS

In the paper, possibilities of application of multi-criteria decision support in choosing an approach to conservation of historical organ have been presented. The work resulted in the renovation of a valuable instrument made by Polish organ-builders. It seems that this methodology may be applied also for a wider range of objects of historical value, although this would require an analysis of the set of criteria under consideration. Another issue requiring an analysis would be the course of action in the case when decision variants are evaluated by a group of experts.

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Leonas Ustinovichius
Galina Ševčenko
Dmitry Kochin

CLASSIFICATION OF REAL ALTERNATIVES AND ITS APPLICATION TO THE INVESTMENT RISK IN CONSTRUCTION

INTRODUCTION

Risk is an integral element of any economic project. It is impossible to avoid, therefore, it is necessary to be able to estimate and minimize it. Any investment in construction can be risky. The basic purpose of risk analysis might be formulated as follows: to give to potential partners of the project the facts on the issue related to making a decision whether to participate in the project and which method to choose so that financial losses are avoided [14].

A superficial assessment of the risk related to capital investment is one of the reasons why investment is not necessarily successful in practice.

Making a reliable qualitative analysis of the investments is a complicated task, since the criteria for assessing the probability of capital recovery have not been established yet. There are many factors that should be taken into account [12; 13]. Each of these factors influences the probability of capital recovery.

Classification is a very important aspect of decision making. It is the process of assigning projects to particular classes. Claims are often made that classes in decision making are determined by individual parameters, i.e. efficiency of technical and technological decisions, project credit value determination etc. It is hardly possible to achieve this process (of assigning projects to particular classes) without employing special techniques in multi-criteria environment [1; 5; 9; 11]. This article presents a verbal method of determining investment risk in construction. The problem under consideration consists in assessing investment projects depending on their level of risk.

Formally, the problem is stated as multicriteria classification. In fact, many different methods for solving multicriteria classification problems are widely known. The ORCLASS method, as an ordinary classification, was one of the first methods designed to solve these kinds of problems [3]. Then more recent methods appeared, such as DIFCLASS [4] and CYCLE methods [6].

A new way to solve the problem is offered – application of the CLARA method (Classification of Real Alternatives) [5]. The method is based on Verbal Decision Analysis approach. In this article methods of verbal analysis are disclosed, their value is analyzed and it is indicated in what cases these methods could be used depending on their productivity. A hierarchical approach for consideration of efficiency indicators is proposed. The efficiency of the method is proved. The procedure of applying the method for the problem in question in practice is described.

1. FORMAL PROBLEM STATEMENT

Investment risk management is a common practice of any bank providing loans for projects. The evaluation of credit risk should be made at various phases of the project [1]. It may be stated that risk management implies that all the procedures should be rigorously followed at any phase of the project, the risk exposure depending on the output and accuracy.

Given:

1. G – a feature, corresponding to the target criterion (e.g. treatment effectiveness).
2. $K = \{K_1, K_2, ..., K_N\}$ – a set of criteria, used to assess each alternative (course of treatment).
3. $S_q = \{k_1^q, ..., k_{w_q}^q\}$ – for $q = 1, ..., N$ – a set of verbal estimates on the scale of criterion K_q, w_q – the number of estimates for criterion K^q; estimates in S_q are ordered based on increasing intensity of the feature G.
4. $Y = S_1 \times ... \times S_N$ – a space of the alternative features to be classified. Each alternative is described by a set of estimates obtained by using criteria $K_1, ..., K_N$ and can be presented as a vector $y \in Y$, where $y = (y_1, y_2, ..., y_N)$, y_q is an index of estimate from set S_q.
5. $C = \{C_1, ..., C_M\}$ – a set of decision classes, ordered based on the increasing intensity of feature G.

A binary relation of strict dominance is introduced:

$$P = \left\{ \begin{matrix} (x, y) \in Y \times Y \big| \ \forall q = 1...N \\ x_q \geq y_q \quad \exists q_0 : x_{q_0} > y_{q_0} \end{matrix} \right\} \tag{1}$$

One can see that this relation is anti-reflexive, anti-symmetric and transitive. It may be also useful to consider a reflexive, anti-symmetric, transitive binary relation of weak dominance Q:

$$Q = \{(x, y) \in Y \times Y \big| \forall q = 1... \ N , \ x_q \geq y_q \ \} \tag{2}$$

Goal: To create, on the basis of the DM's preferences, an imaginary *F*: $Y \to \{Y_i\}, \ i = 1,..., \ M,$ where Y_i – a set of vector estimations belonging to class C_i, satisfying the condition of consistency:

$$\forall x, y \in Y : x \in Y_i, y \in Y_j, (x, y) \in P \Rightarrow i \geq j \tag{3}$$

2. ANALYSIS OF VERBAL DECISION METHODS FOR CLASSIFICATION OF ALTERNATIVES

In this chapter some most frequently used verbal ordinal classification methods are considered. All these methods belong to Verbal Decision Analysis group and have the following common features [2; 9]:

1. The attribute scale is based on verbal description not changing in the process of solution, when verbal evaluation is not converted into the numerical form or score.
2. An interactive classification procedure is performed in steps, where the DM is offered an object of analysis (a course of treatment, for example). An object is presented as a small set of rankings. The DM is familiar with this type of description, therefore he or she can make the classification based on his or her expertise and intuition.
3. When the DM has decided to refer an object to a particular class, the decisions are ranked on the dominance basis. This provides the information about other classes of objects related to it by the relationship of dominance. Thus, an indirect classification of all the objects can be made based on a single decision of the DM.

4. The set of objects dominating over a considered object are referred to as the domination cone. A great number of objects have been classified many times. This ensures error-free classification. If the DM makes an error violating this principle, he or she is shown the conflicting decision on the screen and is prompted to adjust it.
5. In general, a comprehensive classification may be obtained for various numbers of the DM's decisions and phases in an interactive operation. The efficiency of multicriteria classification technique is determined based on the number of questions to the DM needed to make the classification. This approach is justified because it takes into consideration the cost of the DM's time and the need for minimizing classification expenses.

Let us consider several most commonly used methods in more detail.

ORCLASS [4; 6]. This method (Ordinal CLASSification) allows us to build a consistent classification, to check the information and to obtain general decision rules. The method relies on the notion of the most informative alternative, allowing a great number of other alternatives to be implicitly assigned to various classes. ORCLASS takes into account possibilities and limitations of the human information processing system.

Method assessment: The main disadvantage of the method is low effectiveness due to the great number of questions to the DM needed for building a comprehensive classification.

CLARA [5]. This method (CLAssification of Real Alternatives) is based on ORCLASS, but is designed to classify a given subset rather than a complete set of alternatives (Y space). Another common application of CLARA is classification of full set with large number of exclusions, i.e. alternatives with impossible combinations of estimations. In both cases CLARA demonstrates high effectiveness.

DIFCLASS [4]. This method was the first to use dynamic construction of chains covering Y space for selecting questions to DM. However, the area of DIFCLASS application is restricted to tasks with binary criteria scales and two decision classes.

CYCLE [6]. CYCLE (Chain Interactive Classification) algorithm overcomes DIFCLASS restrictions, generalizing the idea of dynamic chain construction to the area of ordinal classification task with arbitrary criteria scales and any number of decision classes. A "chain" means here an ordered sequence of vectors $\langle x_1, \ ..., x_d \rangle$, where $(x_{i+1}, x_i) \in P$ and vectors x_{i+1} and x_i differ in one of the components.

Method assessment: As comparisons demonstrate, the idea of dynamic chain construction allows us to obtain an algorithm close to the optimal by a minimum number of questions for the DM necessary to build a complete classification. The application of ordinal classification demonstrates that problem formalization as well as introduction of classes and criteria structuring allows solution of classification problems by highly effective methods.

The method can be successfully applied to classification of investment projects when the decision classes and the criteria used are thoroughly revised.

3. RISK DETERMINATION AND DESCRIPTION OF THE OPERATIONAL FACTORS OF INVESTMENT PROJECT

After a few iteration series the following final decisions were chosen:

1. a) the lowest risk level,
   b) low risk level.
2. a) satisfactory risk level,
   b) average risk level.
3. a) high risk level,
   b) the highest risk level.

Detailed description of these groups is provided below:

1. Group "evaluation of technical-technological IP risk" is composed of: qualified labour force, supply of construction materials, designing mistakes, progress of the construction work.
2. Group "evaluation of constructional IP risk" is composed of: transport problems, supply problems, production quality, management quality.
3. Group "evaluation of political IP risk" – separate criterion.
4. Group "evaluation of financial IP risk" is composed of: insolvency situations during construction, decrease of project production price in the market, construction expenditure, fluctuations in resource prices.
5. Group "evaluation of ecological IP risk" is composed of: accidents, laws regarding environmental requirements, change in the management attitude towards the project.
6. Group "evaluation of legal IP risk" is composed of: failure to comply with the contracts, inaccurate construction documentation, failure to coordinate the laws, internal and external legal processes.

Next, the classification of the possible investment project risks must be established taking into consideration all levels of their multi-purpose quality descriptions. During that phase the quality of the received results must be checked as well.

First, classification of the factors described on the second level is established. Quality class consists of common evaluations of the first hierarchy level. After classification these common evaluations are filled with concrete contents. Afterwards classification of the first level factors is provided. Final result consists of rules for solving investment project risk evaluation problem.

The DM can establish investment project risk taking into consideration the available classification. It should be noted that only factors from the first hierarchy level might be employed. If difficulties occur while assigning evaluations, the DM creates a second, more accurate level. Moreover, there is a possibility to use the second hierarchy level for separately selected first level factors.

We suggest a way to establish the risk of the construction investment project employing verbal analysis, using CLARA method, which is based on classification that allows evaluating construction investment project by the decision made according to the accurately established classes taking into consideration the respective criteria for risk size evaluation.

The idea of dynamic construction of the links allows for acquiring an algorithm close to the optimal based on the minimal numbers of questions for the DM necessary for establishing the whole classification.

4. CLARA (CLASSIFICATION OF REAL ALTERNATIVES)

A classificator, consisting of risk evaluation criteria and final class decisions, is compiled for establishing investment project risks (Figure 1) [2]. Constructional investment project risk evaluation criteria are provided on the first and second criteria levels.

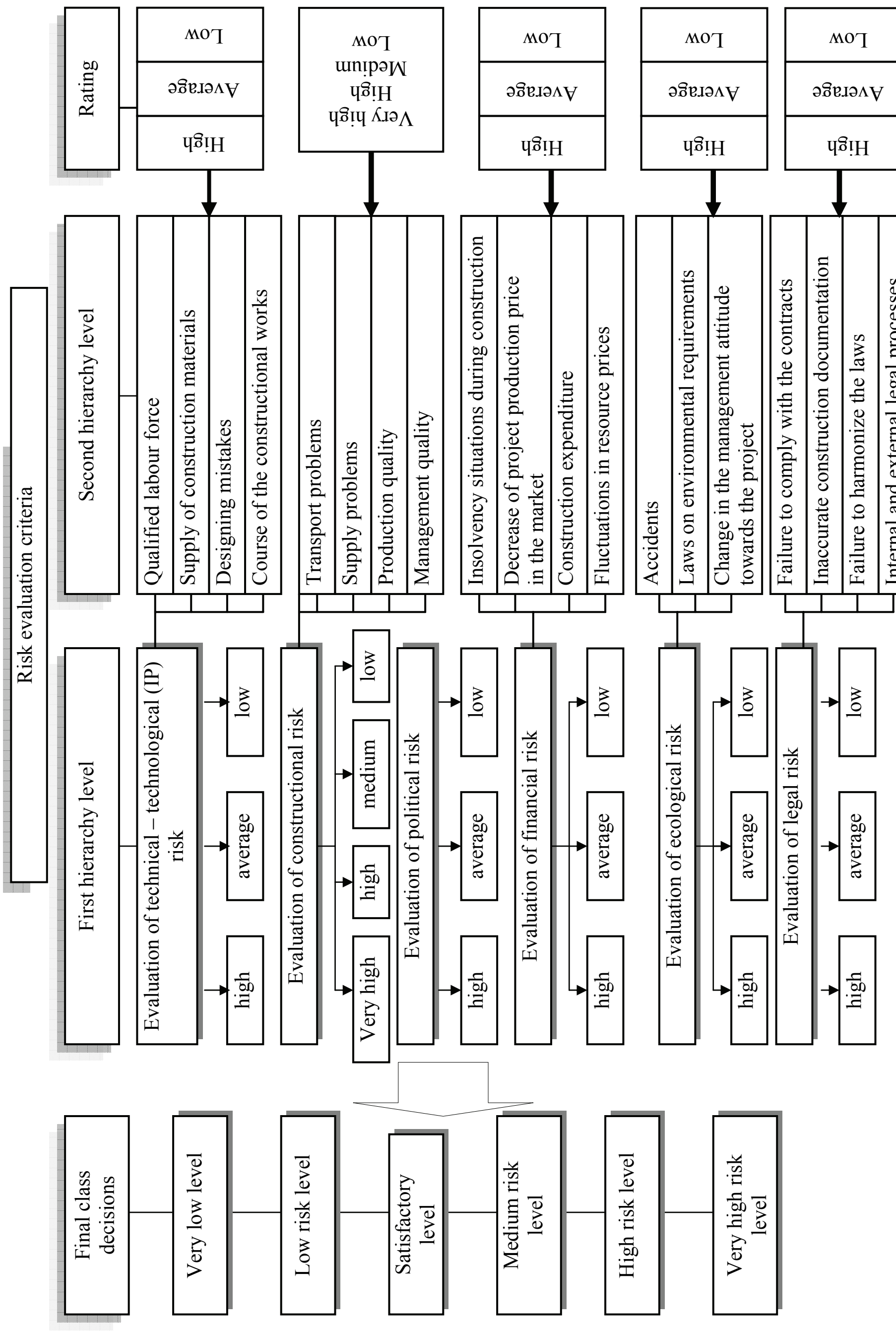

Fig. 1. Classificator

A construction investment project is evaluated by taking into consideration:

- Technical-technological risk
- Constructional risk
- Financial risk
- Political risk
- Ecological risk
- Legal risk

[First hierarchy level criteria]

First hierarchy level is the main one. Construction investment project risk can be evaluated according to the criteria of this level. Each first hierarchy level criterion is assigned an evaluation: low, average, high or very high. When the evaluations are introduced, the result is obtained, i.e. risk levels are established.

These criteria (first level) are not always sufficient for establishment of constructional investment project risk level. Therefore, each first hierarchy level criterion is split into lower level criteria. This way the second hierarchy level is created. Criteria of the second hierarchy level are necessary for performing an accurate analysis (each risk type is analysed) [2].

The following risk evaluation scheme is obtained:

Evaluations of the second hierarchy level criteria $\Rightarrow$ Evaluations of the first hierarchy level criteria $\Rightarrow$ risk level.

Risk level might be established using the composed classificator, but many criteria must be compared. It is a very difficult task for any person, and takes much time. Therefore, it is possible to use the computer program CLARA (classification of real alternatives). This method allows for evaluating construction investment project according to accurately established classes with the respective criteria for risk size evaluation.

Classificator establishment process. Data input into the program

STAGE 1. Evaluation of technical-technological investment project (IP) risk (Figure 2)

For the second hierarchy level evaluation the following criteria are introduced:

- Criterion 1 – qualified labour force.
- Criterion 2 – supply of construction materials.
- Criterion 3 – designing mistakes.
- Criterion 4 – progress of the construction work.

Criteria evaluation classes:

– Class A – high.
– Class B – average.
– Class C – low.

Criteria 1-4 are chosen for evaluation of technical-technological IP risk. While analysing two projects (two alternatives) the expert determines whether the chosen labour force is sufficiently qualified, whether constant supply of materials will be ensured during the construction, what is the estimated progress of work. After the project is has been analysed, it is determined if there are any errors in it.

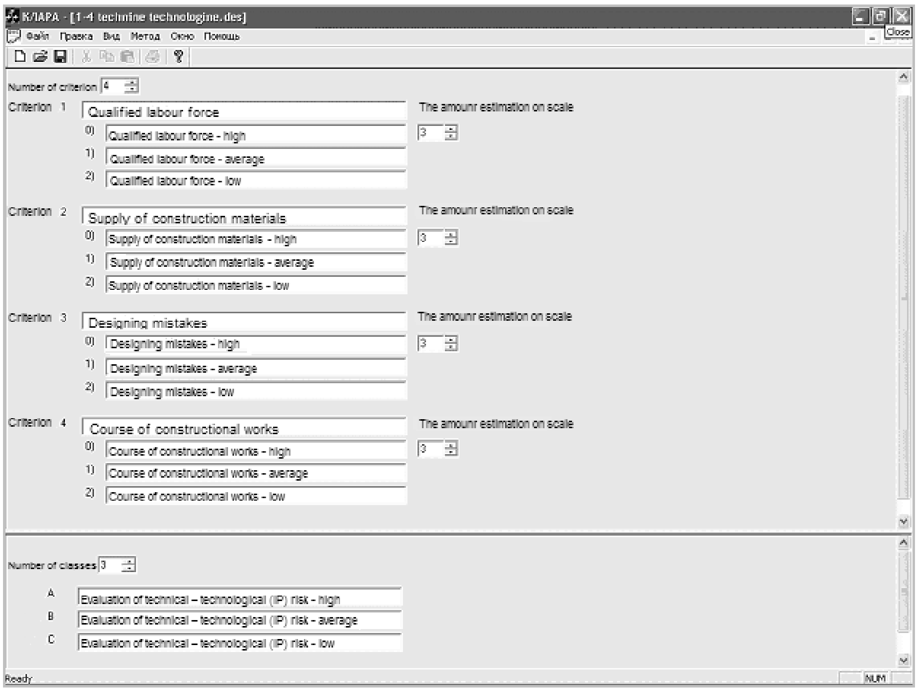

Fig. 2. Criteria for evaluation of technical-technological investment project risk

STAGE 2. Evaluation of constructional investment project risk (Figure 3)

Data input into the program is analogous to the first stage.

– Criterion 1 – transport problems.
– Criterion 2 – supply problems.
– Criterion 3 – production quality.
– Criterion 4 – management quality.

Criteria for evaluating the constructional IP risk are estimated in this stage: during the construction and after the construction.

STAGE 3. Evaluation of financial investment project risk (Figure 4)

Four criteria are used:

– Criterion 1 – insolvency situations during construction.
– Criterion 2 – decrease of project production price in the market.
– Criterion 3 – construction expenditure.
– Criterion 4 – fluctuations in resource prices.

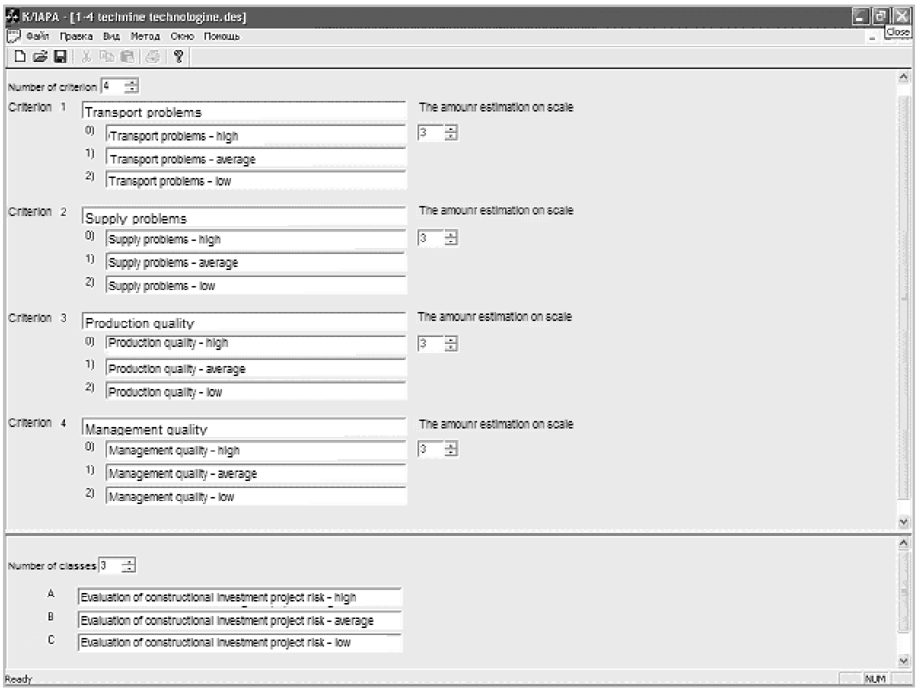

Fig. 3. Criteria for evaluation of constructional investment project risk

STAGE 4. Evaluation of ecological investment project risk

Criteria:

– Criterion 1 – accidents.
– Criterion 2 – laws regarding environmental requirements.
– Criterion 3 – change in the management's attitude towards the project.

STAGE 5. Evaluation of legal investment project risk

Criteria:

- Criterion 1 – failure to comply with the contracts.
- Criterion 2 – inaccurate construction documentation.
- Criterion 3 – failure to coordinate the laws.
- Criterion 4 – internal and external legal processes.

Classification is performed when verbal risk evaluation scheme data are put into the program.

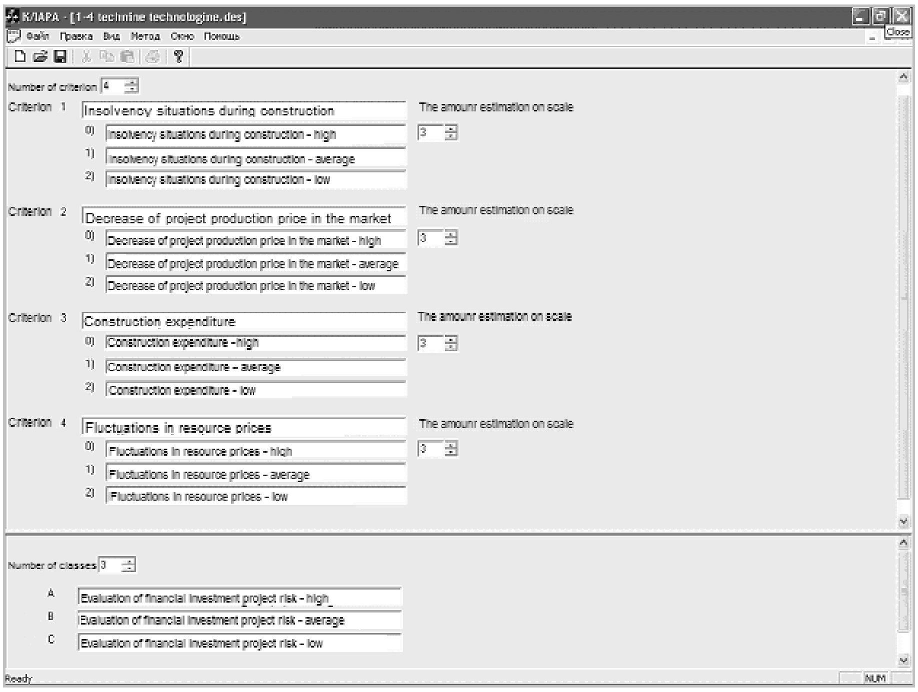

Fig. 4. Criteria for evaluation of financial investment project risk

Implementation of classification in the program

After introducing all the criteria that will be taken into consideration while evaluating two available investment projects, the last stage is performed, i.e. the criteria are compared.

The comparison (Figure 5) is made in the following way: the program selects one evaluation of each criterion and composes their combinations. The expert assigns the available evaluation combination to the respectful class.

For example, if the following combination is taken as input in the program:

1. Qualified labour force – Average.
2. Supply of construction materials – Average.
3. Designing mistakes – High.
4. Progress of the construction work – High.

The expert assigns it to class A – high evaluation.

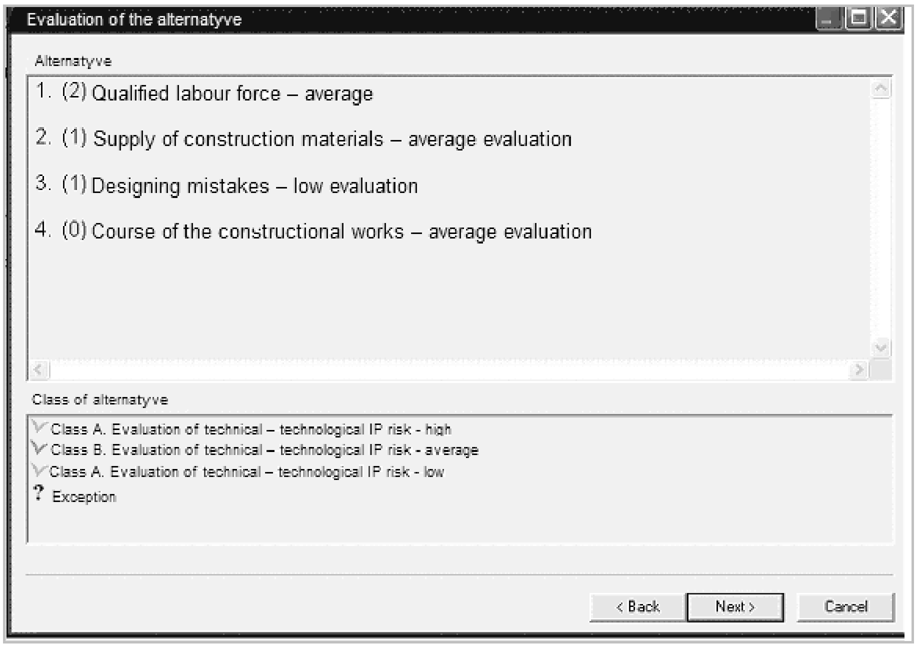

Fig. 5. Evaluation of the alternative

When the assigning is finished, a transfer is made to the next stage (by pushing the button "NEXT"). Another evaluation combination is provided. This is done until all the combinations are assigned to the respective class.

During the work the expert might make a mistake or change his opinion, therefore, contradictions might appear in his answers.

In such case, the program shows a warning that contradictions have occurred and it will ask to confirm the new answer or to change it.

If the program CLARA is used, all the contradictions are eliminated during the work.

After the work is finished, the program saves all the data, performs analysis and shows the number of the given DM questions, the number of classified combinations and the number of eliminated combinations. It also shows the number of the evaluated combinations assigned to classes A, B or C.

Evaluations of all second hierarchy level criteria are established in an analogous way.

In our case, five available files are processed; their use helps to establish the first and the second levels of investment project risk.

Problem solving analysis

Two construction investment project variants are analysed [11].

Description of the first variant [11]. Two-storey housing block with the garret (6 houses) on the nice site outside the city. The holding is in Trakai region, in Čekeliai village, the size of the holding is eleven hectares, the price of the holding, 60 000 Litas. The self-cost price of the real estate is 2 454 130 Lt.

The expert evaluates the available project according to the respective criteria. The evaluations are put into CLARA program data base:

Evaluation of technical-technological IP risk

After analysing the project we obtain the following evaluation:

– Qualified labour force – average evaluation.
– Supply of construction materials – average evaluation.
– Designing mistakes – low evaluation.
– Progress of the construction work – average evaluation.

When these results are input in the program, we find out that the evaluation of technical-technological IP risk is *average – class B.*

Evaluation of constructional IP risk

– Transport problems – very high evaluation.
– Supply problems – very high evaluation.
– Production quality – high evaluation.
– Management quality – high evaluation.

Evaluation of constructional IP risk *very high – class A.*

Evaluation of political IP risk

Evaluation of political IP risk is *low – class C.* This criterion was not evaluated according to separate criteria groups. The expert immediately assigned low evaluation to it.

Evaluation of financial IP risk

– Insolvency situations during construction – low evaluation.
– Decrease of project production price in the market – average evaluation.
– Construction expenditure – average evaluation.
– Fluctuations in resource prices – average evaluation.

Evaluation of financial IP *risk average – class B.*

Evaluation of ecological IP risk

– Accidents – high evaluation.
– Laws on environmental requirements – average evaluation.
– Change in the management's attitude towards the project – high evaluation.

Evaluation of ecological IP risk *high – class A.*

Evaluation of legal IP risk

– Failure to comply with the contracts – high evaluation.
– Inaccurate construction documentation – average evaluation.
– Failure to coordinate the laws – average evaluation.
– Internal and external legal processes – low evaluation.

Evaluation of legal IP risk *average – class B.*

Data base is provided below. It is directly connected with the criteria classification composed in CLARA program. If a person wants to establish the risk level of the construction investment project, he or she must put the evaluations made by the expert into the data base.

We proceed to the second construction investment project evaluation.

Description of the second variant. Two-storey housing block with the garret (6 houses) on the nice site outside the city. The holding is in Molėtai region, Čivyliai village, the size of holding is seven hectares, the price of the holding, 55 000 Litas. The self-cost price of the real estate is 2 554 000 Lt.

The evaluations of the second project are put into CLARA program data base.

Evaluation of technical-technological IP risk

After analysing the project we obtain the following evaluation:

– Qualified labour force – high evaluation.
– Supply of construction materials – average evaluation.
– Designing mistakes – average evaluation.
– Progress of the construction work – average evaluation.

Evaluation of technical-technological IP risk *average – class B.*

Evaluation of constructional IP risk

– Transport problems – high evaluation.
– Supply problems – high evaluation.
– Production quality – average evaluation.
– Management quality – low evaluation.

Evaluation of constructional IP risk *average – class C*.

Evaluation of political IP risk

Similarly to the evaluation of the first project, the evaluation of the political IP risk is *low –_class C.* This criterion was not evaluated according to separate criteria groups. The expert immediately gave it low evaluation.

Evaluation of financial IP risk

– Insolvency situations during construction – high evaluation.
– Decrease of project production price in the market – average evaluation.
– Construction expenditure – low evaluation.
– Fluctuations in resource prices – average evaluation.

Evaluation of financial IP risk *average – class B.*

Evaluation of ecological IP risk

– Accidents – high evaluation.
– Laws on environmental requirements – low evaluation.
– Change in the management's attitude towards the project – low evaluation.

Evaluation of ecological IP risk *average – class B.*

Evaluation of legal IP risk

– Failure to comply with the contracts – average evaluation.
– Inaccurate construction documentation – low evaluation.
– Failure to coordinate the laws – high evaluation.
– Internal and external legal processes – average evaluation.

Evaluation of legal IP risk *average – class B*.

We proceed with the first hierarchy level of the evaluation of the construction investment project.

Final solving analysis

The final analysis is performed according to the evaluations of the first hierarchy level. After the final analysis is performed we get evaluation data of both projects, i.e. we establish their risk levels.

We have six first hierarchy level criteria. Criteria evaluation classes are:

- Class A – the lowest risk level.
- Class B – low risk level.
- Class C – satisfactory risk level.
- Class D – average risk level.
- Class E – high risk level.
- Class F – the highest risk level.

Evaluation combination of the first project according to the second hierarchy level evaluations (Figure 6):

- Evaluation of technical-technological IP risk – average.
- Evaluation of constructional IP risk – very high.
- Evaluation of political IP risk – low.
- Evaluation of financial IP risk – average.
- Evaluation of ecological IP risk – high.
- Evaluation of legal IP risk – average.

Result: according to such evaluations the first construction investment project can be assigned to class B – low risk level.

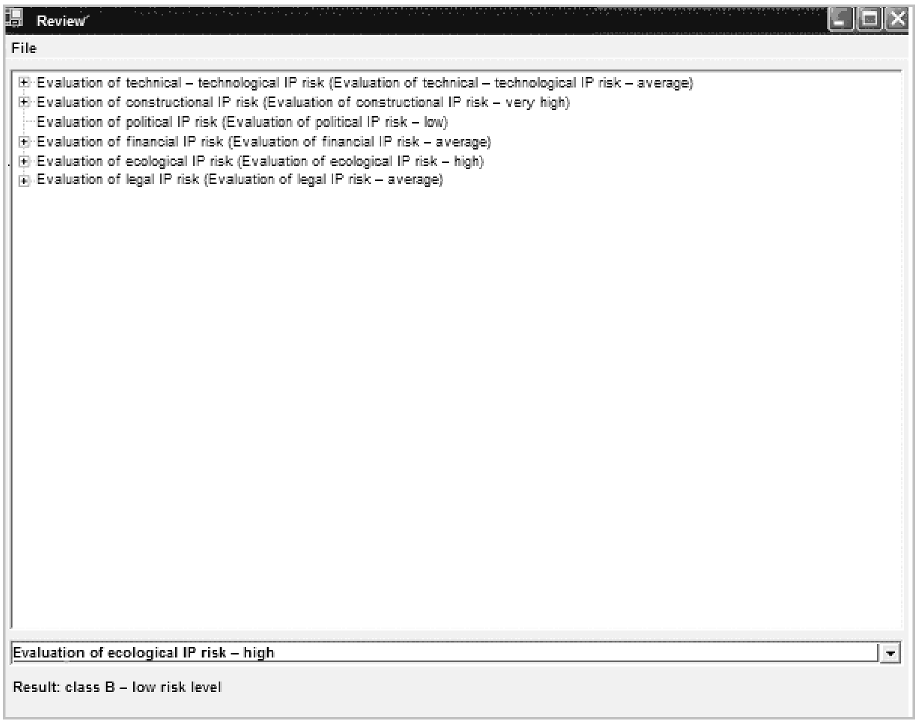

Fig. 6. Data base (first hierarchy level of the first project)

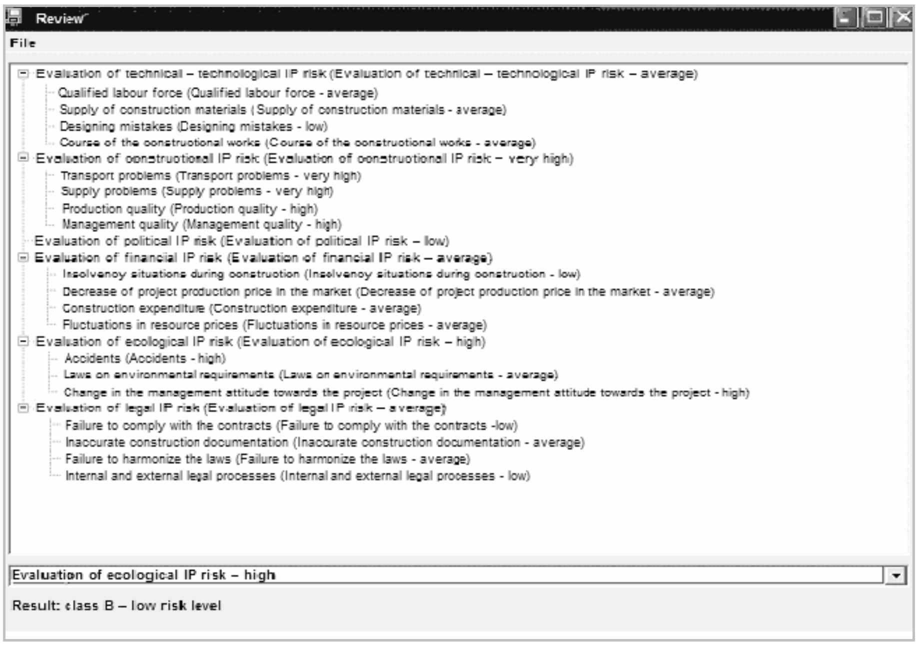

Fig. 7. Data base (second hierarchy level of the first project)

Second project evaluation combination (Figure 8):

– Evaluation of technical-technological IP risk – average.
– Evaluation of constructional IP risk – average.
– Evaluation of political IP risk – low.
– Evaluation of financial IP risk – average.
– Evaluation of ecological IP risk – average.
– Evaluation of legal IP risk – average.

Result: according to such evaluations the second construction investment project is assigned to class C – satisfactory risk level.

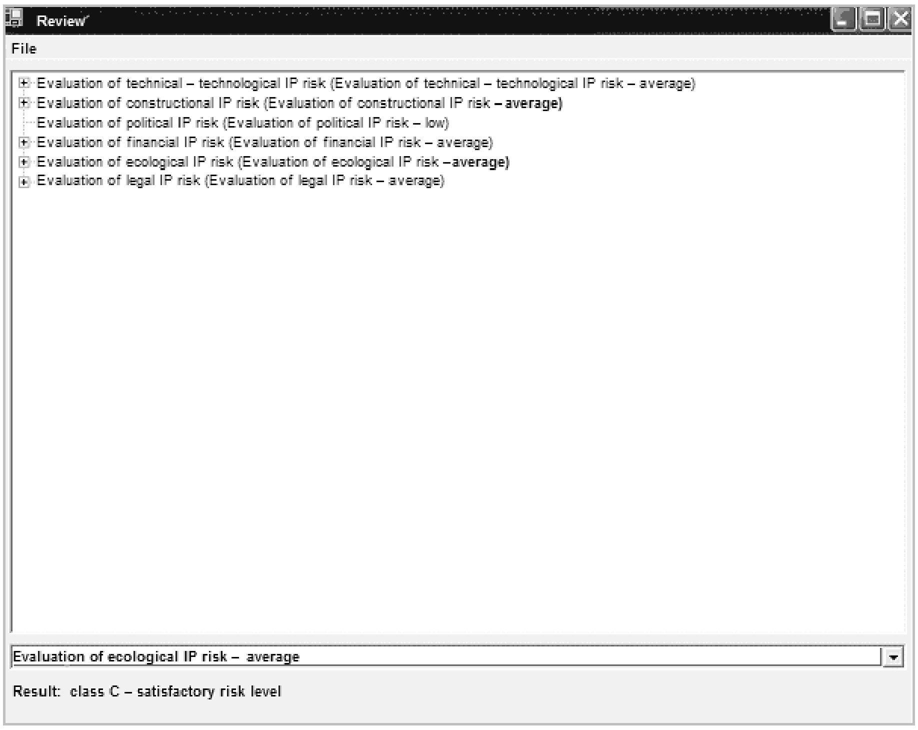

Fig. 8. Data base (first hierarchy level of the second project)

Solution conclusion: after comparing the results obtained we find out that the first project, belonging to class B (low risk level) is less risky than the second project, belonging to class C (satisfactory risk level).

CONCLUSIONS AND SUGGESTIONS

– In practice it is impossible to avoid insufficient and inaccurate information, therefore, unfavourable risky situations occur, the consequences of which can be very damaging to the project. Due to close cooperation of the participants of the project the risk occurring in one stage of the project can transfer to other stages and one type of risk can change into another. This means that chain reaction is characteristic to the risk and it decreases efficiency and safety of any project.

- Various types of risk can be caused by different factors. Classification of risk is determined by efficiency of risk management organisation. Risk classification is understood as risk assignment according to certain features into concrete groups for reaching the set objectives. Conceptually reasoned risk classification allows to define the role of each type of risk in the whole system of all types of risks.
- After reviewing the scientific literature, it is possible to stress that there are many investment project risk evaluation methods, but not all of them are connected by all types of evaluation. This creates difficulties when attempts are made to perform an exhaustive and versatile investment project risk analysis, because often only certain risk evaluation criteria are taken into consideration.
- For establishing the risk of common construction investment project, i.e. for evaluation of all risk factors that influence the efficiency of the project, the available verbal solution analysis methods can be applied. The purpose of these methods is: to determine, in the presence of many problem evaluation criteria, which decision should be made for evaluating the respective problem joining all criteria combinations.
- Investment risk in construction can be evaluated efficiently enough using CLARA method. This method allows to classify all possible construction investment projects presented by evaluations on the predefined criteria into several accurately defined classes reflecting the project risk level. The combination composition idea allows us to obtain an algorithm close to the optimal according to the minimal number of the DM questions.
- The classificator provided in the paper is the main rule for making decisions, evaluating the risk of construction investment projects. It joins factors that influence the probability of risk.
- Criteria of the classifications and the evaluations are introduced into verbal decision analysis support system CLARA, which allows to perform criteria combination classification rather quickly. With all the above-mentioned actions performed, in order to evaluate the risk of the construction investment project it is sufficient to introduce the respective evaluations into the composed program data base and the program will provide the result, that is, the risk level.

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Tomasz Wachowicz

APPLICATION OF MULTIPLE ATTRIBUTE STOCHASTIC DOMINANCE TO SELECTION OF NEGOTIATION STRATEGIES IN E-NEGOTIATIONS

INTRODUCTION

It can be derived from the empirical works [1; 2; 8] and some behavioural models [6; 7] that the negotiation strategy the parties use is one of the most important factors influencing the negotiation process and its outcome. Therefore the determination of the negotiation strategies that allow the negotiating subjects to best satisfy their goals is the major task for the mediator or the negotiation support system in the externally supported negotiations. We propose describing the negotiation situation as a two-person game, the strategies of which correspond to all the possible negotiation strategies that the parties can apply during the negotiation process. This way we find, by determining the solution of the game, the efficient mix of negotiation strategies which maximises the negotiation outcomes for both parties. To compare the payoffs given as the vectors of value distribution we apply the model of multi-attribute stochastic dominance proposed by Zaraś and Martel [11; 12] and to find the game solution we apply the procedure for determining the negotiation set of the game with the combination of the Zaraś and Martel model. The general procedure for determining the negotiation game solution and a numerical example of its application based on the Inspire empirical dataset are also presented.

1. INSPIRE E-NEGOTIATION SYSTEM

We consider bilateral business negotiations conducted via Web, as described in Inspire eNS[1] [5]. Inspire is a simple Internet tool, most frequently used for training and learning negotiations by means of which the hypothetical negotiations between buyer (bicycle manufacturer) and seller

[1] www.inerneg.org/inspire

(bicycle parts producer) can be conducted. It is assumed that the parties wish to sign a new contract and therefore they negotiate four different issues: price of the bicycle parts, time of delivery, time of payment and conditions of spoilage returns. All the issues have predefined values, the only ones that the negotiators can consider during the negotiation process. Inspire helps negotiators evaluate the offers by means of the utility functions that are individually determined in the pre-negotiation phase. In this phase the parties are also asked to fill the pre-negotiation questionnaires in which they describe their psychological profiles, the expectations towards the negotiation atmosphere, the expected behaviour of their opponents and the compromise they are going to achieve.

In the negotiation phase Inspire acts as a communication platform used by the parties to send the offers, arguments and comments. It also presents the history of the process in a graphical way, which helps negotiators analyse and evaluate the scale of concession.

In the post-negotiation phase Inspire analyses the compromise – if achieved – and searches for a better one, if the compromise is outside of the set of Pareto-optimal solutions. In this phase negotiators can also analyse the negotiation dance which is a graphical representation of the concessions made by both parties during the entire negotiation process. Finally, they fill the post-negotiation questionnaires in which they evaluate the negotiation process, their own and their opponents' attitude and behaviour, and the Inspire system itself. In the questionnaires they also describe the negotiation strategy which they applied during the bargaining process. To define the negotiation strategy Inspire users need to evaluate their behaviour on five different platforms: information, persuasion, honesty, exploitation, and co-operation. For each platform they assign a grade between 1 and 5. For example, assigning the grade 1 to the platform of information means that the user consider himself to be extremely informative, while assigning the grade 5 shows him or her to be extremely uninformative.

Gathering of all the detailed information about negotiation processes makes Inspire a powerful tool whose database can be used for many formal analyses. In Section 5 we will use the Inspire data to show an example of the application of the procedure for determining an efficient mix of negotiation strategies.

2. NEGOTIATION GAME

The recent behavioural works in the field of negotiation analysis point to a few factors that influence the negotiation process and its outcome. In summarising the results of these works we can consider the three major determinants of the negotiation outcome, namely: negotiation context, negotiators' psychological profiles and negotiation strategies used by the parties. The negotiation context describes the present and future relationships between the negotiating subjects and the level of conflict [8]. The negotiators' psychological profiles consist of their intrinsic, personal and demographical characteristics [1; 2; 3; 9] like sex, background, experience etc. Some characteristics cannot be changed (e.g. race and background); others can (e.g. emigration or education) but in the vast majority of situations they remain stable during the negotiation process. The negotiation strategies are defined as sets of rules guiding the negotiation process and determining the selection of tactics [6; 7]. Strategies can be perceived as the negotiators' particular ways of acting and behaving during the negotiation process and are purposely selected to fulfil their negotiation goals best.

Since the entire analysis to be conducted is based on the dataset obtained from negotiation experiments conducted in the past, we need to assume that the context of the supported negotiations is the same as the context of the negotiation experiments. Thus only the two remaining factors need to be considered. Therefore the problem under consideration is: how to support two negotiators, both having a particular psychological profile, in determining the efficient mixes of negotiation strategies. The analysis of such case needs to be divided into two separate steps. The psychological profiles of the negotiating subjects have to be identified first. Next, for the negotiators with the specified psychological profiles, the selection of negotiation strategies must be conducted.

The identification of negotiators' psychological profiles comprises a separate analytical problem that can be solved by applying some clustering methods and was previously discussed by Wachowicz [10]. In this paper we assume that we have already identified the psychological profiles of both negotiators and will focus only on the procedure for determining the efficient mix of negotiation strategies.

Since we introduce a mediator (or a software agent) into a negotiation process it is justified to propose a symmetric-prescriptive approach to solve the problem of determining the efficient negotiation strategies. We will construct a two-person negotiation game whose strategies correspond to negotiation strategies which the parties (of specified psychological profiles) can use during the negotiations. If the number of all possible negotiation strategies is too big to consider[2], they can be clustered into a few similarity classes [10]. Each mix of game strategies can be perceived as a separate alternative leading to the outcome described by the payoff value in this matrix cell. The payoffs

[2] With regard to the calculations required.

of such game can be considered as single variables or vectors of variables; they reflect the criteria used by negotiators to evaluate the negotiation outcome. The matrix of payoffs is constructed on the basis of the subset of data applied by the mediator in the analysis[3]. Since the subset of data is relatively large it usually consists of more than one record describing the payoff for each matrix cell. Therefore the payoffs have to be considered not as vectors of deterministic values, but as vectors of value distributions. Solving such game will allow us to find the efficient negotiation strategies for both negotiators.

The general scheme of constructing the negotiation game to support the Inspire users in selection of their negotiation strategies is shown in Figure 1.

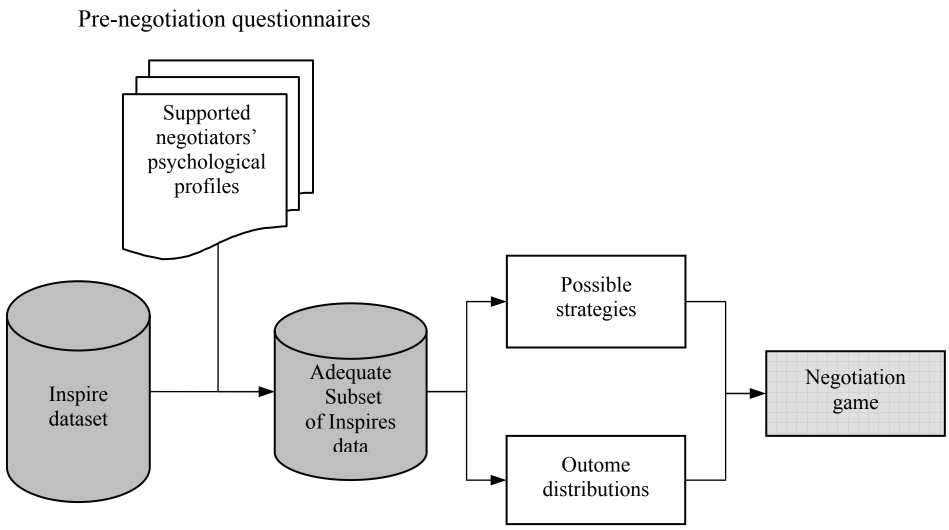

Fig. 1. Construction of the negotiation game for Inspire users

To solve the negotiation game we propose an algorithm of determining the negotiation set of the game developed originally by von Neumann and Morgenstern for the bimatrix game with the utility payoffs [4]. It allows for elimination of the outcomes dominated by individual min-max options of both players from the set of all possible outcomes comprising the game. The elimination prevents from suggesting a mutual agreement which payoffs are worse for one or both the parties that they could assure playing the game non-cooperatively.

[3] The subset of the original dataset consisting of data describing the experiments conducted by negotiators of the same psychological profiles as those of the supported ones.

To compare the alternatives presented as vectors of value distribution we will use stochastic dominance. We propose to apply the Zaraś and Martel model for constructing the global relation of preference [11; 12] which operates with multiple-attribute stochastic dominance. Furthermore, it operates with different types of stochastic dominance which allows finding relation of preference for decision makers with different risk attitudes[4].

We will combine these two procedures to solve the negotiation game, which allows to determine the mix of efficient negotiation strategies. Having identified the efficient mix (or mixes) of strategies the mediator knows the most appropriate way for both negotiators which should suggest to the parties to apply during the negotiation process in order to assure its efficient outcomes.

3. PROCEDURE FOR DETERMINING AN EFFICIENT MIX OF NEGOTIATION STRATEGIES

Let us denote:

A, B – the negotiating parties (negotiator A and negotiator B),

s_A – the negotiation strategy applied by the negotiator A, where $s_A \in S_A$ and S_A is a set of the feasible strategies of the negotiator A ($s_B \in S_B$ is defined similarly for the negotiator B),

$o_A(s_A, s_B)$ – the vector of outcomes obtained by the negotiator A using strategy s_A while the negotiator B is using the strategy s_B, where $o_A(s_A, s_B) \in O_{s_A}$ and O_{s_A} is a set of feasible outcomes for the negotiator A while using the negotiation strategy s_A ($O_{s_A} \in O_A$), ($o_B(s_A, s_B)$ is defined similarly for the negotiator B).

The procedure for solving a two-person negotiation game to find the efficient mix of negotiation strategies for both negotiators can be represented as follows:

1. Determine the worst outcomes $\breve{o}_A(s_A)$ for each game strategy s_A of the negotiator A:

$$\breve{o}_A(s_A) = \Big\{ \breve{o}_A(s_A, s_B) \in O_{s_A} : \underset{s_B \in S_B}{\forall} \underset{o_A(s_A,s_B) \in O_{s_A}}{\neg\exists} \breve{o}_A(s_A,s_B)\, \mathrm{SD}^{\mathrm{p(A)}}_{\mathrm{ZM}}\, o_A(s_A,s_B) \wedge \\ \wedge \underset{o_A(s_A,s_B) \in O_{s_A}}{\exists} o_A(s_A,s_B)\, \mathrm{SD}^{\mathrm{p(A)}}_{\mathrm{ZM}}\, \breve{o}_A(s_A,s_B) \Big\} \tag{1}$$

[4] Risk attitude of Inspire users can be inferred from the pre-negotiation questionnaires.

where:

O_{s_A} – set of feasible outcomes for the negotiator A when he or she uses the strategy s_A, (O_{s_B} is defined analogously for the negotiator B),

$\mathrm{SD}_{\mathrm{ZM}}^{\mathrm{p(A)}}$ – is a relation of preference in the sense of Zaraś and Martel, determined by means of multi-attribute stochastic dominance with weights of the attributes and the concordance threshold assumed by the negotiator A ($\mathrm{SD}_{\mathrm{ZM}}^{\mathrm{p(B)}}$ is defined analogously for the negotiator B).

The outcomes $\breve{o}_A(s_A)$ for all strategies of the negotiator A constitute the set $\breve{O}_A$ of the worst outcomes for this negotiator.

2. Determine the worst outcomes $\breve{o}_B(s_B)$ for each game strategy s_B of the negotiator B:

$$\breve{o}_B(s_B) = \left\{ \breve{o}_B(s_A, s_B) \in O_{s_B} : \underset{s_A \in S_A}{\forall} \; \underset{o_B(s_A,s_B) \in O_{s_B}}{\neg\exists} \; \breve{o}_B(s_A, s_B)\, \mathrm{SD}_{\mathrm{ZM}}^{\mathrm{p(B)}}\, o_B(s_A, s_B) \wedge \right. \\ \left. \wedge \underset{o_B(s_A,s_B) \in O_{s_B}}{\exists} \; o_B(s_A, s_B)\, \mathrm{SD}_{\mathrm{ZM}}^{\mathrm{p(B)}}\, \breve{o}_B(s_A, s_B) \right\} \tag{2}$$

The outcomes $\breve{o}_B(s_B)$ for all strategies of the negotiator B constitute the set $\breve{O}_B$ of the worst outcomes for this negotiator.

3. Identify the best outcomes out of the worst ones from the sets $\breve{O}_A$ i $\breve{O}_B$ of both negotiators such that:

$$\tilde{o}_A = \left\{ \breve{o}_A^*(s_A) \in \breve{O}_A : \underset{\breve{o}_A(s_A) \in \breve{O}_A}{\neg\exists} \; \breve{o}_A(s_A)\, \mathrm{SD}_{\mathrm{ZM}}^{\mathrm{p(A)}}\, \breve{o}_A^*(s_A) \right\} \tag{3}$$

$$\tilde{o}_B = \left\{ \breve{o}_B^*(s_B) \in \breve{O}_B : \underset{\breve{o}_B(s_B) \in \breve{O}_B}{\neg\exists} \; \breve{o}_B(s_B)\, \mathrm{SD}_{\mathrm{ZM}}^{\mathrm{p(B)}}\, \breve{o}_B^*(s_B) \right\} \tag{4}$$

where $\tilde{o}_A$ and $\tilde{o}_B$ are the equivalents of min-max solutions defined by von Neumann and Morgenstern.

4. Eliminate all the outcomes from the set of feasible outcomes that are worse than min-max solutions for each negotiator separately. Thus the sets of acceptable outcomes $\tilde{O}_A$ and $\tilde{O}_B$ for both negotiators are created:

$$\widetilde{O}_A = \{o_A(s_A, s_B) : \widetilde{o}_A \neg \mathrm{SD}_{\mathrm{ZM}}^{\mathrm{p(A)}} \; o_A(s_A, s_B)\} \tag{5}$$

$$\widetilde{O}_B = \{o_B(s_A, s_B) : \widetilde{o}_B \neg \mathrm{SD}_{\mathrm{ZM}}^{\mathrm{p(B)}} \; o_B(s_A, s_B)\} \tag{6}$$

5. Determine the set of acceptable outcomes consisting of outcomes accepted by both negotiators simultaneously:

$$\widetilde{O} = \left\{\left(o_A(s_A, s_B), o_B(s_A, s_B)\right) : o_A(s_A, s_B) \in \widetilde{O}_A \wedge o_B(s_A, s_B) \in \widetilde{O}_B\right\} \tag{7}$$

6. Find the non-dominated outcomes in the set $\widetilde{O}$ with respect to the attributes of both negotiators simultaneously:

$$\begin{aligned}\left(o_A^*, o_B^*\right) = \Big\{\left(o_A^*(s_A, s_B), o_B^*(s_A, s_B)\right) \in \widetilde{O} :\\ : \underset{\left(o_A(s_A,s_B), o_B(s_A,s_B)\right) \in \widetilde{O}}{\neg\exists} \left(o_A(s_A, s_B), o_B(s_A, s_B)\right) \mathrm{SD}_{\mathrm{ZM}}^{\mathrm{p(A,B)}} \left(o_A^*(s_A, s_B), o_B^*(s_A, s_B)\right)\Big\}\end{aligned} \tag{8}$$

where:

$\mathrm{SD}_{\mathrm{ZM}}^{\mathrm{p(A,B)}}$ – is a relation of preference in the sense of Zaraś and Martel, with weights of the attributes and the concordance threshold recalculated with respect to the ones assumed by negotiators *A* and *B*.

The non-dominated outcomes, considered as the solutions of the game, are related to the specific alternatives representing the mixes of game strategies. These mixes correspond to the efficient mixes of negotiation strategies which can be suggested to the supported negotiators.

4. EXAMPLE

We now apply the procedure proposed in Section 3 to the process of determining an efficient mix of negotiation strategies for Inspire users. We assume that the mediator has already identified the psychological profiles of the parties and selected the subset of data to be used in construction of the matrix of payoffs for the negotiation game. We assume further that she or he agreed with the parties on the criteria they will use to evaluate the negotiation outcomes, which are perceived by both parties multi-attributively. These criteria are:

– utility of the agreement (o^1), of importance (weight) equal to 0.5,
– control held over the negotiation process (o^2), weight: 0.3,
– satisfaction from their own performance (o^3), weight: 0.2.

For simplification, the mediator will consider only four different feasible negotiation strategies defined as the four different ways the negotiators can choose on the honesty platform (see Inspire's negotiation strategy definition – section 1)[5]. We number the strategies; thus "1" means strategy of being extremely honest, "2" – honest, "3" – indifferent (neither honest nor dishonest) and "4" – dishonest. The number of records in Inspire's dataset that we will use to describe the distributions of payoffs for different mixes of strategies is given in Table 1.

Table 1

Number of records for the mixes of strategies

		Buyer			
		1	2	3	4
Seller	1	25	29	20	4
	2	25	70	42	21
	3	13	48	39	17
	4	5	11	20	12

Since the matrix of payoffs consists of vectors of value distributions it can not be presented in the traditional game-theoretic way. We will describe it symbolically as the set of alternatives representing the outcomes for each mix of strategies (Table 2).

Table 2

Symbolic representation of the matrix of payoffs

		Buyer			
		1	2	3	4
Seller	1	alternative 1	alternative 2	alternative 3	alternative 4
	2	alternative 5	alternative 6	alternative 7	alternative 8
	3	alternative 9	alternative 10	alternative 11	alternative 12
	4	alternative 13	alternative 14	alternative 15	alternative 16

[5] Originally there were five different possible ways to choose by Inspire users on this platform, but the dataset we use in the analysis consisted of too few observations to construct a matrix of dimensions 5 x 5. Therefore we decided to merge strategies number 4 and 5 into a single strategy.

To find the efficient negotiation strategies for both negotiators the mediator starts the procedure (see Section 3):

1. Determine the worst outcomes $\breve{o}_A(s_A)$ for each game strategy s_A of the negotiator A (seller):

The mediator has to analyse all the alternatives for each feasible strategy of the negotiator A and find the worst ones (giving the worst payoffs for criteria defined by the negotiator A) for each of them. First he analyses the strategy number 1. According to the procedures within the Zaraś and Martel model, he needs to identify the global relation of preferences by determining the stochastic dominances between the alternatives 1, 2, 3 and 4 for every single criterion separately. The results of this procedure are shown in Table 3.

Table 3

Single-attribute stochastic dominances for alternatives constituting the seller's strategy number 1

Attribute 1 – utility				
	alternative 1	alternative 2	alternative 3	alternative 4
alternative 1		TISD2	SSD	SISD
alternative 2	SSD		SSD	SISD
alternative 3				
alternative 4			SSD	
Attribute 2 – control				
	alternative 1	alternative 2	alternative 3	alternative 4
alternative 1		SISD	FSD	SISD
alternative 2	TSD		FSD	SSD
alternative 3				
alternative 4			TSD	
Attribute 3 – satisfaction				
	alternative 1	alternative 2	alternative 3	alternative 4
alternative 1		SISD	SISD	SISD
alternative 2	TSD		FSD	FSD
alternative 3				FSD
alternative 4				

Next, he needs to calculate the explicable (CE) and non-explicable (CN) concordance rates assuming that the seller is risk-averse (see Table 4).

Table 4

Explicable (CE) and non-explicable (CN) concordance rates

Alternatives	1	2	3	4
1 CE 1 CN		0 1	0.8 0.2	0 1
2 CE 2 CN	1 0		1 0	0.5 0.5
3 CE 3 CN	0 0	0 0		0.2 0
4 CE 4 CN	0 0	0 0	0.8 0	

The mediator wants to find an alternative represented by the worst outcome for the highest possible concordance threshold p. Thus he first determines the global relation of preferences for $p = 1$, taking into account the first order, the second order and the third order stochastic dominance only. The relation of preferences is given graphically on Figure 2.

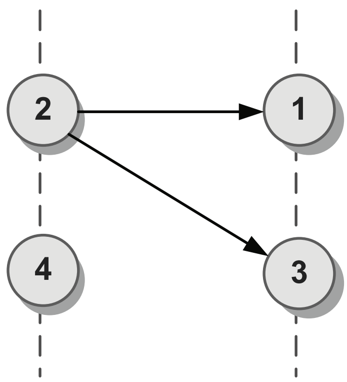

Fig. 2. Relation of preferences for $p = 1$

There are two worst alternatives for the strategy number 1 of negotiator *A*. Since the sum of the explicable and non-explicable concordance rates calculated for the alternative 1 over the alternative 3 is equal to 1, the mediator can explain the relation between these alternatives directly with the negotiator *A*. If the negotiator considers the alternative 1 better than 3, the latter would be chosen as the worst one. The mediator can also lower the concordance threshold

p, if he wants to distinguish a single alternative as the worst one. For the level $p = 0.8$ the alternative 1 dominates the alternative 3 (see Table 4), thus the latter can be considered the worst of all for the strategy number 1. We assume that the mediator considers both alternatives 1 and 3 as equally bad, thus: $\breve{o}_A(1) = \{o_A(1,1), o_A(1,3)\}$. A similar analysis must be provided for all the remaining strategies of the negotiator A. We obtain then:

- for strategy number 2: alternative 7 $\left(\breve{o}_A(2) = o_A(2,3)\right)$,
- for strategy number 3: alternatives 11 and 12 $\left(\breve{o}_A(3) = \{o_A(3,3), o_A(3,4)\}\right)$,
- for strategy number 4: alternative 15 $\left(\breve{o}_A(4) = o_A(4,3)\right)$.

2. Determine the worst outcomes $\breve{o}_B(s_B)$ for each game strategy s_B of the negotiator B (buyer):

The mediator finds the alternatives that give the worst outcomes (evaluated by criteria defined by the negotiator B) for every feasible strategy of the negotiator B. He conducts the same procedure as in step 1 and obtains:

- for strategy number 1: alternatives 9 and 13 $\left(\breve{o}_B(1) = \{o_B(3,1), _B(4,1)\}\right)$,
- for strategy number 2: alternative 10 $\left(\breve{o}_B(2) = o_B(3,2)\right)$.
- for strategy number 3: alternative 11 $\left(\breve{o}_B(3) = o_B(3,3)\right)$,
- for strategy number 4: alternative 16 $\left(\breve{o}_B(4) = o_B(4,4)\right)$.

3. Identify the best outcomes out of the worst in sets $\breve{O}_A$ i $\breve{O}_B$ of both negotiators:

The mediator needs to find the best alternative (giving the best outcomes) out of the six alternatives $\breve{o}_A(s_A)$, defined in step 1 as the worst for each strategy of the negotiator A, that constitute the set $\breve{O}_A$. He determines the global relation of preferences for different concordance thresholds p, looking for the most satisfying order[6]. The mediator obtains the results shown in Table 5.

[6] As the most satisfying order we will consider the one with the highest possible value of concordance threshold p that allows to find a single alternative at the first level of hierarchy or giving the fewest alternatives at the first level of hierarchy.

Table 5

Relations of preferences for different concordance thresholds (negotiator *A*)

Concordance threshold	Order of alternatives – levels of preferences				
	1	2	3	4	5
1	1, 3, 11, 12	7, 15			
1 + CN	1	7	3, 11	12	15

For $p = 1 + \text{CN}$ the mediator obtains $\tilde{o}_A = \breve{o}_A(1) = o_A(1,1)$. Thus the alternative 1 is the best out of the worst ones for the negotiator *A* (the equivalent of min-max option). The $\tilde{o}_B$, corresponding to the best alternative out of the worst ones of the negotiator *B* is determined in a similar way. The global relation of preferences for the alternatives from the set $\breve{O}_B$ is shown in Table 6.

Table 6

Relations of preferences for different concordance thresholds (negotiator *B*)

Concordance threshold	Order of alternatives – levels of preferences		
	1	2	3
1	9, 10, 11	13,16	
1 + CN	10	9, 11	13, 16

For $p = 1 + \text{CN}$ the mediator obtains $\tilde{o}_B = \breve{o}_B(2) = o_B(3,2)$. Thus the alternative 10 is the best out of the worst ones for the negotiator *B*.

4. Eliminate all the outcomes from the set of feasible outcomes that are worse than min-max solutions of each negotiator separately.

The mediator needs to order all the alternatives constituting the game to find the worse ones than min-max options $\tilde{o}_A$ and $\tilde{o}_B$ of both negotiators. First he analyzes the outcomes of the negotiator *A*. For the concordance threshold $p = 0.8 + \text{CN}$[7] he finds ten different alternatives dominated by $\tilde{o}_A$, shown in Table 7 (shaded).

[7] In determining the global relation of preferences we consider the concordance threshold to be acceptable as long as it is greater than 0.5, which means that more than 50% of all criteria were taken into consideration in determining the relations of preferences.

Table 7

Alternatives dominated by $\tilde{o}_A$

		Buyer			
		1	2	3	4
Seller	1	alternative 1	alternative 2	alternative 3	alternative 4
	2	alternative 5	alternative 6	alternative 7	alternative 8
	3	alternative 9	alternative 10	alternative 11	alternative 12
	4	alternative 13	alternative 14	alternative 15	alternative 16

The sets of acceptable outcomes:

$$\tilde{O}_A = \{o_A(1,1); o_A(1,2); o_A(1,4); o_A(2,1); o_A(3,1); o_A(4,2)\}.$$

Next, the mediator analyses the outcomes for the negotiator *B*. For the concordance threshold $p = 0.8 + \mathrm{CN}$ he finds seven different alternatives dominated by $\tilde{o}_B$, shown in Table 8 (shaded).

Table 8

Alternatives dominated by $\tilde{o}_B$

		Buyer			
		1	2	3	4
Seller	1	alternative 1	alternative 2	alternative 3	alternative 4
	2	alternative 5	alternative 6	alternative 7	alternative 8
	3	alternative 9	alternative 10	alternative 11	alternative 12
	4	alternative 13	alternative 14	alternative 15	alternative 16

The sets of acceptable outcomes:

$$\tilde{O}_B = \{o_B(1,1); o_B(1,2); o_B(1,4); o_B(2,1); o_B(2,2); o_B(2,3); o_B(2,4); o_B(3,2); o_B(4,2)\}.$$

5. Determine the set of acceptable outcomes consisting of outcomes accepted by both negotiators simultaneously.

The set $\tilde{O}$ of outcomes accepted by both negotiators simultaneously can be derived from the alternatives which are accepted by the parties individually. If the outcomes of one particular alternative are accepted by the negotiator *A* (that is, they belong to the set $\tilde{O}_A$) and by the negotiator *B* (that is, they belong to the set $\widetilde{O}_B$), they comprise the outcome accepted simultaneously that comprise the set $\widetilde{O}$. The alternatives whose outcomes belong to the set $\widetilde{O}$ are shown in Table 9 (shaded).

Table 9

Alternatives outside the set $\widetilde{O}$

		Buyer			
		1	2	3	4
Seller	1	alternative 1	alternative 2	alternative 3	alternative 4
	2	alternative 5	alternative 6	alternative 7	alternative 8
	3	alternative 9	alternative 10	alternative 11	alternative 12
	4	alternative 13	alternative 14	alternative 15	alternative 16

6. Find the non-dominated outcomes in the set $\widetilde{O}$ with respect to the attributes of both negotiators simultaneously:

The mediator must now find the global relation of preferences for the five alternatives comprising the set $\widetilde{O}$. He analyses the outcomes of both negotiators simultaneously, thus he needs to change weights of all the attributes whose sum must be equal to 1. We assume that the mediator wants to act fairly, thus he decides to divide the initial weights specified by both negotiators by 2. Due to that the criteria of each negotiator will have an equal share in the sum of 1 (both of 0.5).

According to Zaraś and Martel model the mediator needs now to determine the stochastic dominance between the alternatives for every single criterion separately. The results are given in Table 10.

Table 10

Single-attribute stochastic dominances for alternatives from the set $\widetilde{O}$

Attribute 1 – utility of negotiator *A*						Attribute 2 – control of negotiator *A*					
alternative	1	2	4	5	14	alternative	1	2	4	5	14
1		TISD2	SISD		TISD2	1		SISD	SISD		
2	SSD		SISD	TSD	TSD	2	TSD		SSD		
4						4					
5	SSD	SISD	SISD		SISD	5	FSD	FSD	FSD		FSD
14	SSD	SISD	SISD			14	SSD	FSD	SSD		

Attribute 3 – satisfaction of negotiator *A*					
alternative	1	2	4	5	14
1		SISD	SISD		
2	TSD		FSD		
4					
5	FSD	SISD	FSD		SISD
14	SSD	SISD	FSD		

Attribute 4 – utility of negotiator *B*					
alternative	1	2	4	5	14
1				SSD	SISD
2	TISD2			SSD	SISD
4	SSD	SSD		SSD	SISD
5	TISD2				SISD
14				TSD	

Attribute 5 – control of negotiator *B*					
alternative	1	2	4	5	14
1		FSD	FSD	SSD	FSD
2				SSD	FSD
4		TSD		SSD	FSD
5		TISD2			SISD
14					

Attribute 6 – satisfaction of negotiator *B*					
alternative	1	2	4	5	14
1				SISD	FSD
2	TISD1			SISD	FSD
4	FSD	SSD		FSD	FSD
5					SSD
14					

Having analysed the single-attribute dominances, the mediator, using the weights of the attributes, calculates the explicable (CE) and non-explicable (CN) concordance rates (Table 11).

Table 11

Explicable (CE) and non-explicable (CN) concordance rates

Alternative	1	2	4	5	14
1 CE		0.15	0.15	0.4	0.25
1 CN		0.5	0.5	0.1	0.25
2 CE	0.5		0.25	0.65	0.5
2 CN	0.35		0.25	0.2	0.35
4 CE	0.35	0.5		0.5	0.25
4 CN	0.5	0.25		0.25	0.5
5 CE	0.5	0.15	0.25		0.25
5 CN	0.1	0.2	0.25		0.5
14 CE	0.5	0.15	0.25	0.25	
14 CN	0.25	0.35	0.5	0.5	

Based on these concordance rates the mediator affirms that all the alternatives from the set $\widetilde{O}$ are equally good for the parties for the threshold $p = 1$ (there is no alternative that dominates another one). But the mediator can determine the most satisfying global relation of preferences by lowering the concordance threshold to the value of 0.75 and analysing the non-explicable concordance rates. In that case she or he obtains the relation of preferences as shown in Figure 3.

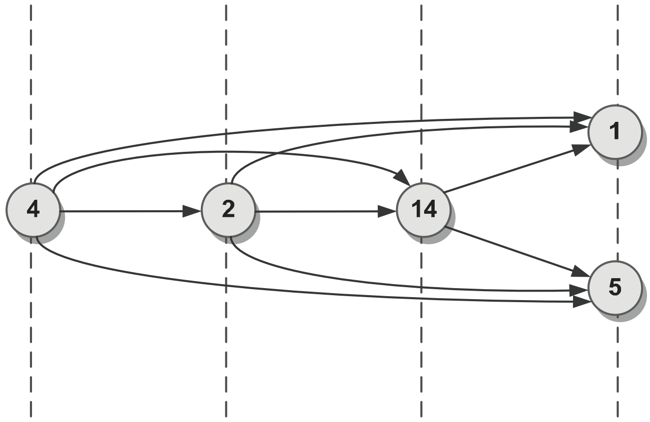

Fig. 3. Global relation of preferences for alternatives from the set $\widetilde{O}$ (p = 0.75 + CN)

Since we have determined the alternative 4 to be the most preferable (the best), we obtain $\left(o_A^*, o_B^*\right) = \left(o_A^*(s_A, s_B), o_B^*(s_A, s_B)\right) = \left(o_A^*(1,4), o_B^*(1,4)\right)$. Hence an efficient mix of negotiation strategies for both parties is $(s_A, s_B) = (1, 4)$. It is the strategy number 1 for the seller (negotiator *A*), which recommends acting in a very honest way, and the strategy number 4 for the buyer (negotiator *B*) – acting in a dishonest way.

Knowing the efficient strategies for the negotiating parties, the mediator has to convince them, using some method of persuasion, to apply such strategies, since they assure the most satisfying and efficient outcomes for both parties.

CONCLUSIONS

From the recent behavioral research and studies we concluded that the strategies applied by the negotiators during the negotiation process are the key factors (along with the negotiation context and negotiators' psychological profiles) determining the outcomes achieved by the parties. Therefore in the paper we proposed the concept of supporting mediator helping with the selection of the negotiation strategies for parties negotiating by means of an e-negotiation system. We suggested a symmetric-prescriptive approach for solving such decision problem. The two-person negotiation game, whose strategies correspond to the feasible negotiation strategies, was constructed. To solve the game we applied the Von Neumann and Morgenstern procedure for determining the negotiation set of the game. It allows for reducing the number of feasible solutions to the individually non-dominated ones. Although the game theory requires the decision problem to be simplified, such a structured problem can be easily implemented as a software module in a negotiation support system.

Since the game payoffs are defined as vectors of value distributions, to compare all feasible alternatives we apply the Zaraś and Martel model for determining the global relation of preferences based on multi-attribute stochastic dominance. It allows for construction of the relation of preferences for decision makers (negotiators) with different risk attitude. By combining these two procedures we obtain the game solution which is a mix of efficient negotiation strategies. The mediator knows then how the parties should act to achieve the efficient outcomes for both and a mutually most satisfying agreement.

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CONTRIBUTING AUTHORS

Tomasz Błaszczyk
Department of Operations Research
The Karol Adamiecki University of Economics
Katowice, Poland
tblaszcz@ae.katowice.pl

Bernhard Böhm
University of Technology
Vienna, Austria
bernhard.boehm@tuwien.ac.at

Rafael Caballero
Department of Applied Economics (Mathematics)
University of Málaga, Spain
rafael.caballero@uma.es

Sydney CK Chu
Department of Mathematics
University of Hong Kong
schu@hku.hk

Cezary Dominiak
Department of Operations Research
The Karol Adamiecki University of Economics
Katowice, Poland
dominiak@ae.katowice.pl

Petr Fiala
Department of Econometrics
University of Economics
Prague, Czech Republic
pfiala@vse.cz

Trinidad Gómez
Department of Applied Economics (Mathematics)
University of Málaga, Spain
Trinidad@uma.es

Mónica Hernández
Department of Applied Economics (Mathematics)
University of Málaga, Spain
m_huelin@uma.es

Markus Hirschberger
Department of Mathematics
University of Eichstätt-Ingolstadt
Eichstätt, Germany

Josef Jablonsky
Department of Econometrics
University of Economics
Prague, Czech Republic
jablon@vse.cz

Dmitry Kochin
Institute for System Analysis
Moscow, Russia

Dorota Kuchta
Institute of Industrial Engineering and Management
Wrocław University of Technology
Wrocław, Poland
dorota.kuchta@pwr.wroc.pl

María Amparo León
Department of Mathematics
University of Pinar del Río,
Pinar del Río, Cuba

Mikuláš Luptáčik
Vienna University of Economics and Business Administration
Vienna, Austria
Mikulas.Luptacik@wu-wien.ac.at

Kaisa Miettinen
Helsinki School of Economics
Helsinki, Finland
miettine@hse.fi

Sigitas Mitkus
Vilnius Gediminas Technical University
Vilnius, Lithuania

Maciej Nowak
Department of Operations Research
The Karol Adamiecki University of Economics
Katowice, Poland
nomac@ae.katowice.pl

Włodzimierz Ogryczak
Warsaw University of Technology, ICCE
Warsaw, Poland
W.Ogryczak@ia.pw.edu.pl

Yue Qi
Terry College of Business
University of Georgia
Athens, Georgia, USA

Jaroslav Ramík
Silesian University
Karvina, Czech Republic
jaroslav.ramik@t-email.cz

Jaideep Roy
Department of Economics and Related Studies
University of York
York, United Kingdom and
LKAEM
Warsaw, Poland
jroy@wspiz.edu.pl

Vaidotas Šarka
Building Technology and Management Department
Vilnius Gediminas Technical University
Vilnius, Lithuania
E_raud@centras.lt

Edita Šarkienė

Building Technology and Management Department
Vilnius Gediminas Technical University
Vilnius, Lithuania
Edita.Sarkiene@adm.vtu.lt

Galina Ševčenko
Vilnius Gediminas Technical University
Vilnius, Lithuania
galina.s@transcom.lt

Sebastian Sitarz
Institute of Mathematics
University of Silesia
Katowice, Poland
ssitarz@ux2.math.us.edu.pl

Honorata Sosnowska
Department of Mathematical Economics
Warsaw School of Economics
Warsaw, Poland and
Center for Economic Psychology and Decision Sciences, LKAEM
Warsaw, Poland
honorata@sgh.waw.pl

Ralph E. Steuer
Terry College of Business
University of Georgia
Athens, Georgia, USA
rsteuer@uga.edu

Tadeusz Trzaskalik
Department of Operations Research
The Karol Adamiecki University of Economics
Katowice, Poland
ttrzaska@ae.katowice.pl

Małgorzata Trzaskalik-Wyrwa
Church Music School
Diocese Siedlce, Poland
mtrzask@chopin.edu.pl

Leonas Ustinovichius
Building Technology and Management Department
Vilnius Gediminas Technical University
Vilnius, Lithuania
leonasu@st.vtu.lt.lt

Tomasz Wachowicz
Department of Operations Research
The Karol Adamiecki University of Economics
Katowice, Poland
wachowic@ae.katowice.pl

Christina SY Yuen
Department of Mathematics
University of Hong Kong, Hong Kong
ChristinaYuen@yahoo.com