MULTIPLE CRITERIA DECISION MAKING

ANNUAL
VOL. 8 (2013)

THE UNIVERSITY OF ECONOMICS IN KATOWICE 2013

ISSN 2084-1531

Publisher of The University of Economics in Katowice
ul. 1 Maja 50, 40-287 Katowice, tel. +48 032 25 77 635, fax +48 032 25 77 643
www.ue.katowice.pl, e-mail: wydawnictwo@ue.katowice.pl

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Vol. 8 2013

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ALGORITHM FOR BI-CRITERIA STOCHASTIC GENERALIZED TRANSPORTATION PROBLEM

Abstract

The Generalized Transportation Problem is a variant of the classical Transportation Problem, where the sum of the amounts of goods delivered to the destination points is different from (usually lower than) the total amount sent from the sources. The Stochastic Generalized Transportation Problem (SGTP) is a version with random demand. We present the Bi-Criteria SGTP and propose an algorithm for determining the set of effective solutions.

Keywords: Bi-criteria stochastic generalized transportation problem, Pareto-optimal solution, stochastic programming, equalization method.

1 Introduction

The Generalized Transportation Problem (GTP) can be used in many real-life applications. It is a special case of the Generalized Flow Problem. The theory of generalized flows and chosen solution methods may be found in Ahuja et al. (1993). A polynomial algorithm for the Generalized Minimum Cost Flow Problem was presented in Wayne (2002). Combinatorial algorithms for the Generalized Circulation Problem were presented in Goldberg et al. (1988). Some issues concerning the generalized networks may be also found in Glover et al. (1972).
The Generalized Transportation Problem was considered e.g. in Balas (1966), Balas and Ivanescu (1964) and Lourie (1964). Anholcer and Kawa (2012) considered application of the two-stage GTP in the logistic networks, where complaints are involved in the distribution process. The connection between complaints ratio and the complexity of the resulting logistic network was studied.

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The significance of generalized flows (in particular of the Generalized Transportation Problem discussed in this paper) follows from the fact that they allow to model the situations when the amount of transported goods changes during the transportation process. It is obviously a useful generalization of ordinary flow (in particular transportation) problems and implies many real-life applications. As mentioned above, Anholcer and Kawa (2012) modeled a distribution network involving complaints. In Ahuja et al. (1993), the authors present several possible applications of generalized flows. For example, they may be used in the modeling of conversions of physical entities in financial, mineral and energy networks. Another application may be machine loading. Yet another possible application discussed by the authors is Land Management Problem. Nagurney et al. (2013) discuss in turn the application of generalized flows in the modeling of supplies of medical materials, food, pharmaceuticals and clothes.

Although in the models one often assumes that the demand is fixed, it is more likely that it will be unpredictable. In real-life applications one may assume at most that the distribution of the demand may be somehow estimated. The Stochastic Generalized Transportation Problem (SGTP) is the generalized version of GTP where the demands are given as random variables with known distributions. In such a case we are interested in minimizing the expected value of the total cost. This way a variant of the Nonlinear Generalized Transportation Problem is obtained. The case of the Stochastic Transportation Problem was discussed e.g. by Williams (1963), Cooper (1977), Holmberg and Jörnsten (1984) and Holmberg (1995). More general version of the Nonlinear Transportation Problem (where any convex costs at the destination points are applicable) were analyzed e.g. by Anholcer (2005, 2008a, 2008b), Sikora (1993) and Sikora et al. (1991). In the last papers the Equalization Method was considered. The convergence of the version for Nonlinear Transportation Problem was proved by Anholcer (2005, 2008a). The convergence of the version for Nonlinear and Stochastic GTP was also proved by Anholcer (2013a, 2013b). In Qi (1985) the Forest Iteration Method for the Stochastic Problem was proposed. Finally, a variant of the latter method, the A-Forest Iteration Method for the Stochastic Generalized Transportation Problem (SGTP) was presented in Qi (1987).

Note, that the stochastic versions of the Generalized Transportation Problem and of the Transportation Problem have the objective functions very similar to the Newsvendors Problem. Actually, the Newsvendor Problem can be considered as a special case of the Stochastic Transportation Problem with one source and one destination point (then, of course, the transportation costs can be treated as a constant and omitted). This is worth noticing as the Newsvendor Problem itself has been known at least from the moment of publication of Edgeworth (1888). Since then, the model was generalized and the solution methods were developed – see e.g. Khouja et al. (1996), Chen and Chuang (2000), Yang et al. (2007), Goto (2013).

In all these papers only the costs were taken under consideration. It is more likely, however, that the Decision Maker will be interested also in the minimization of the total time of the transportation process. This leads us to the bi-criteria problem.

Various versions of the Bi- and Multi-criteria Transportation Problems were analyzed e.g. by Aneja and Nair (1979), Gupta and Gupta (1983), Shi (1995), Li (2000), Basu and Acharya (2002), Khurana and Arora (2011), Kesavarz and Khorram (2011) and Kumar et al. (2012). The Generalized Transportation Problem in the multi-criteria version was studied e.g. by Gen et al. (1999).

In this paper we present a method for determining the set of effective solutions of Bi-criteria Stochastic Generalized Transportation Problem. In section 1 the problem is formulated. In sections 2-5 the algorithm is presented. In section 6 computational experiments are described. Section 7 contains main conclusions and final remarks.

2 Problem formulation

In the ordinary Generalized Transportation Problem, uniform good is transported from m supply points to n destination points. The amount of good delivered to the demand point j from the supply point i is equal to $r_{ij}x_{ij}$, where x_{ij} is the amount of good that leaves the supply point i and r_{ij} is the *reduction ratio*. The unit transportation costs c_{ij} are constant, the demand b_j of every demand point j has to be satisfied and the supply a_i of any supply point i cannot be exceeded. Hence, the model has the following form:

$$\min\left\{f(x) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}\right\},$$

s. t.

$$\sum_{i=1}^{m} r_{ij}x_{ij} = b_j, j = 1, \dots, n,$$

$$\sum_{j=1}^{n} x_{ij} \le a_i, i = 1, \dots, m,$$

$$x_{ij} \ge 0, i = 1, \dots, m, j = 1, \dots, n.$$

In the Stochastic GTP (SGTP), the demands b_j are continuous random variables X_j with density functions φ_j. We will assume that for every $j = 1, \dots, n$ and for every $x > 0$,

$$\varphi_j(x) > 0.$$

The unit surplus cost $s_j^{(1)}$ and the unit shortage cost $s_j^{(2)}$ are defined for every destination point j. As the total amount of good delivered to any destination cannot be less than 0, the function of expected extra cost for destination j takes the form:

$$f_j(x_j) = s_j^{(1)} \int_0^{x_j} (x_j - t)\varphi_j(t)dt + s_j^{(2)} \int_{x_j}^{\infty} (t - x_j)\varphi_j(t)dt,$$

which can be easily transformed to the form:

$$f_j(x_j) = s_j^{(2)}\left(E(X_j) - x_j\right) + \left(s_j^{(1)} + s_j^{(2)}\right) \int_0^{x_j} \Phi_j(t)dt,$$

where Φ_j is the cumulative distribution function of the demand at destination j.

Finally, the SGTP takes the form:

$$\min\left\{f(x) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j(x_j)\right\},$$

s. t.

$$\sum_{i=1}^{m} r_{ij}x_{ij} = x_j, j = 1, \dots, n,$$

$$\sum_{j=1}^{n} x_{ij} \le a_i, i = 1, \dots, m,$$

$$x_{ij} \ge 0, i = 1, \dots, m, j = 1, \dots, n.$$

It is straightforward to see that each function f_j is twice differentiable and strictly convex. This means that the Equalization Method described by Anholcer (2013a) (dedicated to convex functions) may be applied in order to solve this type of problem (see Anholcer, 2013b).

The second criterion that we are interested in is the time. For every pair (i, j) of supply point i and destination point j an integer delivery time t_{ij}, not depending on the amount of transported good, is defined. We assume that all the deliveries may be performed simultaneously. This implies that the transportation process finishes when the last delivery is finished. Thus, the second objective is to minimize the maximum time over all the deliveries. It means that the bi-criteria problem (BSGTP) takes the form:

$$\min\left\{f(x) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j(x_j)\right\},$$

$$\min\left\{t(x) = \max_{x_{ij}>0} t_{ij}\right\},$$

s. t.

$$\sum_{i=1}^{m} r_{ij}x_{ij} = x_j, j = 1, \dots, n,$$

$$\sum_{j=1}^{n} x_{ij} \leq a_i, i = 1, \dots, m,$$
$$x_{ij} \geq 0, i = 1, \dots, m, j = 1, \dots, n.$$

Usually the minima of both objective functions are different. Our goal is to find the set of the effective (Pareto-optimal) solutions.

3 Algorithm – main idea

Let S denote the set of the feasible solutions of BSGTP. The problem may be rewritten as

$$\min f(x),$$
$$\min t(x),$$

s. t.

$$x \in S.$$

Let T_1 be the minimum value of $t(x)$ over S and let T_2 be the minimum value of $t(x)$ over the set of solutions x where $f(x)$ is minimized. Obviously both T_1 and T_2 are integers. The following observation holds true.

Theorem 1

A solution $x' \in S$ of BGSTP is Pareto-optimal if and only if it satisfies the following conditions:

$t(x') = T$ for some integer T, where $T_1 \leq T \leq T_2$,

there exist integers i and j, $1 \leq i \leq m$, $1 \leq j \leq n$, such that $T = t_{ij}$,

x' is the optimal solution of the problem

$$\min f(x),$$

s. t.

$$x \in S,$$
$$t(x) = T.$$

Proof

Assume that the solution x is Pareto-optimal. Then there is no solution x'' such that either $f(x'') \leq f(x')$ and $t(x'') < t(x')$, or $f(x'') < f(x')$ and $t(x'') \leq t(x')$. It follows that $t(x') \geq T_1$ by the definition of T_1. On the other hand $t(x') \leq T_2$, as otherwise by the definition of T_2 there exists a point x'' with $t(x'') = T_2$, where $f(x)$ reaches its minimum value over S and so $f(x'') \leq f(x')$ and $t(x'') < t(x')$, a contradiction. Of course at every point $x \in S$ we have $t(x) = t_{ij}$ for some i and j, $1 \leq i \leq m$, $1 \leq j \leq n$. Finally, x' has to be the optimum of the above problem as otherwise any of its optimal solutions x'' would satisfy $f(x'') < f(x')$ and $t(x'') \leq t(x')$, a contradiction.

Assume now there is a point $x'' \in S$ such that $f(x'') \leq f(x')$ and $t(x'') < t(x')$. As by decreasing the value of T we cannot obtain a better solution to the problem from the condition (iii), this means that $f(x'') = f(x')$ and in consequence $x'' = x'$ (as the objective function is strictly convex, there is a unique optimum for each value of T). Finally assume that there is a point $x'' \in S$ such that $f(x'') < f(x')$ and $t(x'') \leq t(x')$. It would mean that the problem from condition (iii) has a better solution on a subset of its set of feasible solutions than on the whole set of feasible solutions. That is impossible, a contradiction. This implies finally that x' is Pareto-optimal.

The solution strategy is then as follows. We start with finding the solution $x^{(1)}$ minimizing the value of $t(x)$. Let $T_1 = T^{(1)} = t\left(x^{(1)}\right)$. Then in iteration k, given $T^{(k)}$, we define the value of $T^{(k+1)}$ as the smallest t_{ij} such that $t_{ij} > T^{(k)}$. Then we solve the problem from point (iii) of Theorem 1 and find the solution $x^{(k+1)}$. If $x^{(k+1)} \neq x^k$, then it is the next Pareto-optimal solution of BSGTP. Finally, if $x^{(k+1)}$ is also an optimal solution of SGTP, then we stop the procedure, as $x^{(k+1)} = T_2$. In the three following sections we are going to present the details of the algorithm. In section 3 we briefly describe the solution method of the GTP with Time Criterion. In section 4 we present the modified version of the Equalization Method for the SGTP with an additional time constraint. Finally, in section 5 we describe the main algorithm in a detailed way.

4 Generalized Transportation Problem with Time Criterion

The GTP with time criterion has the following form:

$$\min\left\{t(x) = \max_{x_{ij}>0} t_{ij}\right\},$$

s. t.

$$\sum_{i=1}^{m} r_{ij}x_{ij} = x_j, j = 1, \dots, n,$$

$$\sum_{j=1}^{n} x_{ij} \leq a_i, i = 1, \dots, m,$$

$$x_{ij} \geq 0, i = 1, \dots, m, j = 1, \dots, n.$$

It is straightforward to see that the optimal solution of this problem is

$$x_{ij} = 0, i = 1, \dots, m, j = 1, \dots, n,$$

and the minimum value of the objective function is:

$$T_1 = t_{\min}(x) = 0.$$

This solution will be the initial solution for the modified Equalization Method at the first step of the main algorithm. The initial solution in all other steps will be the optimum obtained by the Equalization Method in the previous step.

5 Modified Equalization Method

Given time T, we will solve the problem using the modified version of the Equalization Method (see Anholcer 2013a and 2013b). Let us introduce m additional variables $x_{i,n+1}$. Let $c_{i,n+1} = 0$, $r_{i,n+1} = 1$ for $i = 1, \dots, m$ and let $f_{n+1}(x_{n+1}) \equiv 0$. Then the SGTP with time constraint can be rewritten as :

$$\min\left\{f(x) = \sum_{i=1}^{m}\sum_{j=1}^{n+1} c_{ij}x_{ij} + \sum_{j=1}^{n+1} f_j(x_j)\right\},$$

s. t.

$$\sum_{i=1}^{m} r_{ij}x_{ij} = x_j, j = 1, \dots, n+1,$$
$$\sum_{j=1}^{n+1} x_{ij} = a_i, i = 1, \dots, m,$$
$$x_{ij} = 0, i = 1, \dots, m, j = 1, \dots, n, t_{ij} > T,$$
$$x_{ij} \geq 0, i = 1, \dots, m, j = 1, \dots, n, t_{ij} \leq T.$$

The KKT optimality conditions can be formulated as

$$c_{ij} + r_{ij}f_j'(x_j) \geq u_i, i = 1, \dots, m, j = 1, \dots, n+1, t_{ij} \leq T, x_{ij} = 0,$$
$$c_{ij} + r_{ij}f_j'(x_j) = u_i, i = 1, \dots, m, j = 1, \dots, n+1, t_{ij} \leq T, x_{ij} > 0.$$

The following version of the Equalization Method converges to the KKT point of the SGTP with Time Constraint (the proof is similar to the proofs that can be found in Anholcer (2013a, 2013b), so it will be omitted). As it is a convex programming problem, the resulting point is the optimal solution. We calculate two measures of accuracy, α and $\alpha^\star$ in order to be able to decide if the optimal solution of the SGTP with time constraint is also the optimum of the underlying SGTP with no additional constraints.

Algorithm 1: The Equalization Method for SGTP with time constraint

Input: the initial solution x, the accuracy level ε and the maximum acceptable delivery time *T*.

Output: the optimal solution $x^\star$ and the global accuracy $\alpha^\star$.

1. *Initial solution*. Given the initial solution, calculate the sums of deliveries to every destination point:

$$x_j = \sum_{i=1}^{m} r_{ij}x_{ij}, j = 1, \dots, n,$$

And the partial derivatives:

$$k_{ij} = c_{ij} + r_{ij}f_j'(x_j), i = 1, \dots, m, j = 1, \dots, n,$$
$$k_{i,n+1} = 0, i = 1, \dots, m.$$

Go to step 2.

2. *Checking the optimality*. For every i calculate:

$$v_i = \min\{k_{ij} | j = 1, \dots, n+1, t_{ij} \le T\},$$
$$v_i^\star = \min\{k_{ij} | j = 1, \dots, n+1\},$$
$$w_i = \max\{k_{ij} | j = 1, \dots, n+1, x_{ij} > 0, t_{ij} \le T\} - v_i,$$
$$w_i^\star = \max\{k_{ij} | j = 1, \dots, n+1, x_{ij} > 0,\} - v_i^\star.$$

Let $j^{\star\star}(i)$ be the index j such that $k_{ij} = v_i$ and let $j^\star(i)$ be the index j for which $k_{ij} - v_i = w_i$. Compute:

$$\alpha = \max\{w_i | i = 1, \dots, m\},$$
$$\alpha^\star = \max\{w_i^\star | i = 1, \dots, m\}.$$

Let $i^\star$ be the index i for which $w_i = \alpha$. If $\alpha < \varepsilon$, then STOP. Return the solution and the value of $\alpha^\star$.

Otherwise go to step 3.

3. *Changing the solution*. Let:

$$\delta^-(\lambda) = f'_{j^\star}(x_{j^\star}) - f'_{j^\star}(x_{j^\star} - \lambda)$$

and

$$\delta^+(\lambda) = f'_{j^{\star\star}}(x_{j^{\star\star}} + \lambda) - f'_{j^{\star\star}}(x_{j^{\star\star}}).$$

Let $\lambda^\star$ be the solution to the equation:

$$\delta^-(\lambda) + \delta^+(\lambda) = w_{i^\star}.$$

If

$$\lambda^\star > x_{i^\star j^\star},$$

then set

$$\lambda^\star := x_{i^\star j^\star},$$

Adjust the solution and the derivatives according to the formulae:

$$k_{ij^\star} := k_{ij^\star} - r_{ij^\star}\delta^-(\lambda^\star), i = 1, \dots, m,$$
$$k_{ij^{\star\star}} := k_{ij^{\star\star}} + r_{ij^{\star\star}}\delta^+(\lambda^\star), i = 1, \dots, m,$$
$$x_{i^\star j^\star} := x_{i^\star j^\star} - \lambda^\star,$$
$$x_{i^\star j^{\star\star}} := x_{i^\star j^{\star\star}} + \lambda^\star.$$

Go back to step 2.

Remark: There are special cases, in which we are able to derive a simple formula for the root of the equation:

$\delta^-(\lambda) + \delta^+(\lambda) = w_{i^\star}$

(see Anholcer, 2013a, 2013b for the details). In other cases we use the one-dimensional Newton method to find the value $\lambda^\star$.

6 The main algorithm

The main method has the following form:

Algorithm 2. Method for finding the set of Pareto-optimal solutions of BSGTP.

Input: BSGTP, accuracy level ε.

Output: finite list L of Pareto-optimal solutions sorted by increasing delivery time.

1. *Initial solution*. Solve the GTP with Time Criterion as in Section 3. Initialize list L. Add x at the end of L. Create the list L_T of distinct delivery times t_{ij} in increasing order.
2. *Finding next Pareto-optimal solution*. Let x be the last element of L. Let T be the first element of L_T. Remove T from L_T. Solve the SGTP with Time Constraint using the modified Equalization Method presented in section 4. Let $x^\star$ be the optimal solution obtained and $\alpha^\star$ the global accuracy level. If $x^\star \neq x$, then add $x^\star$ at the end of L.
3. *Checking the stopping criterion*. If $\alpha^\star < \varepsilon$, then STOP, the list L contains all the Pareto-optimal solutions of BSGTP. Otherwise go back to step 2.

The presented method is convergent, which follows from two facts. First, the modified Equalization Method converges to the KKT point. Second, the number of iterations is finite because the number of distinct delivery times is finite.

7 Computational experiments

Several test problems were randomly generated and solved with the proposed method. All the demands have exponential distribution *Exp*(λ), where the values of λ were chosen uniformly at random from the interval <0.5, 0.6). The unit transportation costs were chosen from the interval <5, 10), the surplus costs from the interval <1, 2), the shortage costs from the interval <5, 10), the reduction ratios from the interval <0.8, 0.9) and the supply of each source point from the interval <10, 20). The delivery times were chosen from the set of integers {1, 2, …, 10}. The algorithm was implemented in Java SE and run on a personal computer with Intel(R) Core(TM) i7-2670 QM CPU @2.20 GHz. 1000 randomly generated problems of eight sizes were solved for (*m*, *n*) = (10, 10), (10, 20), (30, 30), (30, 60), (50, 50), (50, 100), (100, 100) and (100, 200) – 8000 test problems in total. The running times in milliseconds (average, standard deviation, minimum and maximum) are presented the Table 1.

Table 1.

Running times in milliseconds

Problem size	10×10	10×20	30×30	30×60	50×50	50×100	100×100	100×200
AVG	4.64	10.81	118.15	401.52	713.55	2733.14	7143.04	34136.65
ST DEV	14.16	23.57	148.17	317.26	615.66	1592.90	3837.50	12030.73
MIN	0.00	0.62	18.70	117.10	125.00	859.00	2185.00	14619.00
MAX	337.74	610.91	2569.40	3304.40	5538.00	11966.00	32605.00	91246.00

As we can see, the algorithm is very fast – the running times are expressed in seconds or even milliseconds in the case of the problems of reasonable size.

8 Final remarks

The algorithm presented above allows to find quickly the set of Pareto-optimal solutions of the Bi-criteria Stochastic Generalized Transportation Problem, where one criterion is the expected total cost of all the deliveries and the other one, the maximum delivery time Given the upper bound on the delivery time, the subproblem takes the form of SGTP with additional Time Constraint. This can be solved by a variant of Equalization Method, that can be shown to be convergent to the KKT point (see Anholcer 2013a, 2013b). As there are only finitely many possible delivery times, also the method dedicated for BGSTP presented above is also convergent. In consequence, we are able to find the set of all the Pareto-optimal solutions in finite time.

While formulating the problem, we made some assumptions. Let us discuss two of them.

First, we assumed that the delivery times must be integers. Although this is likely to be the case in the real world, we can remove this assumption. Even if the times are arbitrary real numbers, their number is finite and so the algorithm is still capable of delivering the list of effective solutions.

Then, we assumed that the density functions have to be positive. This implies that the objective function $f(x)$ is strictly convex and the subproblem, which has the form of SGTP with Time Constraint, has an unique optimal solution. This was in turn used in the proof of Theorem 1. However, we can omit this assumption as we can consider the equivalence relation:

$$x^{(1)} \cong x^{(2)} \Leftrightarrow f\left(x^{(1)}\right) = f\left(x^{(2)}\right).$$

Then, the set of the effective solutions consists of a finite number of the equivalence classes of $\cong$, and the presented method finds the representatives of all those classes. This is of course acceptable in real-life applications.

Aknowledgements

The research was part of the project „Nonlinear optimization in chosen economical applications”. The project was financed by the National Science Center grant awarded on the basis of a decision number DEC-2011/01/D/HS4/03543.

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MULTIPLE CRITERIA DECISION MAKING
Vol. 8 2013

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PORTFOLIO ANALYSIS ON POLISH POWER EXCHANGE AND EUROPEAN ENERGY EXCHANGE

Abstract

The aim of this paper is a comparative analysis of contract electric energy portfolios at Polish Power Exchange (POLPX) and European Energy Exchange (EEX) spot markets. The multi-criteria approach proposed in this paper is based on minimization of the Conditional Value at Risk with the confidence level 0.95 and maximization of portfolio rates of return. The analyzed portfolios have been constructed independently for each power exchange (for investors who are interested to invest on one market only), as well as for POLEX and EEX together (for investors who invest on more than one market) with two criteria.

Keywords: Portfolio analysis, Conditional Value at Risk (CVaR), electric energy spot markets

1 Introduction

The Polish Power Exchange (POLPX) was opened in July 2000. Investors on POLPX may participate in the Day Ahead Market (DAM, spot market), the Commodity Derivatives Market (CDM, future market), the Electricity Auctions, the Property Right Market, the Emission Allowances Market (CO2 spot) and the Intraday Market. All these markets differ with respect to the investment horizon and the commodity traded.

As the result of the merger of the two German power exchanges in Leipzig and Frankfurt the European Energy Exchange AG (EEX) in Leipzig was

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established in 2002. This is one of the European trading and clearing platforms for energy and energy-related products, such as natural gas, CO_2 emission allowances and coal. The EEX consists of three sub-markets (EEX Spot Markets, EEX Power Derivatives and EEX Derivatives Markets) and one Joint Venture (EPEX Spot Market). Moreover, EEX is trying to become the leader among the European Energy Exchanges assuming an active role in the development and integration process of the European market.

The aim of this paper is a comparative analysis of risk on electric energy spot markets. In this paper we propose portfolios based on linear daily rates of return of prices noted on POLPX and EEX from 1st January 2009 to 24th October 2012. We compare risk on these portfolios built independently on two markets and the portfolios of contracts from POLPX and EEX together.

The analyzed portfolios are constructed based on two criteria: minimization of the Conditional Value at Risk (CVaR) with the confidence level 0.95 and maximization of the portfolio rates of return.

2 Methodology

When we make financial decisions, at the same time we take the risk. If we want to estimate the future risk we must measure it. There are many different kinds of risk measures, one of them is downside risk. In these measures we used a well known quantile downside risk measure such as: Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) (Blanco, 1998; Jajuga and Jajuga, 1998; Weron and Weron 2000; Heilpern, 2011):

VaR is defined as such loss of value, which is not exceeded with the given probability α at the given time period Δt, and given by the formula:

$$\mathrm{P}(W_{t+\Delta t} \le W_t - VaR_\alpha(W)) = \alpha \tag{1}$$

where:

W_t - is a present value,

$W_{t+\Delta t}$ - is a random variable, value at the end of duration of investment.

Equation (1) describes VaR_α for short position. VaR_α answers the question: How much money can we lose over time period Δt with probability $1-\alpha$? The VaR quantity represents the maximum possible loss, which is not exceeded with the probability α.

For linear rates of return VaR_α we can write as a percentile of the order α of rates of return for short position:

$$\mathrm{P}(R_t \le VaR_\alpha(R)) = \alpha \tag{2}$$

and for long position:

$$\mathrm{P}(R_t \le VaR_{1-\alpha}(R)) = 1-\alpha \tag{3}$$

where:

$R_t = \dfrac{P_t - P_{t-1}}{P_{t-1}}$ - is a linear rate of return of contract

$\mathrm{P_t}, \mathrm{P_{t-1}}$ are the prices.

Without the assumption of a normal distribution of the rate of return, VaR is a problematic risk measure because it is not coherent (Artzner et al., 1999). It means that VaR for a diversified portfolio can be greater than the sum of VaR values of individual assets. In this sense, the measure, which does not meet the subadditivity requirement, cannot be the basis for portfolio diversification and optimization (Rockafellar and Uryasev, 2000; Rockafellar and Uryasev, 2002). In contrast, CVaR has better properties than VaR. The CVaR quantity is the conditional expected loss given the loss strictly exceeds its VaR. In literature CVaR is also called Expected Shortfall (ES) (Ogryczak and Ruszczyński, 2002; Heilpern, 2011). For short position we can write:

$$CVaR_\alpha(R) = ES_\alpha(R) = E\{R \mid R \ge VaR_\alpha(R)\}. \tag{4}$$

For long position we can write

$$CVaR_{1-\alpha}(R) = ES_{1-\alpha}(R) = E\{R \mid R \le VaR_{1-\alpha}(R)\}. \tag{5}$$

CVaR is defined as the mean of the quantile of worst realizations. The definitions ensure that VaR is never greater than CVaR, so portfolios with low CVaR must have low VaR as well. Pflug (2000) proved that CVaR is a coherent risk measure with the following properties: transition-equivariant, positively homogeneous, convex, monotonic, with stochastic dominance of order 1, and with monotonic dominance of order 2. (Pflug, 2000; Rockafellar and Uryasev, 2000). These properties let us use CVaR in portfolio analysis. Moreover, various numerical experiments and studies considering portfolio optimization with CVaR point out that the minimization of CVaR leads to optimal solutions in terms of VaR (Uryasev, 2000; Rockafellar and Uryasev, 2002).

The portfolio selection model proposed in this paper is based on the two criteria "mean-variance" portfolio problem analyzed by Steuer et al. (2006):

$$\begin{aligned} &\min\{x^T \textstyle\sum x\} \\ &\max \mu^T x \\ &x \in S \end{aligned} \tag{6}$$

which regarding CVaR – downside risk measure for short position is given as follows:

$$\begin{aligned}&\min \boldsymbol{CVaR}_{\alpha}\\ &\max \mu^T x\\ &x \in S\end{aligned} \tag{7}$$

and for long position:

$$\begin{aligned}&\min \boldsymbol{CVaR}_{1-\alpha}\\ &\max \mu^T x\\ &x \in S\end{aligned} \tag{8}$$

where:

$\boldsymbol{CVaR}_{\alpha}$ - *Conditional Value-at-Risk* for portfolio for short position,

$\boldsymbol{CVaR}_{1-\alpha}$ - *Conditional Value-at-Risk* for portfolio for long position,

x - vector of portfolio weights,

μ - vector of contracts means belonging to portfolio,

S - set of acceptable results

$\sum$ - covariance matrix.

Using results of Steuer et al. (2011) the problems (7)-(8) may be expressed in the following form for short position:

$$\begin{aligned}&\min |\boldsymbol{CVaR}_{\alpha} - \mu^T x|\\ &x_{\min} \le x_i \le x_{\max}\\ &\sum_{i=1}^{m} x_i = 1\end{aligned} \tag{9}$$

and for long position:

$$\begin{aligned}&\min |\boldsymbol{CVaR}_{1-\alpha} - \mu^T x|\\ &x_{\min} \le x_i \le x_{\max}\\ &\sum_{i=1}^{m} x_i = 1\end{aligned} \tag{10}$$

In Figure 1 the distance between CVaR calculated for portfolio rate of return and expected rates of return of portfolio is presented.

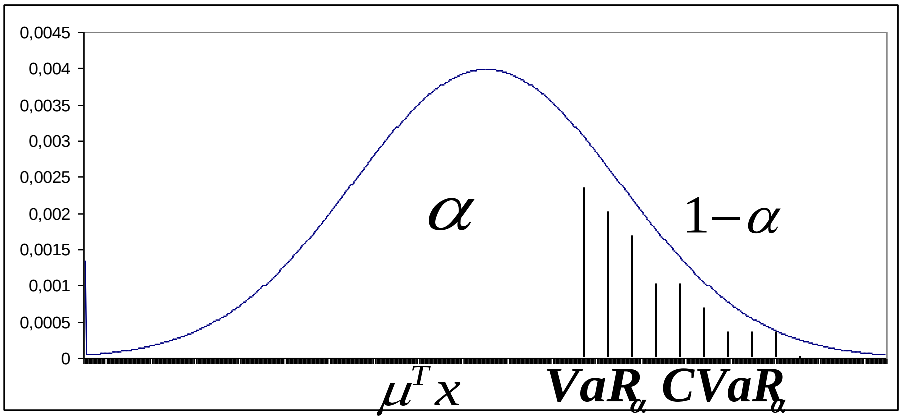

Figure 1. Distribution of rates of return of portfolio

3 Empirical analysis

Investors from spot energy markets make trading decisions with one day horizon of investment. So, to build portfolios from POLPX and EEX we consider daily rates of return of prices from 1st January 2009 to 24th October 2012. We estimate *VaR* and *CVaR* using the historical simulation method with $\alpha = 0.95$. Moreover, because of negative energy prices on EEX, linear rates of return have been applied. In both analyzed markets investors can buy and sell electric energy in 24 independent contracts. Parameters of contract distribution of linear rates of return from spot markets are presented in Table 1. Distribution of contracts is characterized by very high volatility, asymmetry and is leptokurtic. In such situations, the classical risk measures such as variance, which is very sensitive to extreme values and asymmetry, is not appropriate. Furthermore, the values of percentiles and standard deviation of contracts observed on spot markets show that volatility of prices on POLPX is much lower than that on EEX.

In the first step of our risk analysis we built portfolios independently on POLPX and EEX. In table 2 we presented portfolios for investors who take up long position on POLPX. Based on problem (10) we built three different portfolios. In the first portfolio we used restriction $0 \le x_i \le 1$ for portfolio weights. This portfolio consists only of night contracts (see Table 2, and Figure 2). In the next two portfolios the real demand for electric energy in respective hours of the day was taken into consideration ($0 \le x_i \le x_{\max}$). In the second portfolio x_{max} was assumed to be equal to the real demand observed on POLPX for the contract in the studied period, augmented by 5%. In the third portfolio contracts are augmented by 2.5%. Based on these portfolios we can say that investors shouldn't buy electric energy in the hours 7-11, 14 and 17 (compare Figure 2 and Table 2).

Table 1

Distribution parameters of rates of return of contracts on spot market

Contracts	POLPX						EEX					
	Mean	Percentyl 5%	Percentyl 95%	Stand. dev.	Skewness	Kurtozis	Mean	Percentyl 5%	Percentyl 95%	Stand. dev.	Skewness	Kurtosis
1	0.002	-0.09	0.09	0.06	1.46	19.17	0.247	-0.38	0.48	15.70	25.19	987.87
2	0.003	-0.10	0.11	0.07	0.21	5.53	-2.671	-0.52	0.70	83.21	-26.41	754.31
3	0.003	-0.11	0.13	0.07	0.22	5.15	1.780	-0.63	1.04	94.13	29.01	1063.26
4	0.004	-0.12	0.12	0.08	0.13	4.77	-1.510	-0.75	1.46	61.72	-25.02	839.34
5	0.004	-0.13	0.14	0.09	0.51	5.28	0.794	-0.66	1.40	34.26	-5.23	383.43
6	0.007	-0.15	0.20	0.12	1.52	8.64	1.776	-0.63	1.18	83.18	33.42	1215.64
7	0.017	-0.21	0.39	0.20	2.28	8.86	-6.285	-0.77	2.26	144.85	-14.43	286.98
8	0.013	-0.19	0.34	0.17	2.20	9.89	-3.792	-0.62	1.98	101.87	-18.37	458.77
9	0.012	-0.19	0.35	0.16	2.12	11.20	1.062	-0.47	1.34	76.54	23.18	941.68
10	0.01	-0.17	0.31	0.14	1.91	10.86	-0.944	-0.37	0.89	40.03	-37.10	1381.80
11	0.007	-0.15	0.25	0.12	1.39	5.94	0.054	-0.35	0.72	0.42	4.29	32.75
12	0.007	-0.15	0.25	0.12	1.29	5.25	0.041	-0.31	0.60	0.34	4.26	43.19
13	0.007	-0.15	0.23	0.11	1.26	4.99	0.041	-0.32	0.53	0.35	4.36	41.31
14	0.006	-0.15	0.23	0.11	1.12	4.03	-1.007	-0.36	0.75	39.84	-37.30	1391.63
15	0.005	-0.13	0.21	0.10	1.22	3.84	0.757	-0.38	0.90	24.62	37.21	1386.89
16	0.005	-0.12	0.19	0.10	1.32	6.23	-3.779	-0.40	0.93	95.60	-28.04	834.61
17	0.006	-0.13	0.21	0.11	1.48	8.09	0.558	-0.38	0.85	17.18	32.86	1160.54
18	0.006	-0.12	0.20	0.11	1.40	11.77	0.049	-0.32	0.62	0.54	19.27	552.08
19	0.004	-0.11	0.15	0.09	1.08	11.67	0.025	-0.26	0.47	0.27	6.30	102.49
20	0.004	-0.10	0.13	0.08	0.66	10.87	0.017	-0.24	0.35	0.20	2.24	13.34
21	0.003	-0.09	0.11	0.07	1.04	14.80	0.014	-0.22	0.31	0.19	3.36	36.56
22	0.002	-0.08	0.09	0.06	1.36	18.94	0.011	-0.20	0.27	0.15	1.58	8.00
23	0.001	-0.07	0.09	0.05	0.95	6.59	0.011	-0.20	0.24	0.16	2.89	26.03
24	0.002	-0.08	0.10	0.06	5.07	89.16	0.016	-0.25	0.32	0.30	5.31	99.25

In Table 3 we presented portfolios for an investor opening long positions on EEX. Based on problem (10) we built once again three different portfolios. For the portfolio with restriction $0 \le x_i \le 1$ we obtained portfolios without contracts in the hours 1, 2, 8 and 9 (see Table 3 and Figure 3). For every hour during a day we built two portfolios ($0 \le x_i \le x_{\max}$) under the same constraint as for POLPX. Based on these portfolios we can say that investors shouldn't buy electric energy in the hours 1, 6 and 9 (compare Table 3 and Figure 3).

If we compare risk measures by $CVaR_{0,95}$ we can see that the risk on EEX is much higher than the risk on POLPX.

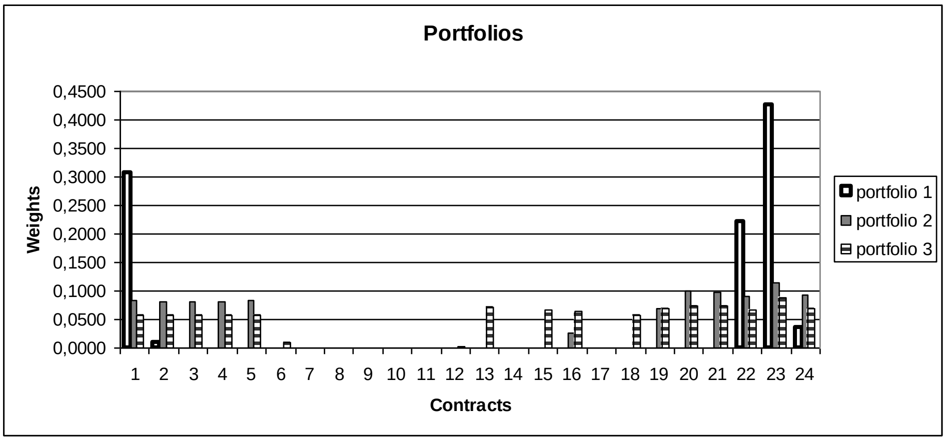

Figure 2. Weights of contracts in portfolios from POLPX

In the next step of the analysis the portfolios based on 48 contracts from POLPX and EEX have been built. Table 4 and Figure 4-6 present results of the optimization problem (10). In general, the risk on EEX is greater than the risk on POLPX, so weights of contracts from POLPX are greater than weights of contracts from EEX, especially for night and early morning hours from 1 to 9. For hours during the day differences between weights are not very significant. Investors who want to buy electric energy in the hour 24 should choose EEX.

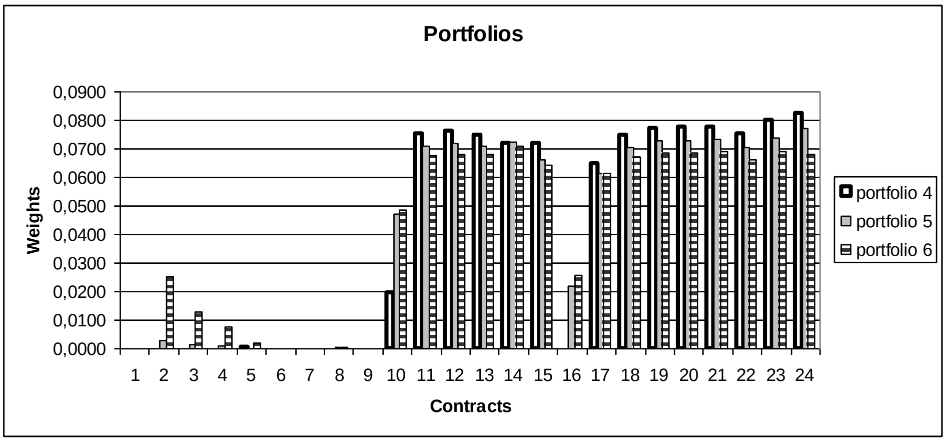

Figure 3. Weights of contracts in portfolios from EEX

Table 2

Portfolios on POLPX

Contracts	Portfolio 1			Portfolio 2			Portfolio 3		
	x	x_{min}	x_{max}	x	x_{min}	x_{max}	x	x_{min}	x_{max}
1	0.3078	0	1	0.0833	0	0.0833	0.0583	0	0.0583
2	0.0104	0	1	0.0817	0	0.0817	0.0567	0	0.0567
3	0	0	1	0.0816	0	0.0816	0.0566	0	0.0566
4	0	0	1	0.0818	0	0.0818	0.0568	0	0.0568
5	0	0	1	0.0824	0	0.0824	0.0574	0	0.0574
6	0	0	1	0	0	0.0835	0.0103	0	0.0585
7	0	0	1	0	0	0.0976	0	0	0.0726
8	0	0	1	0	0	0.0859	0	0	0.0609
9	0	0	1	0	0	0.0904	0	0	0.0654
10	0	0	1	0	0	0.0939	0	0	0.0689
11	0	0	1	0	0	0.0951	0	0	0.0701
12	0	0	1	0	0	0.0969	0.0030	0	0.0719
13	0	0	1	0	0	0.0962	0.0712	0	0.0712
14	0	0	1	0	0	0.0958	0	0	0.0708
15	0	0	1	0	0	0.0922	0.0672	0	0.0672
16	0	0	1	0.0252	0	0.0892	0.0642	0	0.0642
17	0	0	1	0	0	0.0900	0	0	0.0650
18	0	0	1	0	0	0.0919	0.0577	0	0.0669
19	0	0	1	0.0685	0	0.0950	0.0700	0	0.0700
20	0	0	1	0.0989	0	0.0989	0.0739	0	0.0739
21	0	0	1	0.0985	0	0.0985	0.0735	0	0.0735
22	0.2215	0	1	0.0912	0	0.0912	0.0662	0	0.0662
23	0.4256	0	1	0.1137	0	0.1137	0.0887	0	0.0887
24	0.0347	0	1	0.0933	0	0.0933	0.0683	0	0.0683
Objective (6)	0.0934			0.1277			0.1595		
Mmean	0.0016			0.0028			0.0036		
VaR	0.0625			0.0852			0.1115		
CVaR	0.0950			0.1305			0.1630		
Std. Deviation	0.0369			0.0503			0.0605		

In portfolio 7 the restriction $0 \le x_i \le 1$ for portfolio weights was used similar to portfolio 1 (for POLPX) and portfolio 4 (for EEX). For portfolios 8 and 9 x_{max} was assumed in the same way as for the portfolios constructed earlier for POLPX and EEX.

Table 3

Portfolios on EEX

Contracts	Portfolio 4			Portfolio 5			Portfolio 6		
	x	x_{min}	x_{max}	x	x_{min}	x_{max}	x	x_{min}	x_{max}
1	0	0	1	0	0	0.0833	0	0	0.0583
2	0	0	1	0.0030	0	0.0817	0.0252	0	0.0567
3	0.0001	0	1	0.0016	0	0.0816	0.0130	0	0.0566
4	0.0002	0	1	0.0007	0	0.0818	0.0077	0	0.0568
5	0.0005	0	1	0	0	0.0824	0.0017	0	0.0574
6	0.0002	0	1	0	0	0.0835	0	0	0.0585
7	0.0002	0	1	0.0002	0	0.0976	0.0001	0	0.0726
8	0	0	1	0.0005	0	0.0859	0.0006	0	0.0609
9	0	0	1	0	0	0.0904	0	0	0.0654
10	0.0196	0	1	0.0470	0	0.0939	0.0485	0	0.0689
11	0.0754	0	1	0.0711	0	0.0951	0.0678	0	0.0701
12	0.0760	0	1	0.0717	0	0.0969	0.0681	0	0.0719
13	0.0748	0	1	0.0710	0	0.0962	0.0681	0	0.0712
14	0.0718	0	1	0.0724	0	0.0958	0.0708	0	0.0708
15	0.0719	0	1	0.0661	0	0.0922	0.0645	0	0.0672
16	0.0001	0	1	0.0220	0	0.0892	0.0259	0	0.0642
17	0.0649	0	1	0.0614	0	0.0900	0.0612	0	0.0650
18	0.0749	0	1	0.0705	0	0.0919	0.0669	0	0.0669
19	0.0770	0	1	0.0726	0	0.0950	0.0686	0	0.0700
20	0.0775	0	1	0.0731	0	0.0989	0.0688	0	0.0739
21	0.0777	0	1	0.0733	0	0.0985	0.0689	0	0.0735
22	0.0751	0	1	0.0707	0	0.0912	0.0662	0	0.0662
23	0.0798	0	1	0.0738	0	0.1137	0.0692	0	0.0887
24	0.0823	0	1	0.0770	0	0.0933	0.0683	0	0.0683
Objective (6)	0.9805			1.1478			1.4198		
Mean	0.0199			-0.1057			-0.1706		
VaR	0.4453			0.4879			0.5600		
CVaR	1.0004			1.0421			1.2491		
Std. Deviation	0.8526			2.9270			3.7443		

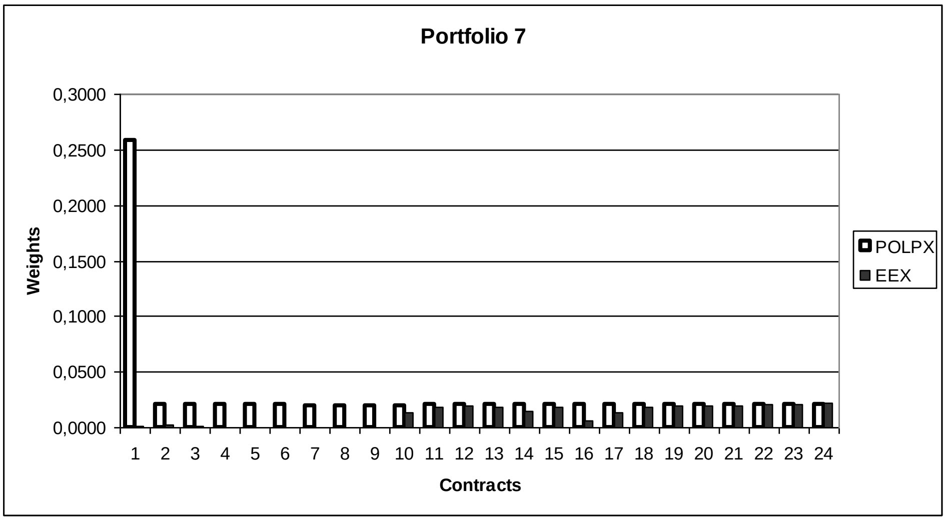

Figure 4. Weights of contracts in portfolio from POLPX and EEX (with the restriction $0 \le x_i \le 1$)

The negative value of portfolios return for POLPX and EEX together (see Table 4) as well as for EEX (see Table 3) can result from negative electricity prices observed on EEX[1].

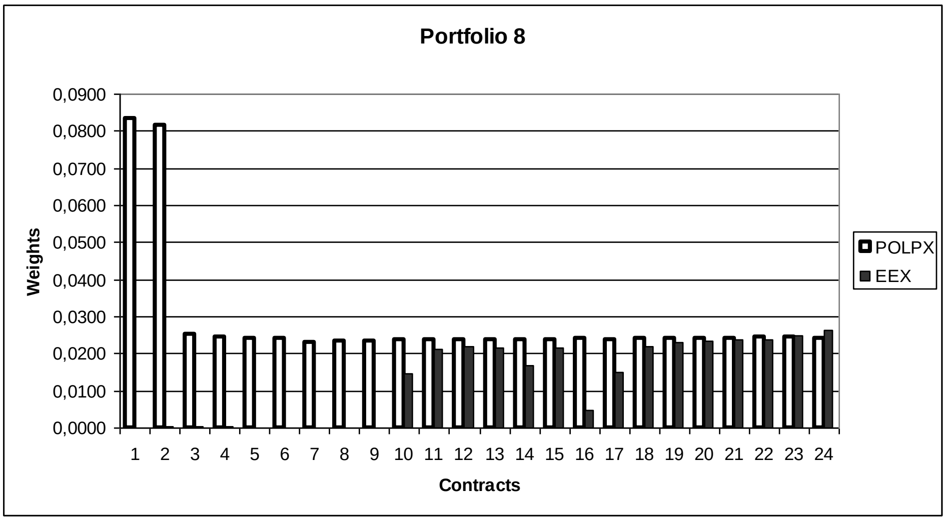

Figure 5. Weights of contracts in portfolio from POLPX and EEX (with the restriction $0 \le x_i \le x_{\max}$ argumented by 5%)

[1] The negative electricity prices ware first observed in 2009 on EEX as a result of demand and supply changes which come independently from price.

Table 4

Portfolios on POLPX and EEX

Contracts	Portfolio 7			Portfolio 8			Portfolio 9		
	POLPX	EEX	x_{max}	POLPX	EEX	x_{max}	POLPX	EEX	x_{max}
1	0.2589	0.0010	1	0.0833	0.0000	0.0833	0.0583	0.0000	0.0583
2	0.0208	0.0022	1	0.0817	0.0004	0.0817	0.0567	0.0000	0.0567
3	0.0207	0.0011	1	0.0252	0.0002	0.0816	0.0566	0.0000	0.0566
4	0.0207	0.0000	1	0.0244	0.0004	0.0818	0.0290	0.0012	0.0568
5	0.0206	0.0000	1	0.0243	0.0000	0.0824	0.0248	0.0000	0.0574
6	0.0204	0.0000	1	0.0240	0.0000	0.0835	0.0245	0.0000	0.0585
7	0.0199	0.0001	1	0.0232	0.0002	0.0976	0.0237	0.0001	0.0726
8	0.0200	0.0000	1	0.0234	0.0001	0.0859	0.0239	0.0002	0.0609
9	0.0201	0.0000	1	0.0235	0.0000	0.0904	0.0240	0.0000	0.0654
10	0.0202	0.0139	1	0.0237	0.0147	0.0939	0.0242	0.0144	0.0689
11	0.0203	0.0185	1	0.0239	0.0213	0.0951	0.0244	0.0217	0.0701
12	0.0203	0.0190	1	0.0239	0.0220	0.0969	0.0244	0.0224	0.0719
13	0.0204	0.0188	1	0.0239	0.0217	0.0962	0.0244	0.0222	0.0712
14	0.0203	0.0152	1	0.0239	0.0169	0.0958	0.0244	0.0169	0.0708
15	0.0204	0.0189	1	0.0239	0.0216	0.0922	0.0245	0.0223	0.0672
16	0.0204	0.0063	1	0.0240	0.0048	0.0892	0.0245	0.0040	0.0642
17	0.0204	0.0139	1	0.0239	0.0148	0.0900	0.0244	0.0145	0.0650
18	0.0204	0.0189	1	0.0240	0.0220	0.0919	0.0245	0.0223	0.0669
19	0.0205	0.0196	1	0.0241	0.0229	0.0950	0.0246	0.0233	0.0700
20	0.0206	0.0200	1	0.0242	0.0234	0.0989	0.0247	0.0239	0.0739
21	0.0206	0.0202	1	0.0243	0.0237	0.0985	0.0249	0.0242	0.0735
22	0.0208	0.0204	1	0.0245	0.0240	0.0912	0.0251	0.0245	0.0662
23	0.0207	0.0210	1	0.0243	0.0250	0.1137	0.0249	0.0253	0.0887
24	0.0206	0.0220	1	0.0243	0.0262	0.0933	0.0248	0.0262	0.0683
Objective (6)	0.3974			0.4129			0.4137		
Mean	-0.0258			-0.0169			-0.0140		
VaR	0.2144			0.2429			0.2476		
CVaR	0.3716			0.3960			0.3997		
Std. Deviation	0.7980			0.7143			0.6682		

4 Conclusion

Concluding, the risk of price changes on EEX is much greater than the analogous risk on POLPX, but, based on two criteria to build the portfolio, the investor should sell electricity on EEX too. For investors, contracts in the night and early morning hours on POLPX are more attractive, but for odd hours contracts on two spot markets give a very similar distance between risk and profit.

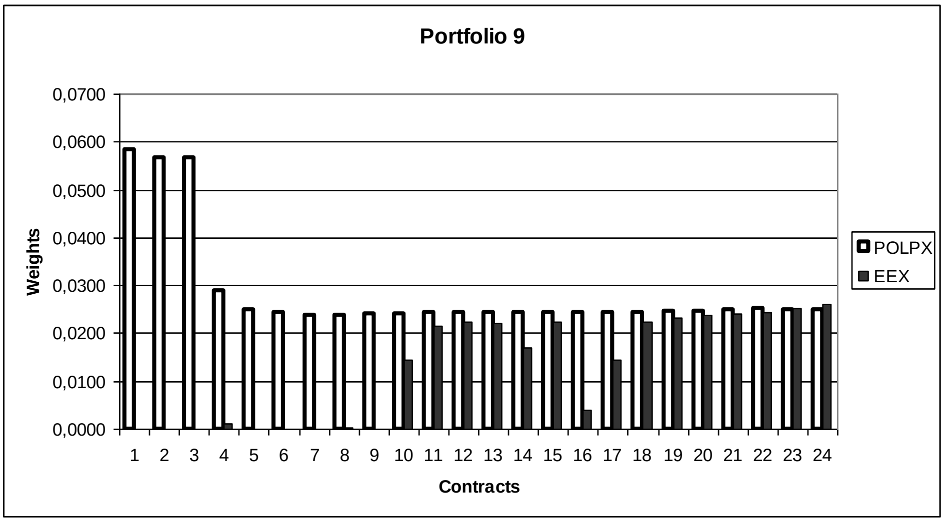

Figure 6. Weights of contracts in portfolio from POLPX and EEX (with the restriction $0 \le x_i \le x_{\max}$ argumented by 2.5%)

Moreover, portfolios constructed for both electricity markets consist of contracts for all hours during the day as opposed to the portfolios built only for POLPX and EEX. From the point of view of retailers, this can be seen as a positive aspect of this approach. Nevertheless, the observed problem of negative portfolio return, caused by negative energy prices on EEX, needs further investigation and analyses.

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M U L T I P L E   C R I T E R I A   D E C I S I O N   M A K I N G
Vol. 8 2013

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COUNTRY MARKET SELECTION IN INTERNATIONAL EXPANSION USING MULTICRITERIA DECISION AIDING METHODS

Abstract

Companies, facing globalization and technological revolution, are constantly forced to search for new ways to grow and develop. A profitable growth strategy can be built through international expansion. Recently it has become one of the few effective ways to develop and enhance the competitiveness of a company in response to the changing dynamics of the global economy.

When a company is expanding its business operations to overseas markets a number of strategic decisions must be taken. The company must define the product it wants to market (what), the country market it wants to enter (where), the timing of entry (when) and the entry mode it wants to use (how). Consequently, the country market selection plays a critical role in shaping the performance of foreign activities and influences the future success of the company. This is a complex and difficult decision, requiring the company to analyse a wide spectrum of factors that affect both investment efficiency and effectiveness. The location selection in international expansion may be determined by general macroeconomic factors, demand factors, socio-political factors or cost factors.

The purpose of this paper is to conduct a simulation of the market selection decision with the help of multi-criteria decision aiding methods based on the perspective of a dynamically developing company that is a leading manufacturer and distributor of hygiene, cosmetic and medical products.

Keywords: International market selection (IMS), preliminary screening, MCDA methods, EXPROM II with veto threshold.

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1 Introduction

These days business is conducted in an increasingly globalized environment characterized by fewer barriers, growing competition and greater opportunities for international expansion.

Before expanding abroad a company must take various strategic decisions. One of them is the choice of international markets worth entering. The identification of promising foreign target markets is a vital issue for the future success of the enterprise as errors committed at this stage can be very costly.

A wide range of factors need to be considered by a firm choosing new markets, including economic, political and cultural elements. Hence, this issue is a multi-criteria decision-making problem and it can be solved with the help of MCDA methods.

The aim of this paper is to apply multi-criteria decision aiding methods to the problem of market selection. This paper will illustrate the usefulness of these methods with a real-life example of a leading manufacturer and distributor of hygiene, cosmetic and medical products in its quest for new markets. The scope of the paper is limited to a preliminary screening analysis based on external factors represented by a set of macro-level indicators. The internal factors relating to the firm, its resources, international experience, applied competitive and functional strategies are beyond the scope of this analysis.

This article consists of an introduction, a conclusion and five sections. In the first section the market selection process in international expansion is described. In the second section the first stage of the assessment of foreign markets, namely preliminary screening, is presented in more depth. The third and the fourth sections, in turn, include the description of the case study and the solutions obtained as a result of applying the MCDA methods. Finally, in the fifth section the results of the sensitivity and robustness analysis are shown.

2 The market selection process in international expansion

Starting business operations in overseas markets is a complex and complicated process. It requires decisions about many related but distinct issues. The consequences of these decisions may have a significant influence on the success and performance of a firm.

First, the company must clearly articulate the reasons why it wants to be involved in international business. Objectives and goals of international expansions are an essential prerequisite for the entire international market entry strategy. Second, a firm must define the product or products it wants to deliver to a foreign market. At this point it is essential to determine the distinctive features of the product in terms of price, quality and other characteristics and verify whether they are still unique in non-domestic markets. Managers also have to consider the global product life cycle as well as the scope and costs of adapting products to foreign markets (Root, 1994). Third, a company must identify the target market or markets in which it wants to offer its product. This is known as international market selection (IMS). More detailed information about this stage will be presented in the following section of the paper. Subsequently, a firm must choose the entry mode it wants to use. This is referred to in the literature as the entry mode decision. At this stage a firm is choosing between non-equity entry modes such as exporting, licensing, franchising, management contracts, turnkey contracts or subcontracting, and equity-based entry modes which result in establishing a company in the host country that is either partly or wholly owned. Finally, a company must also determine the timing of entry. All five decisions are elements of an international market entry strategy – a comprehensive plan that is to contribute to the entry of a firm's products, resources and capabilities into a foreign country. The issue of international entry strategy has been widely addressed in the literature; however, most studies usually focus on the analysis of the strategy elements individually.

In the literature, several approaches to the international market entry strategy have been identified (Root, 1994; Kotler, 2005; Stonehouse et al., 2001). Root recommends a model consisting of five elements: (1) the choice of a target product/market, (2) the objective and goals in the country, (3) the choice of entry mode, (4) the marketing plan, and (5) the control system. There is a logical sequence involving the above-mentioned elements, however the model does not exclude feedback loops that make the strategy a continuing and open-ended process in a short-time horizon (Root, 1994). Kotler also views the international market entry strategy as a process composed of five stages, but he defines some stages slightly differently. His proposed framework covers the following stages: (1) the decision about international market expansion, (2) market selection, (3) selection of entry mode, (4) the marketing plan, and (5) the marketing organization (Kotler, 2005). Stonehouse's model suggests four stages involving: (1) the decision about international market expansion, (2) the overview of the international environment in search of opportunities and threats, (3) market selection, and (4) selection of entry mode (Stonehouse et al., 2001). While analyzing international market entry strategy, it should be emphasized that each individual decision is central to successful overseas expansion. The models that have been presented confirm the complexity of the process and illustrate the broad decision-making set related to overseas expansion.

The selection of foreign markets is one of the most critical decisions in international market entry strategy. According to the models presented above a company identifies the target market in which it wants to launch its product before it selects the entry mode. A firm must choose its target market from a wide range of national markets. The national markets often differ markedly in terms of market size, income, level of development, language, culture, religion, political and economic stability, social aspects and many other important dimensions. The diversity and complexity of market opportunities is huge, hence the market selection is a complicated process that should be well thought through.

In the literature, several market selection models have been proposed (Root, 1994; Koch, 2001; Kumar et al., 1994; Cavusgil, 1985). They attempt to formalize the decision-making process. IMS is usually seen as a sequential process where each stage is aimed at progressively eliminating the less attractive markets in order to arrive at the selection of the prospective target market at the end of the process. The systematic approach to IMS is crucial in the context of a decision that involves assimilating a huge amount of information from many diverse and complex markets.

Most of the models illustrating international market selection view the process of assessing overseas markets as composed of three stages such as preliminary screening (or screening), in-depth screening (or identification) and final selection (or selection) (Koch, 2001; Kumar et al., 1994; Root, 1994; Cavusgil, 1985). Preliminary screening identifies the prospective target markets for subsequent in-depth analysis. At this stage, companies use set of macro-level indicators to eliminate countries that do not meet their objectives. More detailed information about preliminary screening is provided in the next section of this paper. During the identification stage, the attractiveness of the industry is evaluated. A firm gathers industry-specific information such as market size and growth, level of competition, entry barriers and market segments in order to create a short-list of high-market-potential countries. During the final selection stage, the company focuses on firm-specific information. It analyses profitability, assessing forecasts of revenues and cost, compatibility with the existing portfolio taking into consideration company objectives and goals, resources and strategies (in fact, all stages should bear in mind company objectives and goals, resource constraints and the adopted expansion strategy). The final selection should highlight the country market which best matches company objectives.

The nature of the market selection process (related to analyzing the large number and diversity of foreign markets) means that the existing literature is fairly consistent in describing the desirable features of market selection models. IMS models should be flexible, comprehensive and cost-effective (Papadopoulos and Martín Martín, 2011).

There are two basic approaches to the selection of international markets: expansive and contractible (Root, 1994; Schroeder, 2007; Albaum and Duerr, 2008). In the expansive approach the company favours new markets that have the least psychic distance from those in which it operates. The selection of markets is based on similarities among markets in terms of their political, economic and social nature. The contractible approach takes as its starting point a global perspective including all national markets. It involves a systematic screening of all country markets in order to eliminate the less attractive ones and focus in greater depth on those which are more promising (Albaum and Duerr, 2008).

3 Preliminary screening of country markets

As mentioned earlier, preliminary screening is the first stage in international market selection models, making it a critical success factor for the entire selection process. It helps to identify prospective target markets that warrant further investigation (Root, 1994). According to Root, this approach tries to minimize two possible errors. First, it reduces the chance of ignoring countries that offer good prospects for a company's generic product by applying a preliminary screening process to all countries. Second, it minimizes the risk of spending too much time investigating countries that are poor prospects by focusing on low cost and widely available quantitative data and a relatively quick and simple screening technique to eliminate a large number of unattractive countries from the subsequent in-depth analysis. In addition, Root emphasizes that this preliminary screening should identify promising target countries without regard to entry mode. However, these two decisions are closely related and should not be discussed separately. Some scholars suggest even that they should be a part of one decision process (Koch, 2001).

Root also suggests that companies identifying potential markets should begin the selection process with the total set of available countries. Cooper and Kleinschmidt state that companies which adopt this approach realize more rapid export growth than those which limit their choice to a few alternatives (Cooper and Kleinschmidt, 1985). These conclusions are not obvious for all companies. There are companies that still tend to select the target market without systematic analysis. It occurs particularly among smaller firms (Papadopoulos et al., 2002). They tend to start their international expansion by entering neighbouring countries in response to unsolicited orders. This behaviour is consistent with the internationalization theory based on stage models. According to the Uppsala model, internationalization is a sequential and successive process (Johanson and Wiedersheim-Paul, 1975; Johanson and Vahlne, 1977). The firm tends to gradually increase its involvement in foreign operations, starting from geographically and culturally close markets. It is only later (with greater knowledge and experience) that firms tend to enter markets characterized by successively greater psychic distance, in most cases greater geographical distance. According to the Uppsala model the market selection process is mainly based on psychic distance, which dictates where a firm will market its product. There is no need for a systematic approach that would allow a firm to analyze the total set of available countries. The general reasons for selecting potential markets without applying a preliminary screening process to all countries include the limited experience of managers in export research, difficulties in collecting data and the lack of a proven effective approach which would include in its framework the huge diversity and complexity of current markets (Papadopoulos and Denis, 1988).

An effective preliminary screening process can only be implemented if it is possible to identify potential markets by comparing and evaluating country characteristics (Russow and Okoroafo, 1996). The criteria for the country evaluation must be defined before the screening process starts. It is an essential element of the screening stage because in fact it has a direct impact on the screening results. There is no agreement among scholars on which criteria should be used and how they should be measured. The lists of suggested criteria that are available in the literature are based on the respective author's perception of what criteria would be most suitable in a given situation (Russow and Okoroafo, 1996). They are directly related to the objectives and goals of a firm's international expansion and vary from one form of entry to another. They depend on what exactly a firm wants to achieve with its involvement in international business. The criteria applied will vary according to whether a company is driven by market seeking, resource seeking, efficiency seeking or strategic assets seeking motives (Dunnig, 1993)[1]. The criteria suggested will also differ depending on whether a company has chosen export activities or the investment route. However, as mentioned earlier, Root emphasizes that we should screen markets without regard for entry mode.

A literature review of screening criteria indicates that market size and the level of economic development were the most frequently suggested criteria by both international business theory (Vernon, 1966; Dunning, 1988; Porter, 2001) and the marketing literature (Samli, 1977; Root, 1994; Douglas et al., 1982; Gaston-Breton and Martín Martín, 2011, Sheng and Mullen, 2011; Cavusgil, 1997; Natarajarathinam and Nepal, 2012; Sakarya et al., 2007; Whitelock and Jobber, 2004). In addition, international business theory frequently emphasizes the importance of endowment factors (factors of production) as determinants of potential opportunities (Vernon, 1966; Dunning, 1988; Porter, 2001). However, the number of applied criteria supporting the assessment process is significantly wider. Some studies use other and more detailed criteria such as: market growth rate (Cavusgil, 1997; Natarajarathinam and Nepal, 2012; Kumar et al., 1994), market intensity, commercial/physical infrastructure, economic freedom, market receptivity (Cavusgil, 1997; Sheng and Mullen, 2011), country risk (Natarajarathinam and Nepal, 2012;), political stability (Whitelock and Jobber, 2004), geographic distance (Sheng and Mullen, 2011; Whitelock and Jobber, 2004), cultural distance (Sakarya et al., 2007; Sheng and Mullen, 2011; Whitelock and Jobber, 2004), language differences (Sheng and Mullen, 2011;

[1] The motives presented are directly related to equity-based modes (foreign direct investment), however, assuming that equity-based modes are the most advanced forms of entry modes, the group of motives should also include motives related to simpler non-equity modes.

Whitelock and Jobber, 2004), religious differences (Sheng and Mullen, 2011) or government attitude to FDI and trade barriers (Whitelock and Jobber, 2004).

To keep preliminary screening process low-cost, simple and flexible, most models use a macro approach, focusing on general country factors (Cavusgil, 1997; Cavusgil et al., 2004; Sheng and Mullen, 2011; Samli, 1977; Papadopoulos et al., 2002). However, there are some models that contain industry- or product-specific approaches (Douglas et al., 1982; Root, 1994; Whitelock and Jobber, 2004; Sakarya et al., 2007; Kumar et al., 1994). Root suggests using direct estimates of market size for the candidate product by projecting actual sales data or by projecting the apparent consumption or imports of the product. Sakarya et al. recommends competitive strength of the specific industry in the potential market and customer receptiveness to the products of the foreign industry and its country of origin. Whitelock and Jobber include stable competitive environment in their formalized statistical analysis to select the appropriate foreign market.

While the criteria are identified, there are another two issues to discuss. The first issue relates to the indicators used to measure the applied criteria. Again, theory offers a large array of indicators that help to measure the screening criteria used for selecting the most appropriate foreign market. As in the case of the screening criteria, the author's perception is a key influence in defining the list of indicators used in a market assessment. It seems that there is a need for a standardized variable set that could make the process less subjective in some aspects (Papadopoulos et al., 2002).

The second critical issue that surfaces during the evaluation process is the question of weighting the different criteria. There is no agreement on how to assign weights to the criteria to reflect their relative importance (Russow and Okoroafo, 1996; Papadopoulos et al., 2002). Some studies recommend an approach that weighs all criteria equally; others state that some criteria may be more important than others. Cavusgil suggests the application of a Delphi process involving international business professionals and educators in order to determine the relative importance of each criterion (Cavusgil, 1997). However, there is no general description of how to assign weights to the criteria.

The international marketing literature contributes two main approaches to identifying target markets: clustering and ranking. Both methods are recommended for evaluating and selecting potential markets during an initial country screening process. Cluster methods group countries on the basis of similarities along commercial, economic, political and cultural dimensions. The similarities are aimed at helping managers compare countries and identifying potential synergies among markets (Cavusgil et al., 2004). The approach assumes that firms prefer to enter countries from the same cluster in which they have been operating successfully. They can use already accumulated knowledge and experience from similar markets and apply this to other markets in the same cluster (Johanson and Vahlne, 1990). The second group of methods ranks countries by order of preference. Markets are evaluated according to one or more criteria (Sakarya et al., 2007). The ones with the highest score should be chosen for further analysis. The approach might give managers an aggregate measure of market attractiveness that might be customized by them according to their own preferences and priorities by assigning weights to dimensions or by adding new dimensions (Cavusgil et al., 2004). Both methods have been recognized as important tools for analyzing a large number of countries with heterogeneous markets, however, both should only be used at the preliminary market assessment stage.

4 Methodology

The present study illustrates the application of multi-criteria decision aiding methods in the preliminary market assessment process. It is based on the example of one of the leading manufacturers and suppliers of sanitary articles, cosmetics and medical devices to the global market. This is an enterprise with 100% Polish capital, composed of 49 companies including 19 manufacturing companies (in Poland, Russia, Ukraine and India), 24 trading companies (in 14 European countries, India and USA) and 6 service (medical and IT technology) companies (in Poland and Russia). It employs over 7.4 thousand people and sells its products in more than 70 countries worldwide (they are available in Europe, Asia, Africa, America and Australia). Thanks to the firm's own Research and Development Centre, which cooperates closely with scientific institutions, its products are manufactured using the most recent technologies. This helps the company to compete successfully in the highly competitive markets in which it operates[2].

A concise history of the firm, emphasizing especially its foreign operations, is presented in the Table 1.

The present simulation of an initial country selection refers to a project already carried out by the company during the period from 2002 to 2005, namely the investment made in India. Consequently, our study involves the verification of a choice made in the past.

It is assumed that the main reason why the company wanted to go abroad was to access new markets. In addition we suppose that the company was willing to run its operations in the foreign market using an equity-based mode as it had already had a relatively high level of experience of operating subsidiaries abroad.

Table 1

[2] Information about the company comes from its website: http://www.tzmoglobal.com/en_GLO (9 March 2013).

Company's history in brief

Years	Event
1950s	The company is established as a state-owned enterprise. Dressing material is produced for the Ministry of National Defence and the Central Mining Office Supply. Production is set to shut down after completing the order but thanks to the high quality of work further orders appear. The company begins conquering foreign markets: products are sold in **European, African and Asian countries**.
1990s	The company is privatised – a joint-stock company is created by individuals (Polish citizens): the employees of the company and representatives of the academic and medical environment. Since the end of the 1990s the company is entitled to mark its products with the European CE safety mark.
2000s	In the early 2000s the company opens a hospital in Poland which – since 2007 –has been serving as a modern polyclinic. Since the beginning of 2000s it has also been providing a sterilization service for hospitals. Production of hygiene products in the newly built plants in the East market starts – in 2003 in **Russia** and in the first quarter of 2004 in **Ukraine**. In 2002 the company establishes a joint-venture company with its Indian partner. A new factory in **India** begins manufacturing hygiene and medical products in 2005. At the end of 2000s it gains the CE mark for medical production. In 2004 the company builds a modern logistic centre in Poland (it serves as a central distribution warehouse). The following year a training, marketing and logistics centre is opened in **Germany**. Another logistics centre is founded in 2007 in **Romania**. In 2008 new business units are established in Poland (e.g. a films and laminates production plant and a clean room for medical production). At the end of 2000s the company starts business activity in North America – it establishes its headquarters in the **United States**.

Source: http://www.tzmo-global.com/en_GLO/companyHistory (9 March 2013).

After considering the various alternatives we have selected 20 countries as the target market set in order to illustrate the model. The pre-selection was necessary to reduce the number of countries for the application of the multi-criteria decision aiding method. The applied criteria were operationalized through a 'total population' indicator. We selected only countries with total population over 50 million for the analysis. However, Germany and Russia were removed from the list, because the company was already operating in these two markets. Furthermore, the Democratic Republic of Congo was eliminated from the list because of the Second Congo War (known as the Great War of Africa) lasting from August 1998 until its official end in July 2003.

Measurement data was collected from three publicly available secondary data sources: The World Bank, The Hofstede Centre and The Heritage Foundation.

As the project which is subject to analysis was completed during the period 2002-2005, we used data from year 2001 for the calculation.

Variables were identified through the literature review and based on information about the company. We selected 15 economic, cultural, social and political variables to assess the markets' attractiveness. The market's attractiveness is measured through the ten fundamental dimensions that are represented by the selected 15 variables. Table 2 shows the 15 variables used to reflect the ten dimensions of the model, along with a description and the corresponding measurement units.

Table 2

Market potential assessment factors (dimensions, measures, units and descriptions)

Dimension	**Measures (units)**	**Indicator Description**
Market Size (ability to sell products)	Total population (number of inhabitants)	The total population
	Urban population (number of inhabitants)	People living in urban areas as defined by national statistical offices
Market growth (ability to sell products in the future)	GDP growth rate (annual %)	Annual percentage growth rate at market prices based on constant local currency. Aggregates are based on constant 2000 U.S. dollars. GDP is the sum of gross value added by all resident producers in the economy plus any product taxes and minus any subsidies not included in the value of the products. It is calculated without making deductions for depreciation of fabricated assets or for depletion and degradation of natural resources
Economic Development (low productivity of the local companies)	Electric power consumption (kWh per capita)	The production of power plants and combined heat and power plants less transmission, distribution, and transformation losses and own use by heat and power plants
Quality of life (ability to sell luxury products to fulfil basic needs)	Life expectancy at birth (years)	The number of years a newborn infant would live if prevailing patterns of mortality at the time of its birth were to stay the same throughout its life
	Improved sanitation facilities (% of population with access)	The percentage of the population with at least adequate access to excreta disposal facilities that can effectively prevent human, animal, and insect contact with excreta
Infrastructure (ability to	Road density (km of road per	Ratio of the length of the country's total road network to the country's land area in

Dimension	**Measures (units)**	**Indicator Description**
organize production and distribution of products)	100 sq. km of land area)	2003 or 2004 (Thailand – in 2006). The road network includes all roads in the country: motorways, highways, main or national roads, secondary or regional roads, and other urban and rural roads.
	Internet users (per 100 people)	People with access to the worldwide network.
Market Intensity (ability to satisfy unfulfilled needs)	GDP per capita (GDP per capita constant 2000;US$)	Gross domestic product divided by midyear population.
Market receptivity (ability to export and import products/materials and semi-products)	Trade (% of GDP)	The sum of exports and imports of goods and services measured as a share of gross domestic product.
Cultural Distance (differences in culture preventing the flow of information from and to the market)	Cultural distance (index)	Based on four cultural dimensions (power distance, individualism/collectivism, masculinity/femininity and uncertainty avoidance; long-term/short-term orientation was not included because for a few countries data was not available). Index calculated as Euclidean distance from Poland in accordance with the formula used by Morosini et al. (1998)
Factors of production (access to resources which are not available at home or have higher quality and/or lower cost)	Cotton production (in thousand bales)	Production of cotton; the base years 2003/2004
	Labour force (number of persons)	People ages 15 and older who meet the International Labor Organization definition of the economically active population: all people who supply labour for the production of goods and services during a specified period
Investment Climate (ease of doing business)	Foreign direct investment net inflows (% of GDP)	Foreign direct investment are the net inflows of investment to acquire a lasting management interest (10 percent or more of voting stock) in an enterprise operating in an economy other than that of the investor. It is the sum of equity capital, reinvestment of earnings, other long-term capital, and short-term capital as shown in the balance of payments. Data show net inflows (new investment inflows less

Dimension	**Measures (units)**	**Indicator Description**
		disinvestment) in the reporting economy from foreign investors, and is divided by GDP
	Economic freedom (index)	Overall score (property rights, freedom from corruption, fiscal freedom, government spending, business freedom, labour freedom, monetary freedom, trade freedom, investment freedom, financial freedom)

Source: World Bank, The Hofstede Centre, The Heritage Foundation

To rank countries from the best to the worst from the point of view of international expansion the EXPROM II method (Diakoulaki and Koumoutsos, 1991) with veto threshold has been applied. It is based on the notion of ideal and anti-ideal solutions and enables the decision-maker to rank alternatives on a cardinal scale. Thanks to the introduction of the veto threshold (see Appendix) the technique is partly compensatory (a really bad score on one criterion cannot be compensated with a good score on another).

We decided to employ this method because it is considered to be a user-friendly one – all steps can be quite easily explained to the decision-maker as they are neither very complex nor mathematically challenging. Moreover, this technique allows us to obtain a complete pre-order of the alternatives to which the points are assigned in the final solution. This form of the final solution is recognized as being convincing for the potential users of MCDA methods.

To check the impact of changes in the weights of evaluation criteria on the final rankings of countries we have established five different vectors of weighting coefficients. The first two vectors were determined arbitrarily by the present authors, the third one was created with the help of Hinkle's method, which is also called the 'resistance to change' grid (Hinkle, 1965; Rogers and Bruen, 1998), and the fourth one used the AHP method (Saaty, 2006; Saaty and Vargas, 1991). In the last approach all measures were assumed to be equally important. The authors also established the values of indifference (q), preference (p) and veto (v) thresholds. The model of preferences for the decision-making problem is presented in the Table 3.

5 Results

Table 4 provides a summary of the results obtained by applying the EXPROM II technique with veto threshold using 5 different vectors of weighting coefficients.

Table 3

Model of preferences

Measure	Max /min	Vectors of weighting coefficients					q	p	v
		I	II	III	IV	V			
Total population	max	0,089	0,100	0,120	0,136	0,067	10 ml	100 ml	500 ml
Urban population	max	0,089	0,100	0,120	0,136	0,067	5 ml	50 ml	300 ml
GDP growth rate	max	0,107	0,133	0,120	0,113	0,067	0,5	1	10
Electric power consumption	min	0,071	0,067	0,060	0,04	0,067	100	1000	10000
Life expectancy	max	0,036	0,033	0,010	0,02	0,067	3	10	25
Sanitation facilities	max	0,036	0,033	0,010	0,02	0,067	5	10	25
Road density	max	0,036	0,033	0,010	0,02	0,067	5	50	200
Internet users	max	0,036	0,033	0,010	0,02	0,067	5	10	40
GDP per capita	min	0,071	0,067	0,060	0,04	0,067	500	5000	30000
Trade	max	0,107	0,067	0,120	0,113	0,067	5	15	50
Cultural distance	min	0,107	0,100	0,120	0,113	0,067	5	10	40
Labour force	max	0,054	0,067	0,060	0,057	0,067	5 ml	20 ml	100 ml
Cotton production	max	0,054	0,067	0,060	0,057	0,067	10	100	20000
FDI net inflows	max	0,054	0,050	0,060	0,057	0,067	0,5	3	10
Economic freedom	max	0,054	0,050	0,060	0,057	0,067	5	10	40

The rankings presented in Table 4 show the sensitivity of the solutions to the changes in the vectors of weights as the modifications of the parameter values led to alterations in countries' rankings.

The different rankings of the countries obtained are not in agreement. However, in spite of that it is possible to determine the set of countries which are the best, taking into account their attractiveness as the target of international expansion for the company considered (China and Thailand), the set of countries which are quite good as the values of net flows determined for them are in all cases positive (Brazil, France, India, Italy, Mexico, Philippines, the United States and Vietnam) and the set of countries which are the worst (Bangladesh, Ethiopia, Indonesia, Iran, Nigeria, Pakistan and Turkey). Egypt, Japan and the United Kingdom may be regarded as controversial since in some cases the values of net flows determined for them are positive and in some cases – negative.

Table 4

Rankings of the countries obtained using EXPROM II method
with veto threshold and 5 different vectors of weights

No.	EXPROM II with veto threshold					No.
	Vector no. 1	Vector no. 2	Vector no. 3	Vector no. 4	Vector no. 5	
1	China	China	China	China	Thailand	1
2	Thailand	Thailand	Thailand	Thailand	China	2
3	Brazil	Brazil	Brazil	Brazil	Italy	3
4	Italy	Italy	Mexico	Mexico	Brazil	4
5	France	France	Italy	Italy	France	5
6	Mexico	India	France	India	Mexico	6
7	Philippines	Mexico	India	France	Vietnam	7
8	Vietnam	Vietnam	Vietnam	Vietnam	Philippines	8
9	India	Philippines	Philippines	Philippines	India	9
10	United States	United States	United States	United States	United States	10
11	Japan	Egypt, Arab Rep.	Japan	Japan	Japan	11
12	Egypt, Arab Rep.	Japan	Egypt, Arab Rep.	Egypt, Arab Rep.	United Kingdom	12
13	United Kingdom	United Kingdom	Indonesia	Indonesia	Egypt, Arab Rep.	13
14	Indonesia	Iran, Islamic Rep.	United Kingdom	United Kingdom	Turkey	14
15	Iran, Islamic Rep.	Indonesia	Iran, Islamic Rep.	Iran, Islamic Rep.	Iran, Islamic Rep.	15
16	Turkey	Bangladesh	Turkey	Turkey	Indonesia	16
17	Bangladesh	Turkey	Nigeria	Nigeria	Bangladesh	17
18	Nigeria	Nigeria	Bangladesh	Bangladesh	Nigeria	18
19	Ethiopia	Pakistan	Ethiopia	Pakistan	Pakistan	19
20	Pakistan	Ethiopia	Pakistan	Ethiopia	Ethiopia	20

To sum up, taking into account all the results we have obtained, the following countries are recommended for further analysis (in-depth screening and final selection) – excluding China and Thailand: Brazil, France, India, Italy, Mexico, Philippines, the United States and Vietnam.

6

7 Sensitivity and robustness analysis

In the first step of the analysis the ranges of variations of indifference and preference thresholds, which do not result in modification of the rankings obtained with the help of the EXPROM II method with veto threshold applying the second and the fifth vector of weighting coefficients, were determined using optimization tools integrated with Excel. The analysis was carried out separately for each of the thresholds provided that they satisfy the following condition: $0 \le q_k \le p_k \le v_k$. The results are displayed in Tables 5 and 6. They indicate that the rankings obtained are not very sensitive to variations of the values of the thresholds.

Table 5

Ranges of variations of the indifference and preference thresholds values
in the case of EXPROM II method with veto threshold applying
the second vector of weighting coefficients

Measure	q min*	q original	q max**	p min*	p original	p max**
Total population	0	10 ml	24,46 ml	91,48 ml	100 ml	113,21 ml
Urban population	0	5 ml	8,05 ml	42,08 ml	50 ml	53,28 ml
GDP growth rate	0	0,5	1	0,87	1	1,49
Electric power consumption	0	100	381,94	100	1000	2048,70
Life expectancy	0,33	3	9,20	3,63	10	14,20
Sanitation facilities	0	5	10	7,22	10	12,34
Road density	0	5	11,50	43,74	50	67,24
Internet users	1,47	5	10	6,87	10	18,29
GDP per capita	0	500	1691,50	2734,65	5000	6370,28
Trade	0	5	15	7,59	15	16,24
Cultural distance	0	5	9,51	7,50	10	13,50
Labour force	0	5 ml	7,90 ml	5 ml	20 ml	26,16 ml
Cotton production	0	10	100	10	100	382,69
FDI net inflows	0	0,5	1,86	2,28	3	3,75
Economic freedom	0	5	8,79	5	10	13,13

* Values are rounded up to the nearest hundredth.
** Values are rounded down to the nearest hundredth.

To check the impact of the method applied on the final rankings of the countries we have employed two other outranking techniques, namely the PROMETHEE II method with veto threshold (Górecka and Pietrzak, 2012) and the modified ELECTRE III method (Górecka, 2009). The results obtained with the help of them are presented in Tables 7 and 8. In both cases five aforementioned vectors of weighting coefficients have been applied to show the influence of changes in the weights of evaluation criteria on the final rankings of countries examined.

Table 6

Ranges of variations of the indifference and preference thresholds values in the case of EXPROM II method with veto threshold applying equal weights

(the fifth vector of weighting coefficients)

Measure	q min	q original	q max	p min*	p original	p max**
Total population	0	10 ml	100 ml	95,88 ml	100 ml	500 ml
Urban population	0	5 ml	50 ml	47,46 ml	50 ml	300 ml
GDP growth rate	0	0,5	1	0,88	1	1,61
Electric power consumption	0	100	1000	100	1000	2474,98
Life expectancy	0	3	10	6,35	10	10,39
Sanitation facilities	0	5	10	9,33	10	12,47
Road density	0	5	50	33,53	50	52,76
Internet users	0	5	10	5	10	25,96
GDP per capita	0	500	5000	4466,96	5000	7885,30
Trade	0	5	15	14,47	15	17,26
Cultural distance	0	5	10	5	10	10,51
Labour force	0	5 ml	20 ml	19,68 ml	20 ml	38,00 ml
Cotton production	0	10	100	10	100	116,57
FDI net inflows	0	0,5	3	1,40	3	3,04
Economic freedom	0	5	10	5	10	10,39

*Values are rounded up to the nearest hundredth.
**Values are rounded down to the nearest hundredth.

Once again, it can be easily noticed that the rankings obtained do not differ much from each other. Hence, it is possible to determine the set of countries which are the best, taking into account their attractiveness as the target of international expansion for the considered company (China and Thailand in the case of the PROMETHEE II method with veto threshold; Brazil, China and Mexico in the case of the modified ELECTRE III method), the set of countries which are quite good (Brazil, France, India, Italy, Mexico, Philippines, the United States and Vietnam in the case of the PROMETHEE II method with veto threshold[3]; France, India, Indonesia, Italy, the United States, Thailand and Vietnam in the case of the modified ELECTRE III method[4]) and the set of countries which are the worst (Bangladesh, Ethiopia, Indonesia, Iran, Nigeria, Pakistan and Turkey in the case of the PROMETHEE II method with veto

[3] The values of net flows determined for them are in all cases positive.

[4] The differences between the number of countries outranked by them and the number of countries that outranks them are in all cases non-negative.

threshold[5]; Bangladesh, Egypt, Ethiopia, Iran, Pakistan and Turkey in the case of the modified ELECTRE III method[6]).

Table 7

Rankings of the countries obtained using PROMETHEE II method
with veto threshold and 5 different vectors of weights

No.	PROMETHEE II with veto threshold					No.
	Vector no. 1	Vector no. 2	Vector no. 3	Vector no. 4	Vector no. 5	
1	Thailand	China	Thailand	China	Thailand	1
2	China	Thailand	China	Thailand	China	2
3	Brazil	Brazil	Brazil	Brazil	Italy	3
4	Italy	Italy	Mexico	Mexico	Brazil	4
5	France	France	France	Italy	France	5
6	Mexico	Mexico	Italy	France	Mexico	6
7	Philippines	India	India	India	Vietnam	7
8	Vietnam	Vietnam	Philippines	Vietnam	Philippines	8
9	India	Philippines	Vietnam	Philippines	India	9
10	United States	United States	United States	United States	United States	10
11	Japan	Egypt, Arab Rep.	Japan	Japan	Japan	11
12	Egypt, Arab Rep.	Japan	Egypt, Arab Rep.	Egypt, Arab Rep.	United Kingdom	12
13	United Kingdom	Iran, Islamic Rep.	Indonesia	Indonesia	Egypt, Arab Rep.	13
14	Indonesia	United Kingdom	United Kingdom	United Kingdom	Turkey	14
15	Iran, Islamic Rep.	Indonesia	Iran, Islamic Rep.	Iran, Islamic Rep.	Indonesia	15
16	Turkey	Turkey	Turkey	Turkey	Iran, Islamic Rep.	16
17	Nigeria	Bangladesh	Nigeria	Nigeria	Bangladesh	17
18	Bangladesh	Nigeria	Bangladesh	Bangladesh	Nigeria	18
19	Ethiopia	Pakistan	Ethiopia	Pakistan	Pakistan	19
20	Pakistan	Ethiopia	Pakistan	Ethiopia	Ethiopia	20

[5] The values of net flows determined for them are in all cases negative.
[6] The differences between the number of countries outranked by them and the number of countries that outranks them are in all cases negative.

Furthermore, it should be noted that rankings obtained with the help of three different MCDA techniques are similar. This observation can be confirmed by the Spearman rank correlation coefficients presented in Table 9. These coefficients, calculated separately for each of five vectors of weights considered, indicate the existence of strong correlation dependencies between the obtained orderings of the countries.

Table 8

Rankings of the countries obtained using
modified ELECTRE III method and 5 different vectors of weights

No.	Modified ELECTRE III					No.
	Vector no. 1	Vector no. 2	Vector no. 3	Vector no. 4	Vector no. 5	
1	Brazil	Brazil	Brazil	Brazil	Brazil	1
2	China	China	China	China	China	2
3	Mexico	Mexico	Mexico	Mexico	Mexico	3
4	India	India	India	India	Italy	4
5	Indonesia	Indonesia	United States	United States	Thailand	5
6	Vietnam	Vietnam	Vietnam	Indonesia	France	6
7	Thailand	France	Thailand	Vietnam	India	7
8	France	United States	France	France	Indonesia	8
9	Italy	Thailand	Indonesia	Thailand	United States	9
10	United States	United Kingdom	Nigeria	Italy	Japan	10
11	United Kingdom	Italy	Italy	Japan	Philippines	11
12	Japan	Japan	Japan	Philippines	Vietnam	12
13	Philippines	Philippines	Philippines	United Kingdom	United Kingdom	13
14	Nigeria	Egypt, Arab Rep.	United Kingdom	Nigeria	Bangladesh	14
15	Egypt, Arab Rep.	Iran, Islamic Rep.	Egypt, Arab Rep.	Egypt, Arab Rep.	Turkey	15
16	Iran, Islamic Rep.	Turkey	Iran, Islamic Rep.	Iran, Islamic Rep.	Egypt, Arab Rep.	16
17	Turkey	Bangladesh	Turkey	Turkey	Iran, Islamic Rep.	17
18	Bangladesh	Nigeria	Bangladesh	Bangladesh	Pakistan	18
19	Ethiopia	Ethiopia	Pakistan	Pakistan	Nigeria	19
20	Pakistan	Pakistan	Ethiopia	Ethiopia	Ethiopia	20

To sum up, the analysis performed has illustrated that the solutions obtained are quite robust to changes in the values of the parameters of the preference model. It has also shown that the rankings of the countries are not very sensitive to choice of the decision-aiding technique.

Taking into account all the results of the research conducted, the following countries are recommended for further analysis (in-depth screening and final selection): Brazil, China, France, India, Italy, Mexico, Thailand, the United States and Vietnam. The Philippines has been removed from this list as according to the results obtained with the help of the modified ELECTRE III method it does not belong either to the set of countries which are the best or to the set of countries which are quite good from the point of view of their attractiveness as the target of international expansion for the company [7] considered.

Table 9

Spearman rank correlation coefficients

Vector no. 1			
Method	EXPROM II	PROMETHEE II	Modified ELECTRE III
EXPROM II	1,0000	0,9970	0,8748
PROMETHEE II	0,9970	1,0000	0,8718
Modified ELECTRE III	0,8748	0,8718	1,0000
Vector no. 2			
Method	EXPROM II	PROMETHEE II	Modified ELECTRE III
EXPROM II	1,0000	0,9955	0,8195
PROMETHEE II	0,9955	1,0000	0,8150
Modified ELECTRE III	0,8195	0,8150	1,0000
Vector no. 3			
Method	EXPROM II	PROMETHEE II	Modified ELECTRE III
EXPROM II	1,0000	0,9955	0,8711
PROMETHEE II	0,9955	1,0000	0,8598
Modified ELECTRE III	0,8711	0,8598	1,0000
Vector no. 4			
Method	EXPROM II	PROMETHEE II	Modified ELECTRE III
EXPROM II	1,0000	0,9985	0,8699
PROMETHEE II	0,9985	1,0000	0,8662
Modified ELECTRE III	0,8699	0,8662	1,0000
Vector no. 5			
Method	EXPROM II	PROMETHEE II	Modified ELECTRE III
EXPROM II	1,0000	0,9985	0,8831
PROMETHEE II	0,9985	1,0000	0,8966
Modified ELECTRE III	0,8831	0,8966	1,0000

[7] According to the solutions obtained using modified ELECTRE III method the Philippines belongs to the set of countries which are quite bad from the point of view of international expansion since the differences between the number of countries outranked by it and the number of countries that outranks it are in all cases non-positive.

8 Conclusions

In reality, the firm that formed the basis of our analysis of its international expansion has chosen India. It is not grossly at variance with the results we have obtained due to the fact that India is in the group of countries selected for further analysis. Within this group – as expected – emerging markets predominate as these countries are experiencing a brisk rate of economic growth and industrialization, leading to improved standards of living. They represent long-term market potential and sourcing opportunities because they offer access to young consumers with purchasing power as well as to cheaper resources. Unfortunately in the analysis conducted within the framework of this article the costs of resources were not taken into consideration as the necessary data were not available.

Additionally, it has to be emphasized that the outcomes of the analysis strongly depend on the dimensions and measures that are used in it. We do not know which criteria were applied at the time of conducting the original assessment in the company concerned. It is possible that India could have been ranked higher, had a different set of criteria been used.

Finally, it is worth mentioning that the MCDA methods based on the outranking relation (e.g. EXPROM II with veto threshold) can be used to solve the market selection problem for international expansion. In fact, applying these methods can enhance the evaluation process and improve decision-making since the assumptions on which they are based are in line with reality.

Appendix. Exprom method with veto threshold

EXPROM is a modification and extension of PROMETHEE method[8] that was proposed in Diakoulaki and Koumoutsos, 1991. It is based on the notion of ideal and anti-ideal solutions and enables the decision-maker to rank alternatives on a cardinal scale. Assuming that all criteria are to be maximized, the values of the ideal and anti-ideal solutions are defined as follows:

ideal alternative: $f_k(\mathrm{a}^*) = \max\limits_{\mathrm{a}^i \in A} f_k(\mathrm{a}^i)$;

anti-ideal alternative: $f_k(\mathrm{a}_*) = \min\limits_{\mathrm{a}^i \in A} f_k(\mathrm{a}^i)$[9],

where $A = \{\mathrm{a}^1, ..., \mathrm{a}^m\}$ is a finite set of m alternatives and $F = \{f_1, f_2, ..., f_n\}$ is a set of n criteria examined.

[8] The idea of PROMETHEE methodology is presented in Brans and Vincke, 1985 and description of PROMETHEE techniques can be found in Brans et al., 1986.

[9] The values can be also defined independently from the examined alternatives, representing – in the case of the ideal solution – some realistic goals and in the case of the anti-ideal solution – the situation that should be avoided.

After introducing the veto threshold to EXPROM method the procedure of ordering alternatives consists of the following steps:

1. Calculation of concordance indices for each pair of alternatives $(\mathrm{a}^i, \mathrm{a}^j)$:

$$c(\mathrm{a}^i, \mathrm{a}^j) = \sum_{k=1}^{n} w_k \varphi_k(\mathrm{a}^i, \mathrm{a}^j),$$

where: $\sum_{k=1}^{n} w_k = 1$,

$$\varphi_k(\mathrm{a}^i, \mathrm{a}^j) = \begin{cases} 1, \ \mathit{if} \quad f_k(\mathrm{a}^i) - f_k(\mathrm{a}^j) > p_k, \\ \dfrac{f_k(\mathrm{a}^i) - f_k(\mathrm{a}^j) - q_k}{p_k - q_k}, \ \mathit{if} \quad q_k < f_k(\mathrm{a}^i) - f_k(\mathrm{a}^j) \le p_k, \\ 0 \ \ \mathit{otherwise}, \end{cases}$$

w_k – coefficient of importance for criterion f_k,

$f_k(\mathrm{a}^i)$ – evaluation of alternative a^i with respect to criterion f_k,

q_k – indifference threshold for criterion f_k,

p_k – preference threshold for criterion f_k.

2. Calculation of discordance indices for each pair of alternatives $(\mathrm{a}^i, \mathrm{a}^j)$ and for each criterion:

$$d_k(\mathrm{a}^i, \mathrm{a}^j) = \begin{cases} 1, \ \mathit{if} \quad f_k(\mathrm{a}^j) - f_k(\mathrm{a}^i) > v_k, \\ \dfrac{f_k(\mathrm{a}^j) - f_k(\mathrm{a}^i) - p_k}{v_k - p_k}, \ \mathit{if} \quad p_k < f_k(\mathrm{a}^j) - f_k(\mathrm{a}^i) \le v_k, \\ 0 \ \ \mathit{otherwise}, \end{cases}$$

where v_k – veto threshold for criterion f_k.

3. Calculation of credibility indices for each pair of alternatives $(\mathrm{a}^i, \mathrm{a}^j)$:

$$\sigma(\mathrm{a}^i, \mathrm{a}^j) = c(\mathrm{a}^i, \mathrm{a}^j) \prod_{k \in D(\mathrm{a}^i, \mathrm{a}^j)} \frac{1 - d_k(\mathrm{a}^i, \mathrm{a}^j)}{1 - c(\mathrm{a}^i, \mathrm{a}^j)}$$

where: $D(\mathrm{a}^i, \mathrm{a}^j) = \{k : d_k(\mathrm{a}^i, \mathrm{a}^j) > c(\mathrm{a}^i, \mathrm{a}^j)\}$.

4. Determination of strict preference indices for each pair of alternatives $(\mathrm{a}^i, \mathrm{a}^j)$:

$$\pi(\mathrm{a}^i,\mathrm{a}^j)=\nu(\mathrm{a}^i,\mathrm{a}^j)\cdot\sum_{k=1}^{n}w_k\pi_k(\mathrm{a}^i,\mathrm{a}^j),$$

where:

$$\nu(\mathrm{a}^i,\mathrm{a}^j)=\begin{cases}1, & gdy \quad \forall k: d_k(\mathrm{a}^i,\mathrm{a}^j)\le c(\mathrm{a}^i,\mathrm{a}^j),\\ 0, & gdy \quad \exists k: d_k(\mathrm{a}^i,\mathrm{a}^j)> c(\mathrm{a}^i,\mathrm{a}^j),\end{cases}$$

$$\pi_k(\mathrm{a}^i,\mathrm{a}^j)=\begin{cases}\dfrac{f_k(\mathrm{a}^i)-f_k(\mathrm{a}^j)-p_k}{f_k(\mathrm{a}^*)-f_k(\mathrm{a}_*)-p_k}, & if \quad \varphi_k(\mathrm{a}^i,\mathrm{a}^j)=1,\\ 0 \ otherwise. & \end{cases}$$

5. Calculation of total preference index for each pair of alternatives $(\mathrm{a}^i,\mathrm{a}^j)$:

$$\omega(\mathrm{a}^i,\mathrm{a}^j)=\min\{1;\ \sigma(\mathrm{a}^i,\mathrm{a}^j)+\pi(\mathrm{a}^i,\mathrm{a}^j)\}.$$

6. Calculation of outgoing flow $\phi^+(a^i)$ and incoming flow $\phi^-(a^i)$ for each alternative:

$$\phi^+(\mathrm{a}^i)=\sum_{j=1}^{m}\omega(\mathrm{a}^i,\mathrm{a}^j)$$

$$\phi^-(\mathrm{a}^i)=\sum_{j=1}^{m}\omega(\mathrm{a}^j,\mathrm{a}^i).$$

In EXPROM I a final partial ranking is obtained as follows:

$$\begin{cases}\mathrm{a}^iP\mathrm{a}^j, \quad if & \begin{cases}\phi^+(\mathrm{a}^i)>\phi^+(\mathrm{a}^j) \quad i \quad \phi^-(\mathrm{a}^i)<\phi^-(\mathrm{a}^j) \quad or\\ \phi^+(\mathrm{a}^i)=\phi^+(\mathrm{a}^j) \quad i \quad \phi^-(\mathrm{a}^i)<\phi^-(\mathrm{a}^j) \quad or\\ \phi^+(\mathrm{a}^i)>\phi^+(\mathrm{a}^j) \quad i \quad \phi^-(\mathrm{a}^i)=\phi^-(\mathrm{a}^j);\end{cases}\\ \mathrm{a}^iI\mathrm{a}^j, \quad if & \phi^+(\mathrm{a}^i)=\phi^+(\mathrm{a}^j) \quad i \quad \phi^-(\mathrm{a}^i)=\phi^-(\mathrm{a}^j);\\ \mathrm{a}^iR\mathrm{a}^j, \quad if & \begin{cases}\phi^+(\mathrm{a}^i)>\phi^+(\mathrm{a}^j) \quad i \quad \phi^-(\mathrm{a}^i)>\phi^-(\mathrm{a}^j) \quad or\\ \phi^+(\mathrm{a}^i)<\phi^+(\mathrm{a}^j) \quad i \quad \phi^-(\mathrm{a}^i)<\phi^-(\mathrm{a}^j);\end{cases}\end{cases}$$

where P, I and R stand for preference, indifference and incomparability, respectively.

In EXPROM II a final complete ranking is constructed according to the descending order of the net flows $\phi(a^i)$, where $\phi(\mathrm{a}^i)=\phi^+(\mathrm{a}^i)-\phi^-(\mathrm{a}^i)$.

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M U L T I P L E   C R I T E R I A   D E C I S I O N   M A K I N G
Vol. 8 2013

RE-CALCULATION OF HAPPY PLANET INDEX USING DEA MODELS

Josef Jablonsky*

Abstract

Happy Planet Index (HPI) is an aggregated index that measures the extent to which each nation produces long and happy lives per unit of environmental input. The HPI uses global data on life expectancy, experienced well-being, and ecological footprint to rank countries. The last HPI report was published in 2012 and it contains data for 151 countries from all continents. The aim of the paper is to re-calculate the HPI using DEA models and other multiple criteria decision making techniques and compare the results obtained results. MCDM methods evaluate alternatives (countries) according to the set of criteria with respect to given preferences. Most of them allow ranking of alternatives according to aggregated indices defined by various methods. DEA models compare the countries with the best performers in the data set and measure the efficiency of transformation of multiple inputs into multiple outputs. Even though they are based on different principles than MCDM methods they allow ranking of evaluated units according to their efficiency or super-efficiency scores. The paper analyzes both methodological approaches and compares their results.

Keywords: Data envelopment analysis, MCDM, Happy Planet Index, efficiency

1 Introduction

There are many attempts to compare the level of development of world countries from different points of view. The best-known and oldest characteristic is the human development index (HDI) which has been published by the United Nations Development Programme (UNDP) since 1990. It is an aggregated measure that is based on four criteria: life expectancy at birth, adult literacy rate, combined enrolment ratio, and GDP per capita.

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A multiple criteria decision making problem (MCDM) consists in the selection of a “best” (compromise) alternative or, more generally, ranking of all alternatives. In a narrow sense one of the characteristics of the MCDM problem is the presence of the decision maker’s preferences that can be given in several quite different ways. The most common way how the DM’s expresses his/her preferences is the selection of the set of criteria and the specification of their weights. In a broader sense MCDM problems are any problems where a set of alternatives is evaluated with respect to the given set of criteria. This set of criteria and their weights can be determined by a discussion in a group of DMs or by any authority or institution. In this case the DM is not present in the construction of the final solution but the problem remains an MCDM one. The calculation of the HDI belongs to a broad group of problems of the above-mentioned nature. Several attempts to re-calculate the HDI were published in the past. They are based either on using other MCDM methodology than the one used by the UNDP, or on data envelopment analysis (DEA) models. An interesting attempt to re-calculate the HDI using DEA models can be found e.g. in Mahlberg and Obersteiner (2001) and Despotis 2005).

One of the newest global indicators of countries is the Happy Planet Index (HPI). This index was introduced in 2006 by the New Economic Foundation (NEF); it is a global measure of sustainable human well-being and environmental impact. The value of the HPI is influenced positively by the level of experienced well-being and life expectancy and negatively by the ecological footprint. The aim of the present paper is to discuss a possibility of recalculation of the HPI using MCDM and DEA models and to compare the results obtained with standard methodology. Section 2 of the paper contains detailed information about the calculation of the HPI. Section 3 formulates DEA models suitable for analysis of the HPI and Section 4 presents results obtained by various modelling approaches. The final section contains conclusions and directions for future research.

2 Happy Planet Index

The information about the HPI and its calculation are taken from the HPI 2012 Report (Abdallah et al., 2012). The HPI is an efficiency measure which expresses the level which long and happy lives achieve per unit of environmental impact. It is based on the following data sources:

1. *Life expectancy at birth* (further denoted *x*). This figure expresses the number of years an infant born in that country could expect to live if prevailing patterns of age-specific mortality rates at the time of birth in the country stay the same throughout the infant’s life. The calculation of HPI 2012 uses data published in 2011 Human Development Report.
2. *Experienced well-being* (*y*). The data for the average level of well-being in the countries are taken from the survey of the Gallup World Poll which uses samples of around 1000 individuals aged more than 15 years from

each country. They assign grades from 0 (worse living conditions) to 10 (best living condition) and the final well-being country index is a simple average of all responses.

3. *Ecological footprint* (z). It is a measure expressed in g ha (so called global hectares) per capita of human demand on nature. Ecological footprint index measures the amount of land required to sustain the country's consumption pattern. It includes the land required to provide the renewable resources that people use, the area occupied by infrastructure, and the area necessary to absorb CO_2 emissions. More information about calculation of this composite index can be found in Boruckea et al. (2013).

In general, the HPI is calculated as the ratio:

$$\text{HPI} = \frac{x\,.\,y}{z}.$$

The HPI cannot be simply calculated using the formula above because x, y, and z are given on different scales with different variances. This formula is only very general and for comparison purposes the HPI is calibrated to reach values from 0 to 100. The calculation is divided into two steps:

Calculation of the Happy Life Years index (w). This index composes first two elements of the HPI, i.e. life expectancy and well-being, as follows:

$$w = \frac{x(y+\alpha)}{10+\alpha},$$

where α = 2.93 is a constant added to y (experienced well-being) to unify the level of variance of both characteristics.

Calculation of the HPI. In the second stage, the constant γ = 5.67 is subtracted from w to ensure that the country with an average well-being score of 0 or a life expectancy of 25 or lower achieves the HPI score of 0, and the constant β = 4.38 is added to ecological footprint to ensure that its coefficient variance is equal to that of index w. Finally, the HPI scores are calculated according to the following formula:

$$\text{HPI} = \frac{\delta(w-\gamma)}{z+\beta},$$

where δ = 7.77 is the constant that ensures that the country with average well-being score of 10, average life expectancy of 85 years, and ecological footprint of 1.78 g ha per capita (equivalent to one planet living) achieves the HPI score of 100.

3 DEA models for HPI re-calculation

DEA models are a general tool for evaluation of efficiency and performance of the set of decision making units. The re-assessment of the HPI or other indices of world countries is a very specific problem. The main aim of the present paper is to propose a DEA based methodology for the calculation of the HPI and to compare the results with commonly used methodology based on a simple aggregation of the criteria.

Let us suppose that the set of decision making units (DMUs) contains n elements. The DMUs are evaluated by m inputs and r outputs with input and output values x_{ij}, $i = 1,2,\dots,m$, $j = 1,2,\dots,n$ and y_{kj}, $k = 1,2,\dots,r$, $j = 1,2,\dots,n$, respectively. The efficiency score θ_q of the DMU$_q$ can be expressed as the weighted sum of outputs divided by the weighted sum of inputs with weights reflecting the importance of single inputs/outputs v_i, $i = 1,2,\dots,m$ and u_k, $k = 1,2,\dots,r$ as follows:

$$\theta_q = \frac{\sum_{k=1}^{r} u_k y_{kq}}{\sum_{i=1}^{m} v_i x_{iq}}.$$

The conventional CCR DEA model formulated by Charnes et al. (1978) consists in the maximization of the efficiency score θ_q of the DMU$_q$ subject to constraints that efficiency scores of all other DMUs are lower than or equal to 1. The linearized form of this model with output orientation is as follows:

Minimize

$$\theta_q = \sum_{i=1}^{r} v_i x_{iq}$$

$$\text{subject to} \quad \sum_{k=1}^{m} u_k y_{kq} = 1, \tag{1}$$

$$\sum_{k=1}^{r} u_k y_{kj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1,2,\dots,n,$$

$$u_k, v_i \ge \varepsilon, \qquad k = 1,2,\dots,r, i = 1,2,\dots,m.$$

If the optimal value of model (1) $\theta^*_q = 1$ then the DMU$_q$ is CCR efficient and it is lying on the CCR efficient frontier, otherwise $\theta^*_q > 1$ and the unit is not efficient. The value θ^*_q expresses the rate of increase of outputs needed to reach the efficient frontier.

Model (1) is the CCR output oriented model with the assumption of constant returns to scale. The appropriate model with variable returns to scale (VRS) is as follows:

Minimize

$$\theta_q = \sum_{i=1}^{r} v_i x_{iq} + \mu$$

subject to

$$\begin{aligned}
&\sum_{k=1}^{m} u_k y_{kq} = 1, && \\
&\sum_{k=1}^{r} u_k y_{kj} - \sum_{i=1}^{m} v_i x_{ij} + \mu \le 0, && j = 1,2,\ldots,n, \\
&u_k, v_i \ge \varepsilon, && k = 1,2,\ldots,r, i = 1,2,\ldots,m, \\
&\mu - \text{free.} &&
\end{aligned} \tag{2}$$

Many other modifications of the conventional DEA models have been formulated in the literature. One of the most interesting is the slack based model (SBM) – Tone (2001) which is formulated as follows:

Minimize

$$\rho_q = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^- / x_{iq}}{1 + \frac{1}{r}\sum_{k=1}^{r} s_k^+ / y_{kq}},$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = x_{iq}, && i = 1,2,...,m, \\
&\sum_{j=1}^{n} y_{kj}\lambda_j - s_k^+ = y_{kq}, && k = 1,2,...,r, \\
&\lambda_j \ge 0, \quad j = 1, 2,..., n. && \\
&s_k^+, s_i^- \ge 0, \quad i = 1, 2,..., m, k = 1, 2,..., r, &&
\end{aligned} \tag{3}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n)$ is a vector of weights of the DMUs, $\mathbf{s}^+ = (s_1^+, s_2^+,..., s_r^+)$ is a vector of surplus variables, $\mathbf{s}^- = (s_1^-, s_2^-,..., s_m^-)$ is a vector of slack variables and ρ_q is efficiency score of the DMU_q. Tone's model is a non-radial model that measures the efficiency using relative slack and surplus variables only. The efficiency score ρ_q equals 1 for efficient units (all slack and surplus variables equal 0) and is lower than 1 for inefficient ones. The model (3) is not linear in objective function but can be simply transformed into a LP problem – see Tone, (2001) for more details.

Most of the DEA applications consider solely inputs or resources used by a DMU and desirable outputs that are the results of input utilization. In this case higher values of outputs lead to higher efficiency (when a fixed level of inputs is used). Nevertheless, this assumption is rarely acceptable and one or several outputs in the model are undesirable (e.g. environmental impact, pollutions, tax payments, etc.). Various approaches were proposed in the past for dealing with undesirable outputs. The easiest way is to transform the undesirable output into a desirable one by subtracting the original values from a given upper (worse) bound. In evaluation of the HPI there are two main desirable outputs: life expectancy and well-being, and one undesirable output: ecological footprint.

Conventional DEA models, e.g. the model (2), optimize the efficiency of the evaluated unit using adjustment of the weights of the inputs and outputs. The weights are limited by the infinitesimal constant ε (e.g. 10^{-8}) only. That is why some of the weights may be reaching their lower bounds, i.e. they are very small which may be unacceptable for decision makers. Various ways of restricting weights in DEA models have been proposed. This question is very important because inappropriate restrictions can easily lead to infeasible solutions of the model.

Another important task in applications of DEA models for ranking of DMUs consists in ranking of efficient units. All efficient units have maximum efficiency score 1 and cannot be ranked using the conventional models. For discrimination among them various so called super-efficiency models were proposed in the past. More information about them can be found e.g. in Cooper at al. (2000) and Tone (2002).

In the numerical experiments described in the next section of the paper there were applied all modifications of the DEA models mentioned above, i.e. models with weight restrictions, super-efficiency models and models with undesirable outputs.

4 DEA and MCDM analysis of the HPI

As described above, the HPI consists of three indicators (criteria). The calculation of this index can be regarded as a conventional MCDM problem. In the numerical experiments described below we use WSA and TOPSIS. Their common feature is that these three methods do not require any additional information from DM except weights of criteria. Apart from that, various modifications of DEA models are applied. All calculations using DEA models were performed on the modified data set that assigns 0 to basal and 1 to ideal alternative. MCDM methods in our experiments use the original data set described in detail in Table 1. This table contains, apart from the three main indicators, information about GDP because it was used as an additional indicator in some calculations presented below (the ideal value for GDP is set up to \$ 60 000/cap and to each of the few countries with a higher value of GPD a maximum value, i.e. 1, is assigned). Numerical experiments are performed using the software package *Sanna* which implements MCDM methods, and the *DEA Excel Solver* (Jablonsky and Dlouhy, 2010). Both applications can be downloaded from the author's web page. The results of the numerical experiments for both modeling approaches are described below.

Table 1

Characteristics of the original data set

	Life expectancy [years]	Well-being [points]	Ecological footprint [g ha/cap]	GDP [$/cap]
Minimum	47.8	2.81	0.54	347
Lower quartile	63.1	4.38	1.39	2308
Median	73.2	5.18	2.13	8274
Upper quartile	76.5	6.22	4.26	20 545
Maximum	83.4	7.77	11.68	86 124
Mean value	69.83	5.39	3.07	14 582
Standard dev.	9.77	1.17	2.16	16 168
Basal	25	0	12	0
Ideal	85	10	0	60 000

The *Sanna* application implements most of the MCDM methods. For the numerical experiments two simple methods are used. The weights of all three criteria for all methods are supposed to be identical, i.e. 1/3. This corresponds to the practice used in the original definition of the HPI. The applied methods are:

1. WSA, which uses a simple linear utility function for the aggregation of preferences.
2. TOPSIS, which uses a different way of normalization of the original criterion matrix; that is why no prior normalization is necessary. The main idea of this method is a minimization of the distances from both basal and ideal alternatives.

Table 2 presents the results obtained by both approaches. Due to the limited space only rankings of the first and the last three countries (according to the original HPI definition) are presented together with results for the Czech Republic, Slovakia and Poland. The information in Table 2 is completed by the average and maximum differences in rankings obtained by the appropriate method.

Table 2

MCDM re-calculation

HPI	Country	WSA	TOPSIS
1	Costa Rica	1	1
2	Vietnam	8	12
3	Colombia	6	5
:			
71	Poland	63	55
89	Slovakia	78	72
92	Czech Republic	82	101
:			
149	Qatar	151	151
150	Chad	149	140
151	Botswana	150	143
Average difference		10,8	12,1
Maximum difference		34	44

Table 3 shows the same information as the previous table but it contains results obtained by the application of DEA models. Due to the limited space only five different experiments are presented. They are described as follows:

1. DEA model (1) with one dummy input (identical for all countries) and three outputs. Efficient countries are ranked according to their super-efficiency measure (Andersen and Petersen model).
2. DEA model (1) with weight restrictions. The weights can be restricted in different ways – either by absolute lower and/or upper bounds or by their ratios. The results in the second column correspond to the relative restrictions – all pairs of weights can differ by 50% of their values only.
3. SBM model (3). The efficient countries are ranked according to the SBM super-efficiency measures – see (Tone, 2002).
4. The fourth column contains results obtained using the common set of weights (CSW) – see (Despotis, 2005). The weights of three outputs are the results of the following linear optimization model:
   Minimize
   $$z = \sum_{j=1}^{n} d_j / n$$
   subject to (4)

$$\sum_{k=1}^{r} u_k y_{kj} + d_j = \theta_j, \qquad j = 1,2,\ldots,n,$$
$$u_k \geq \varepsilon, d_j \geq 0, \qquad k = 1,2,\ldots,r,\ j = 1,2,\ldots,n,$$

where θ_j is the efficiency score of the DMU$_j$. The model minimizes the sum of deviations from efficiency scores using the weights of the outputs. The optimal weights of the model (4) are: u_1 = 0.465, u_2 = 0.157, u_3 = 0.588. They are applied in a similar way as when the WSA method is used.

5. The data set was extended by the fourth output (GDP per capita) and the impact of this change was analyzed. The conventional DEA model (2) was applied to the extended data set.

Table 3

Re-calculation using DEA models

HPI	Country	CCR DEA	CCR w WR	SBM	CSW	CCR +GDP
1	Costa Rica	1	1	1	1	3
2	Vietnam	8	4	5	2	10
3	Colombia	17	8	17	8	16
:						
71	Poland	108	74	75	74	112
89	Slovakia	125	93	91	95	126
92	Czech Rep.	105	89	100	103	108
:						
149	Qatar	95	149	151	151	58
150	Chad	139	148	144	142	139
151	Botswana	151	151	147	144	141
Average difference		29.6	9.2	12,5	13,7	30,0
Maximum difference		105	33	103	43	136

The results presented in Table 2 and 3 can be explained from several points of view. The main conclusions are:

1. The MCDM methods based on similar principles as the original definition give similar results even though the differences in rankings for some countries are quite large.
2. DEA models without weight restrictions are hardly usable for the given problem. This is because the efficiency score is based on optimal

weights of the evaluated units which can differ significantly (very small values for some criteria and large values for the others).

3. DEA models with weight restrictions give much better results than models without them (in our case even better than WSA and TOPSIS methods).
4. The extension of the model by the fourth output (GDP) does not affect significantly the results.
5. The application of a common set of weights is a compromise between the conventional DEA model and the WSA method. The results are quite close to WSA method.

5 Conclusions

The selection of a compromise alternative or ranking of alternatives in the case of multiple criteria depends not only on the DM's preferences but it is also influenced by the application of a suitable methodology. The final result depends on the DM's preferences and the selection of the method for the analysis. Unfortunately it is very difficult or even impossible to determine the most appropriate method for a given problem. That is why it can be interesting to apply various evaluation methods and compare their results. One of the aims of the present paper was to compare the original definition of the HPI with two MCDM methods and several DEA models.

The results presented in the previous sections show that the ranking of a large number of alternatives according to few criteria depends not only on the weights of the criteria but also on many other factors. The simple CCR DEA model with one dummy input and all the remaining criteria as outputs does not give acceptable results in comparison to the standard HPI definition. The differences in rankings are very high, which results from the nature of the DEA models that optimize weights of the outputs to maximize the efficiency of the evaluated unit. This can lead to unacceptably high differences in weights of the output pairs. Much more results are obtained when the model with weight restrictions is applied. Then the final ranking gets closer to the original HPI very significantly. These experiments show that the DEA models can be used to define the final ranking of countries (or other alternatives) according to given criteria.

Future research in this field is open. There are many country indices related to various areas of human activity. The data used for their calculation are often given with a certain level of uncertainty; to work with them, a methodology for dealing with imprecise data is needed. Other directions of research can involve real discretionary and/or non-discretionary inputs instead of one dummy input.

Acknowledgements

The research is supported by the Czech Science Foundation – project No. P403/12/1387, and by IGA project No. F4/11/2013.

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MULTIPLE CRITERIA DECISION MAKING
Vol. 8 2013

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PREFERENCE-DRIVEN MULTIOBJECTIVE OPTIMIZATION USING ROBUST ORDINAL REGRESSION FOR CONE CONTRACTION

Abstract

We present a new interactive procedure for multiobjective optimization problems (MOO), which involves robust ordinal regression in contraction of the preference cone in the objective space. The most preferred solution is achieved by means of a systematic dialogue with the decision maker (DM) during which (s)he specifies pairwise comparisons of some non-dominated solutions from a current sample. The origin of the cone is located at a reference point chosen by the DM. It is formed by all directions of isoquants of the achievement scalarizing functions compatible with the pairwise comparisons of non-dominated solutions provided by the DM. The compatibility is assured by robust ordinal regression, i.e. the DM's statements concerning strict or weak preference relations for pairs of compared solutions are represented by all compatible sets of weights of the achievement scalarizing function. In successive iterations, when new pairwise comparisons of solutions are provided, the cone is contracted and gradually focused on a sub-region of the Pareto optimal set of greatest interest. The DM is allowed to change the reference point and the set of pairwise comparisons at any stage of the

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method. Such preference information does not need much cognitive effort on the part of the DM. The phases of preference elicitation and cone contraction alternate until the DM finds at least one satisfactory solution, or there is no such solution for the current problem setting.

Keywords: Multiobjective optimization, robust ordinal regression, interactive procedure, preference elicitation, cone contraction

1 Introduction

In multiobjective optimization (MOO), several objectives compete for the best compromise. Identification of a small subset of non-dominated solutions (sometimes reduced to a singleton) that, according to the preferences of the decision maker (DM) yield the best compromise among the conflicting objectives, is the main task of interactive multiple objective optimization (IMO). IMO procedures are composed of two alternating stages: optimization and decision making (see, e.g. Vanderpooten and Vincke, 1997). The stage of decision making, or, more precisely, preference elicitation, consists in the exchange of information between the method and the DM. The method provides the DM with a sample of candidate solutions and the DM returns some critiques of these solutions, which permits to generate in the next optimization stage a new sample that better fits the DM's preferences. One of the major advantages of the IMO is that it aids the DM in improving her/his knowledge about the problem statement, its potential solutions, possible tradeoffs and existing limitations.

A review of interactive procedures shows that reference point methods (RPMs) are gaining importance. In the recent years, one has been able to observe a growing interest in the development of theoretical foundations of the RPMs (see, e.g. Branke et al., 2008; Ogryczak, 2001; Wierzbicki, 1999) as well as a large variety of real-world applications (see, e.g. Granat and Guerriero, 2003)). A reference point is a vector composed of desirable or acceptable values of the objective functions, so-called aspiration levels, represented by a point in the objective space. Given a set of non-dominated solutions, which, in the objective space, are called non-dominated points or the Pareto frontier, the DM is interested in getting a non-dominated point located either as close as possible to the reference point (when the reference point appears infeasible) or as far as possible from the reference point (when the reference point appears feasible). Thus, the reference point is projected onto the set of non-dominated points with the aim of producing solutions which are most preferred to the DM. The result of this projection depends on the weights of the achievement scalarizing function that measures the distance in the objective space between a reference point and non-dominated points. The direction in which the distance is measured depends on the weights assigned to the objective functions. As some projection directions may lead to more desirable non-dominated points to the DM than others, the most straightforward way for browsing interesting regions of the Pareto frontier consists in incorporation of preference information into weights of the achievement function. As far as interaction with the DM is concerned, the recently proposed RPMs present to the DM a sample of non-dominated points at each decision making stage, and expect her/him to state some crucial evaluation of the proposed points, e.g., multiple objective comparisons of some pairs of non-dominated points. Assessment of a preference model reflecting such holistic preferences necessitates looking for the rational basis through which the desired pairwise comparisons were made.

A method that would combine the aforementioned features, i.e. interactive elicitation of preferences consisting of co-ordinates of a reference point, pairwise comparisons of some non-dominated points from a current sample, and incorporation of the DM's preferences into the weights in the achievement scalarizing function, would have many desirable properties of MOO techniques. This motivation has driven our work on a new interactive method designed for the exploitation of the Pareto frontier (PF) in view of searching for the best compromise non-dominated point (Pareto-optimal solution in the decision space). The first version of our method has appeared recently (see Kadziński and Słowiński, 2012). In this method, the identification of the most preferred solution is achieved by means of a systematic dialogue with the DM during which (s)he specifies pairwise comparisons of some non-dominated points from a current sample. Within the method, statements concerning strict or weak preference relations for pairs of points are represented by a compatible form of the achievement scalarizing function (ASF). The preferences are translated into inequalities between distances of compared points from the current reference point. Subsequently, a corresponding set of constraints on the weights of objectives in the ASF is formulated, which ensures that points compared by the DM are compared by the function in the same way. The directions of the isoquants of all compatible ASFs create a cone in the objective space. The origin of the cone is located at the current reference point specified by the DM. Consequently, the preference model used in the method is a set of ASFs compatible with the currently available preference information, rather than only a single compatible ASF. Since we are considering all ASFs compatible with the pairwise comparisons provided, and not just a single ASF as in the traditional methods, our approach can be seen as an inherent part of the robust ordinal regression paradigm (see, e.g. Greco et al., 2008, 2011). In successive iterations, when still new pairwise comparisons of non-dominated points are provided, the cone is contracted and gradually focused on a subregion of the Pareto frontier of greatest interest. The DM is allowed to change the reference point at any decision making stage of the method. The phases of preference elicitation and cone contraction alternate until the DM is satisfied by the compromise yielded by the values of objective functions of at least one non-dominated solution, or until the DM states that there is no such compromise solution for the current problem setting, or until some other stopping criterion is satisfied. The idea of "cone contraction" comes from IMO procedures originally proposed by Steuer (1978), Steuer and Choo (1983), Jaszkiewicz and Słowiński (1992), and Kaliszewski (1994), however, in our method, the preference information provided by the DM, and the way of translating it into constraints contracting the cone, are very different from the previous methods – the preference information has the form of holistic pairwise comparisons of some non-dominated points, and the cone contraction proceeds via robust ordinal regression.

This paper adapts the original proposal of Kadziński and Słowiński (2012) to the conference presentation, omitting many technical details and focusing on the methodological aspect of the procedure. The paper is organized as follows. In Section 2, we introduce notation and concepts used in the paper, including a formal statement of the problem, definition of the non-dominated solutions and points, and characteristics of the ASF. In Section 3, we describe the IMO procedure based on cone contraction via robust ordinal regression. In Section 4, we illustrate this procedure using an exemplary three-objective optimization problem. The final section contains conclusions.

2 Concepts: Definitions and Notation

The *general multiple-objective programming problem* is formulated as:

$$Minimize\ \{f_1(x), f_2(x), \ldots, f_k(x)\},\ \text{subject to } x \in S,$$

where $x = [x_1, \ldots, x_n]$ is a vector of decision variables from the nonempty feasible region $S \subset \mathbb{R}^n$, and $f_1, \ldots, f_k$, with $k \geq 2$ are conflicting objective functions $f_i : \mathbb{R}^n \to \mathbb{R}$, that we want to minimize simultaneously. We assume, without loss of generality, that all objective functions are characterized by decreasing directions of preference, i.e., less is preferred to more. Let us denote the set of indices of the considered objectives by $I = \{1, 2, \ldots, k\}$. This problem can also be formulated as:

$$Minimize: \; z, \text{ subject to } z \in Z,$$

where $z = [z_1 = f_1(x), \ldots, z_k = f_k(x)]$ is a vector of objective function values, and Z is an image of the set S in the objective space $\mathbb{R}^k$, $Z = f(S)$, $f : \mathbb{R}^n \to \mathbb{R}^k$.

To avoid switching between x and z when speaking about solutions of a MOO problem, we will use x to design a solution (non-dominated, dominated, feasible, etc.) understood either as a vector in the decision space, or as its image vector (point) in the objective space. The context in which x is used makes it clear whether we mean a solution in the decision space or a point in the objective space; e.g., when speaking about a distance between a non-dominated solution x and the reference point $\bar{z}$, we mean a distance in the objective space, or when speaking about preferential comparison of non-dominated solutions x^1 and x^2, we mean comparison of their images in the objective space, as ASFs and the preference cone are considered in this space.

Non-dominated solutions

In multiple objective optimization no unique optimal solution usually exists, but a set of options with different trade-offs, i.e. such that none of their components can be improved without deterioration of some other components. Formally, a decision vector $x \in S$ is called *non-dominated* (Pareto-optimal, efficient) if and only if there is no other $y \in S$ such that y is at least as good as x with respect to all objectives, and strictly better for at least one objective, i.e. $f_i(y) \leq f_i(x)$, for all $i \in I$, and there exists $j \in I$, for which $f_j(y) < f_j(x)$. The set of all non-dominated solutions is called the *non-dominated set* and denoted by $P(S)$. In the objective space, $P(S)$ is also called *Pareto frontier*.

Reference point

To measure the quality of non-dominated points, the DM may define some desired objective function values, which constitute a *reference point* denoted by $\bar{z} = \{\bar{z}_1, \ldots, \bar{z}_k\}$. Most often, reference points correspond to objective values that the DM would like to achieve (*aspiration levels*), or that should at least be achieved, according to the DM (*reservation levels*). The reference point may be feasible or not.

Achievement scalarizing function

Achievement scalarizing function is used to project a reference point onto the set of non-dominated solutions. ASF is often defined as (see Wierzbicki, 1982):

$$s(x, \lambda, f) = max_i\{\lambda_i(f_i(x) - \bar{z}_i)\} + \rho \sum_{i=1}^{k} (f_i(x) - \bar{z}_i), \tag{1}$$

where $\lambda = [\lambda_1, \ldots, \lambda_k]$ is a weighting vector, $\lambda_i > 0,\ i = 1, \ldots, k$, and $\rho > 0$ is an augmentation multiplier (sufficiently small positive number). By giving a slight slope to the contours of the scalarizing function, one avoids weakly non-dominated solutions. Without this slope, the contours (isoquants of the scalarizing function) have the shape of orthogonal cones (see Figure 1). Note that if the scales of objectives differ substantially, to avoid problems with significantly different weights $\lambda_i,\ i = 1, \ldots, k$, one should use ASF defined as (see Wierzbicki, 1986):

$$s(x, \lambda, f) = max_i\{\lambda_i(f_i(x) - \bar{z}_i)\} + \rho \sum_{i=1}^{k} \lambda_i(f_i(x) - \bar{z}_i). \tag{2}$$

Note that in RPMs, an ASF is switching from minimization to maximization of the distance between non-dominated solutions and the reference point when the reference point changes from an infeasible one to a feasible one. Thus, e.g., for a infeasible reference point, the smaller the value of the ASF for a given weighting vector, the smaller the distance between a feasible solution and the reference point, i.e. the more this solution is preferred to the DM.

3 Interactive Robust Cone Contraction Method

In this section, we present the IMO procedure based on cone contraction via robust ordinal regression. It is designed for preference-driven exploration of the non-dominated set $P(S)$ of the MOO problem. Thus, we assume that this set, its proper representation or approximation, is generated prior to the right procedure, using some non-interactive parametric or evolutionary (EMO) technique.

In the course of the interactive procedure, the DM specifies pairwise comparisons of some non-dominated solutions from a current sample. More precisely, in the *q-th* iteration the preference information concerns the direction of a strict $\succ$ or weak $\succsim$ preference relation between two solutions x^1

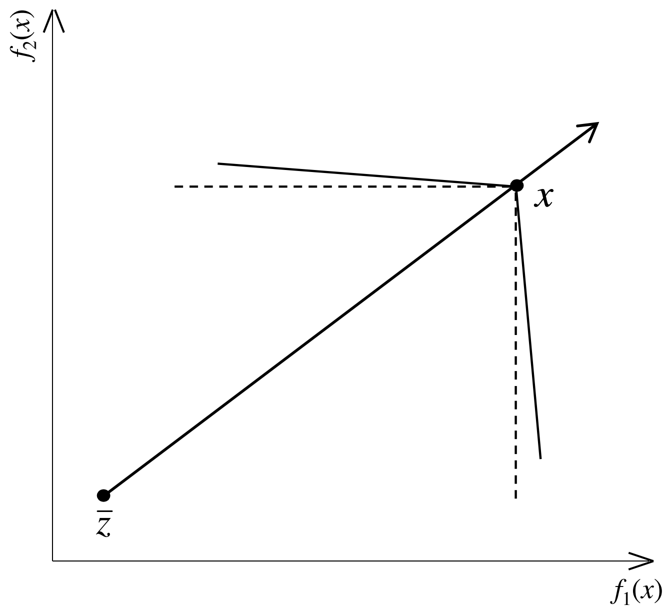

Figure 1. Direction of the isoquant of an achievement scalarizing function

and x^2 chosen from the subset $P(S)_q \subseteq P(S)$ delimited in the previous step $(P(S)_q \subset P(S)_{q-1})$. In this way, the DM specifies some examples of holistic judgments, which requires a relatively small cognitive effort from her/him.

Within the method, the preference information provided is represented by a compatible form of the ASF. The incorporation of the DM's preferences into weights in the achievement scalarizing function is achieved by the formulation of the suitable inequalities. The directions of the isoquants of all compatible ASFs create a convex polyhedral cone in the objective space, with the origin at the current reference point. When new pairwise comparisons are performed in the subsequent iterations, the cone is contracted, and, consequently, the region of the non-dominated solutions which are supposed to better fit the DM's preferences is constrained. The desired effect is to reduce the set of compatible ASFs with each new piece of preference information, and in this way to focus on a subregion of the non-dominated set that better corresponds to the DM's preferences. The phases of preference elicitation and contraction of the cone alternate until the DM has found the most preferred solution, or until (s)he concludes that there is no satisfactory solution for the current problem setting.

The steps of the proposed interactive robust cone contraction method are summarized below:

S1. Compute a representation of the non-dominated set $P(S)_0$ of the considered multiple objective optimization problem.

S2. Ask the DM to specify the starting reference point $\bar{z}_0$.

S3. Update the index of the current step $q := q + 1$ (at the beginning $q = 0$).

S4. Present the set $P(S)_q$ to the DM.

S5. **If** the DM feels satisfied with at least one solution found in the set $P(S)_q$, **then** the procedure stops. **If** (s)he concludes that no compromise point exists, or some other stopping criteria are satisfied, **then** the procedure stops without finding the satisfactory solution. **Otherwise**, continue.

S6. **If** the DM wants to backtrack to one of the previous iterations and continue from this point, **then** go to **S4** of the chosen iteration.

S7. **If** the DM wants to change the reference point, **then** ask her/him to provide a new one, $\bar{z}_q$. **Otherwise** $\bar{z}_q = \bar{z}_{q-1}$

S8. Ask the DM to provide preference information in the form of pairwise comparisons of two solutions chosen from $P(S)_q$ (let us assume that in each iteration x^1 will represent a solution preferred to x^2, i.e., $x^1 \succ x^2$ or $x^1 \succsim x^2$).

S9. Formulate constraints on the weights of the compatible ASFs, which compare the solutions x^1 and x^2 in the same way as the DM.

S10. Form a set $P(S)_{q+1}$ by leaving only those solutions from $P(S)_q$ that are inside the area delimited by the cone formed by all the directions of isoquants of the compatible achievement scalarizing functions.

S11. **If** $P(S)_{q+1}$ is empty, or $x^1 \notin P(S)_{q+1}$, or $x^2 \in P(S)_{q+1}$ (in case $x^1 \succ x^2$), **then** inform the DM about inconsistency and go back to **S4**.

S12. Go to **S3**.

Three points of the procedure need to be commented in more detail. The first point concerns some restrictions on the location of the reference point (for discussion of S7, see Subsection 3.1). The second point concerns the way we obtain the weights of the compatible ASFs (for a discussion of

S9, see Subsection 3.2). The third one deals with checking which solutions should still be considered as potential “best choices” in the next iteration (for discussion of S10, see Subsection 3.3).

3.1 Location of a Reference Point

In each iteration, the DM may specify the reference point which constitutes the origin of the cone indicating the non-dominated solutions that correspond to the DM’s preferences. However, its location is subject to some restrictions. In particular, at the initial stages of the interaction, when the DM’s knowledge about the shape of the Pareto frontier is rather poor, the reference point should be at least as good as the utopia point. This guarantees that all solutions are included within the considered cone, and thus, each of them can become the best compromise. This is reasonable because all non-dominated solutions are incomparable when no preference information is provided.

In the subsequent stages, when the DM’s knowledge about the existing solutions improves, the DM may move the reference point. In this way, (s)he could indicate a more promising subregion and eliminate from further consideration the non-dominated solutions situated outside the new cone. Hence, the desired aspiration or reference objective levels which form the reference point should be selected so that a subregion of non-dominated set covered by the new cone is non-empty. In fact, when considering a finite set of non-dominated solutions representing the Pareto frontier, a rational DM needs to indicate the reference point which is not worse than some non-dominated solutions. Thus, the specified levels should correspond to the best objective values in the promising subregion.

3.2 Inferring Achievement Scalarizing Functions Compatible with Preference Information

Consider the pairwise comparison of solutions $x^1 \succsim x^2$. In this section, we will show how to represent this comparison by constraints on the weights of the compatible ASFs. These constraints contract the cone, which represents the currently available preference information. In this way, we are able to indicate a subset of non-dominated solutions which satisfy the preferences expressed by the DM.

Pairwise comparison $x^1 \succsim x^2$ implicates that the distance from the reference point $\bar{z}$ to the solution x^1 is not greater than the distance from $\bar{z}$ to the solution x^2, i.e., $s(x^1, \lambda, f) \leq s(x^2, \lambda, f)$. Considering ASF in form (2), this inequality leads to the following alternative of k systems of linear inequalities:

$$[\lambda_1(f_1(x^1)-\bar{z}_1)+\rho\sum_{i=1}^{k}\lambda_i(f_i(x^1)-f_i(x^2))\leq\lambda_1(f_1(x^2)-\bar{z}_1)\vee$$

$$\vee\ \lambda_2(f_2(x^1)-\bar{z}_2)+\rho\sum_{i=1}^{k}\lambda_i(f_i(x^1)-f_i(x^2))\leq\lambda_1(f_1(x^2)-\bar{z}_1)\ \vee\ \ldots$$

$$\ldots\ \vee\ \lambda_k(f_k(x^1)-\bar{z}_k)+\rho\sum_{i=1}^{k}\lambda_i(f_i(x^1)-f_i(x^2))\leq\lambda_1(f_1(x^2)-\bar{z}_1)]\ \wedge$$

$$\wedge\ [\lambda_j(f_j(x^1)-\bar{z}_j)+\rho\sum_{i=1}^{k}\lambda_i(f_i(x^1)-f_i(x^2))\leq\lambda_2(f_2(x^2)-\bar{z}_2),$$
$$\text{for some } j=1,\ldots,k]\ \wedge\ \ldots$$
$$\ldots\ \wedge[\lambda_j(f_j(x^1)-\bar{z}_j)+\rho\sum_{i=1}^{k}\lambda_i(f_i(x^1)-f_i(x^2))\leq\lambda_k(f_k(x^2)-\bar{z}_k),$$
$$\text{for some } j=1,\ldots,k].$$

Knowing $f_i(x^1), f_i(x^2)$, $\bar{z}_i$, $i=1,\ldots,k$, and ρ, we obtain the set of constraints on the weights that contract the cone. Note that weights which satisfy the above set of constraints need to be nonnegative, i.e. $\lambda_i\geq 0$, $i=1,\ldots,k$. For the strict preference ($x^1\succ x^2$), we replace weak inequalities with strict inequalities. Since all weights λ_i, $i=1,\ldots,k$, are used in each inequality, it is impossible, in general, to reduce the system above by indicating that some inequalities hold for all possible vectors of weights or none of them. Such an analysis is possible for the ASF having the form (1). In this case, the considered alternative of k systems of linear inequalities has the following form:

$$\lambda_j(f_j(x^1)-\bar{z}_j)+\rho\sum_{i=1}^{k}(f_i(x^1)-f_i(x^2))\leq\lambda_p(f_p(x^2)-\bar{z}_p),$$
$$\text{for some } j=1,\ldots,k,$$

for all $p=1,\ldots,k$ and $\lambda_i\geq 0,\ i=1,\ldots,k$. Thus, unlike the case of an ASF in the form (2), here each inequality involves only one pair of weights since the augmentation factor ($\rho\sum_{i=1}^{k}(f_i(x^1)-f_i(x^2))$) is constant.

3.3 Passing Solutions to the Next Iteration

There are two equivalent ways of checking whether a solution x from $P(S)_q$ should be left in $P(S)_{q+1}$ and still considered to be the potential best compromise. One of them consists in checking whether the weights of the ASF corresponding to the direction determined by x satisfy the set of conditions defined in Subsection 3.2. These conditions delimit the cone so that ASFs with the isoquants going in the directions of the solutions which are inside the cone compare reference solutions in the same way as the DM does. The other way consists in a direct verification that an ASF with the set of weights λ^x compares solutions in the same way as the DM does. Thus, it is sufficient to check whether $s(x^1, \lambda^x, f) < s(x^2, \lambda^x, f)$, if the DM stated that $x^1 \succ x^2$, or $s(x^1, \lambda^x, f) \leq s(x^2, \lambda^x, f)$, if (s)he claimed $x^1 \succsim x^2$. If it is the case, $x \in P(S)_q$ is left in $P(S)_{q+1}$. Otherwise, x is excluded from the set of solutions which are still considered to be the potential best compromise.

4 Illustrative Example

In this section, we illustrate the way our method supports the DM in solving a MOO problem, and we give examples of possible interactions. We consider a MOO problem that involves three objectives to be minimized. The non-dominated solutions satisfy the following condition $f_1(x) + f_2(x) + f_3(x) = 0.5$ (like in Three-Objective Test Problem DTLZ1 (Zitzler et al., 2000)). We consider the subset $P(S)_0$ composed of 66 non-dominated solutions (see Table 1 and Figure 2). The initial reference point is situated at the point $[0.0, 0.0, 0.0]$. Since the scales of the objectives are the same, we will use the ASF in the form (1).

Obviously, solutions in $P(S)_0$ are incomparable, unless preference information is expressed by the DM. In this perspective, (s)he provides a first comparison: $x^{33} = [0.15, 0.10, 0.25] \succ x^{55} = [0.30, 0.15, 0.05]$. Note that x^{33} is evaluated better than x^{55} on objectives f_1 and f_2, whereas it is worse on the third objective f_3. Therefore, the cone formed by the directions of isoquants of all ASFs compatible with the statement $x^{33} \succ x^{55}$, is a sum of the following two cones. The first is formed by the directions of ASFs which ensure that solutions included in this cone would be evaluated better on objective f_1 to recompense for weakness on objective f_3, whereas the other cone is formed by the directions of ASFs which guarantee that the advantage of evaluation on f_2 would allow to recompense for a relatively worse evaluation on f_3. To be precise, the constraints on the weights of

Table 1

The representative set of non-dominated solutions $P(S)_0$

	$f_1(x)$	$f_2(x)$	$f_3(x)$		$f_1(x)$	$f_2(x)$	$f_3(x)$
x^1	0.00	0.00	0.50	x^{34}	0.15	0.15	0.20
x^2	0.00	0.05	0.45	x^{35}	0.15	0.20	0.15
x^3	0.00	0.10	0.40	x^{36}	0.15	0.25	0.10
x^4	0.00	0.15	0.35	x^{37}	0.15	0.30	0.05
x^5	0.00	0.20	0.30	x^{38}	0.15	0.35	0.00
x^6	0.00	0.25	0.25	x^{39}	0.20	0.00	0.30
x^7	0.00	0.30	0.20	x^{40}	0.20	0.05	0.25
x^8	0.00	0.35	0.15	x^{41}	0.20	0.10	0.20
x^9	0.00	0.40	0.10	x^{42}	0.20	0.15	0.15
x^{10}	0.00	0.45	0.05	x^{43}	0.20	0.20	0.10
x^{11}	0.00	0.50	0.00	x^{44}	0.20	0.25	0.05
x^{12}	0.05	0.00	0.45	x^{45}	0.20	0.30	0.00
x^{13}	0.05	0.05	0.40	x^{46}	0.25	0.00	0.25
x^{14}	0.05	0.10	0.35	x^{47}	0.25	0.05	0.20
x^{15}	0.05	0.15	0.30	x^{48}	0.25	0.10	0.15
x^{16}	0.05	0.20	0.25	x^{49}	0.25	0.15	0.10
x^{17}	0.05	0.25	0.20	x^{50}	0.25	0.20	0.05
x^{18}	0.05	0.30	0.15	x^{51}	0.25	0.25	0.00
x^{19}	0.05	0.35	0.10	x^{52}	0.30	0.00	0.20
x^{20}	0.05	0.40	0.05	x^{53}	0.30	0.05	0.15
x^{21}	0.05	0.45	0.00	x^{54}	0.30	0.10	0.10
x^{22}	0.10	0.00	0.40	x^{55}	0.30	0.15	0.05
x^{23}	0.10	0.05	0.35	x^{56}	0.30	0.20	0.00
x^{24}	0.10	0.10	0.30	x^{57}	0.35	0.00	0.15
x^{25}	0.10	0.15	0.25	x^{58}	0.35	0.05	0.10
x^{26}	0.10	0.20	0.20	x^{59}	0.35	0.10	0.05
x^{27}	0.10	0.25	0.15	x^{60}	0.35	0.15	0.00
x^{28}	0.10	0.30	0.10	x^{61}	0.40	0.00	0.10
x^{29}	0.10	0.35	0.05	x^{62}	0.40	0.05	0.05
x^{30}	0.10	0.40	0.00	x^{63}	0.40	0.10	0.00
x^{31}	0.15	0.00	0.35	x^{64}	0.45	0.00	0.05
x^{32}	0.15	0.05	0.30	x^{65}	0.45	0.05	0.00
x^{33}	0.15	0.10	0.25	x^{66}	0.50	0.00	0.00

compatible ASFs in the first iteration are the following:

$$\{[\lambda_1 > 5/6 \cdot \lambda_3] \ \vee \ [\lambda_2 > 5/3 \cdot \lambda_3]\} \ \wedge \ \{\lambda_i \geq 0, \ i = 1, 2, 3\}.$$

The transition from the formulated inequalities to the cone formed by the directions of all compatible ASFs in the three-dimensional objective space is presented in Figure 3. The set of solutions which are inside the cone is:

$$P(S)_1 = \{x^1, \ldots, x^{20}, x^{22}, \ldots, x^{28}, x^{31}, \ldots, x^{35}, x^{39}, x^{40}, x^{41},$$

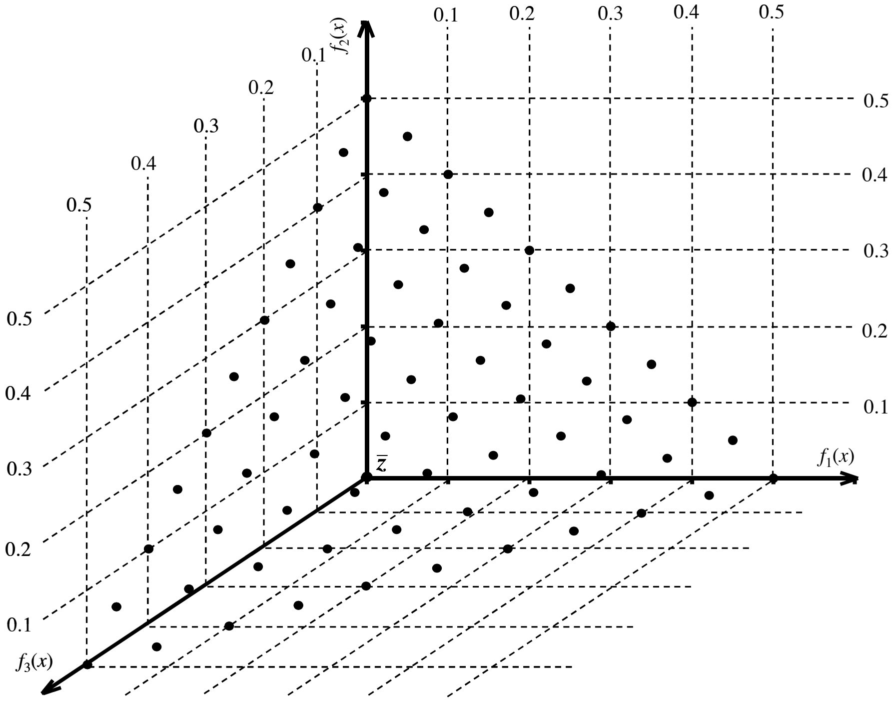

Figure 2. The set of non-dominated solutions $P(S)_0$ before the first iteration

$$x^{46}, x^{47}, x^{52}, x^{53}, x^{57}, x^{58}, x^{61}, x^{64}\}.$$

Thus, in the second iteration, the DM needs to consider 43 solutions out of the initial 66 ones.

To make the remaining solutions more comparable, the DM states that $x^{41} = [0.20, 0.10, 0.20] \succ x^{13} = [0.05, 0.05, 0.40]$. Note that x^{41} is better than x^{13} only on the third objective, while being worse on the other two. Consequently, the constraints on the weights of the ASFs compatible with the pairwise comparison provided in the second iteration are the following:

$$\{[\lambda_3 > 1/2 \cdot \lambda_1] \ \wedge \ [\lambda_3 > 1/4 \cdot \lambda_2]\} \ \wedge \ \{\lambda_i \geq 0, \ i = 1, 2, 3\}.$$

Taking into account the outcomes of the previous iteration, we could present the cone formed by the directions of compatible ASFs which guarantee that $x^{33} \succ x^{55}$ and $x^{41} \succ x^{13}$ as in Figure 4. The set of non-dominated solutions situated inside the contracted cone consists of 10 solutions:

$$P(S)_2 = \{x^{11}, x^{20}, x^{27}, x^{28}, x^{33}, x^{34}, x^{35}, x^{41}, x^{53}, x^{58}\}.$$

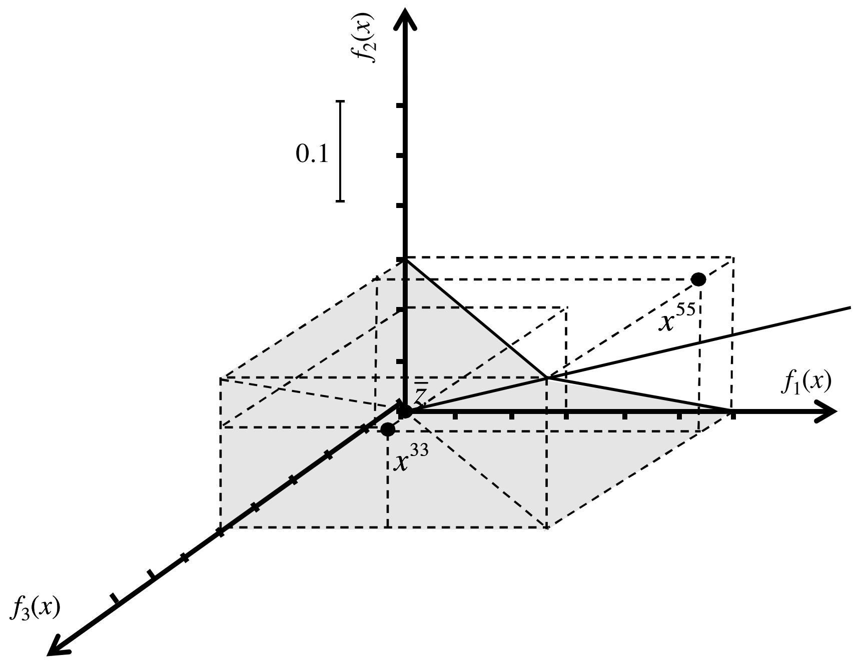

Figure 3. The directions of ASFs compatible with a pairwise comparison provided in the first iteration

Knowing the evaluations of the solutions which are still perceived as the potential best compromise solutions, the DM decides to change the reference point to $\bar{z} = [0.15, 0.10, 0.10]$. Consequently, the set of considered solutions is limited to $\{x^{33}, x^{34}, x^{35}, x^{41}\}$ (see Figure 5). The DM states that $x^{35} = [0.15, 0.20, 0.15] \succ x^{34} = [0.15, 0.15, 0.20]$. In this way, (s)he prefers a solution with a slightly better evaluation on f_3 than a solution with a slightly better evaluation on f_2. Since within the contracted cone there is only one solution (see Figure 5), it is presented to the DM as the one that best satisfies her/his indirectly provided preferences.

5 Conclusions

The major advantage of the presented interactive robust cone contraction method is the organization of the search over the non-dominated set through pairwise comparisons of solutions from the current sample and suitable moving of the reference point by the DM, which may be inspired by the knowledge gained by her/him in the course of the interactive process. The

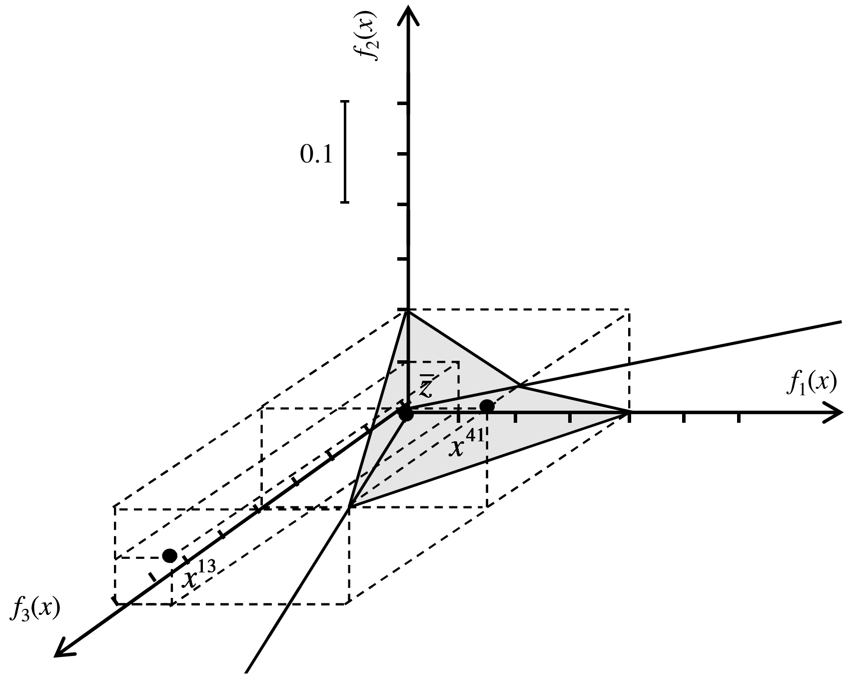

Figure 4. The directions of ASFs compatible with a pairwise comparison provided in the second iteration

motivation for employing the achievement scalarizing function came from its suitability for producing different solutions by weighting the marginal differences between attainable values of objective functions and respective co-ordinates of the current reference point. This permits to get control over the process of solving a MOO problem through an appropriate formulation of constraints on the weights.

Within the presented procedure, the DM is required to provide preferences composed of understandable and not very demanding holistic judgments. According to psychologists, people feel more confident exercising their decisions rather than explaining them directly in terms of values of some preference model parameters. Since the process of selecting a single, most preferred solution is organized by contraction of a cone in the objective space, the DM can easily observe the consequences of one's decisions and learn about the nature of the problem. Moreover, as in every iteration the set of still considered solutions is being delimited and its intuitive representation is presented to the DM, (s)he is able to build a conviction about what is possible in this psychologically convergent process.

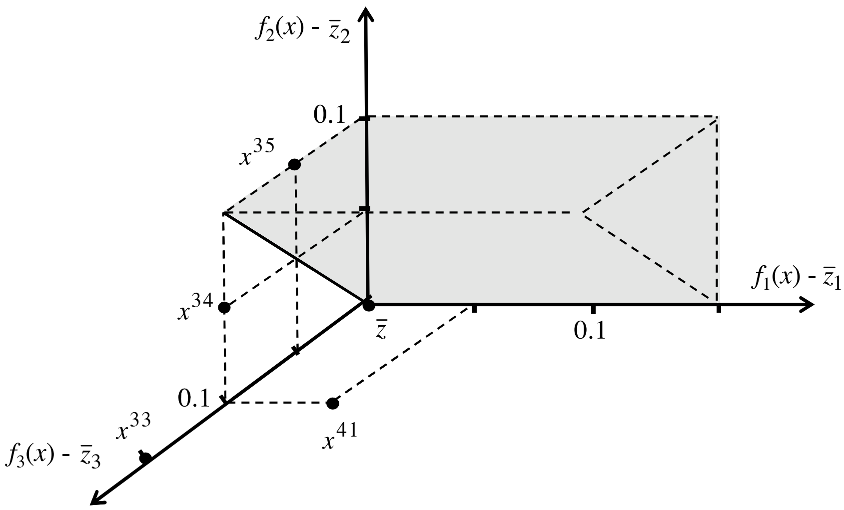

Figure 5. The directions of compatible ASFs after changing the reference point to $\bar{z} = [0.15, 0.10, 0.10]$ and accounting for a pairwise comparison provided in the third iteration

Acknowledgments

The authors wish to acknowledge financial support from the Ministry of Science and Higher Education, grant 91-516/DS-MLODA KADRA (first author) and DS-91-543/2013 (second author).

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DECISION MAKER'S PREFERENCES, AIRPORT GATE ASSIGNMENT PROBLEM AND MULTIOBJECTIVE OPTIMISATION

Abstract

We present an application of a methodology we developed earlier to capture a decision maker's preferences in multiobjective environments to a notorious problem in the realm of Air Traffic Management, namely the Airport Gate Assignment Problem.

The problem has been modelled as an all-integer optimisation problem with two criteria.

We have implemented this methodology into the commercial solver CPLEX and also into an Evolutionary Multiobjective Optimisation algorithm and we have solved with them a numerical instance of the Airport Gate Assignment Problem for a couple of decision making scenarios.

Keywords: preference capture, airport gate assignment, exact optimization computations, evolutionary optimization computations

1 Introduction

Nowadays Multiple Criteria Decision Making (MCDM) problems are most often solved interactively (Miettinen, 1999; Ehrgott, 2005; Kaliszewski, 2006). Interactive decision making processes reflect best the natural dynamics of the Decision Maker (DM) problem recognition, accumulation of knowledge about

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interplay of problem driving factors and DM's ability to reveal preferences about outcomes of different factor patterns.

Along this line, a methodology has been proposed to capture DM's preferences in the course of interactive decision making processes (Kaliszewski, 2004, 2006) which subsumes two classic MCDM methods, namely the *weighting method* and the *reference point method*. This methodology is quite general. It is applicable to any MCDM problem which uses Multiobjective Optimisation (MO) as the underlying formal model. Moreover, the methodology is independent of optimisation methods and solvers used to solve MO. Over time this methodology has been coupled with MCDM methods based on approximate calculations of efficient outcomes (Kaliszewski, 2006), evolutionary computations (Miroforidis, 2008, 2010; Kaliszewski, 2008; Kaliszewski, Miroforidis, 2009, 2012b; Kaliszewski et al., 2012), and classical optimisation calculations (Kaliszewski, Miroforidis, 2012a).

This work is an extension of an earlier research reported in Kaliszewski, Miroforidis (2012a). There we presented a model for a notorious problem of Air Traffic Management, namely the Airport Gate Assignment Problem (AGAP) (Dorndorf et al., 2007). The rationale behind the model was to assist air ground services in a small or medium size airport in assigning flights to gates under conflicting criteria. We showed that under the concept of time windows the problem can be effectively decomposed into a series of smaller problems and we argued that under such a decomposition the deterioration of optimality of solution, if any, is not very significant. To illustrate how the preference capture methodology works it was tested on a small instance of AGAP where efficient assignments were derived by enumeration. This, however, raised a question of scalability of the model to practical sizes. Moreover, the ability of optimisation software, academic and commercial, to cope with such problems in the preference capture methodology environment was still an open question.

To perform optimisation calculations, in this paper we use commercial software, namely the CPLEX package – a leader of many benchmarks. By this we attempt to convey the message that the methodology we developed couples well with commercial optimisation solvers able to handle effectively medium- and large-scale problems. This in turn paves the way for scalability of MCDM problems into the realm of such sizes.

We complement this work by mirroring CPLEX computations by an Evolutionary Multiobjective Optimisation (EMO) algorithm implemented specifically for that task. Our intention was to draw preliminary comparability conclusions on the workload and scalability with respect to those two distinct optimisation paradigms.

The outline of the paper is as follows. In the next section we present an adaptation of a model of AGAP adequate for small airports and presented in Kaliszewski, Miroforidis (2012a). In the subsequent section, for the sake of completeness, we give a concise description of our multiple criteria decision making methodology (for a more detailed treatment cf. Kaliszewski et al., 2012). Lastly, we apply this methodology to a small but illustrative instance of the presented AGAP model with the help of the CPLEX package and our custom-coded EMO solver, and we comment briefly on the suitability of these solvers for medium- and large-scale instances of AGAP. We conclude with some remarks on possible directions of future research.

2 AGAP for a Small Airport

The problem under consideration is to assign incoming flights to airport gates in time horizon Θ. If at a given time there is no gate to serve a flight that flight (the corresponding plane) can be directed to wait for a gate or it can be served at once on the airport apron. Waiting times and the number of flights served on the apron are best if both are equal to zero but in the case of airport overload they are in an obvious conflict.

We assume that the airport under consideration:

1) is small, so gate assignments have no significant impact on passenger walking distance,
2) all gates can serve any flight,
3) there are no constraints on neighbour gate operations,
4) any flight can wait to be served at a gate for time T at most and after that time it is served on the apron.

2.1 The Model

A flight j, $j = 1, \dots, n$,t is characterised by arrival time a_j and ground operation time (for short: ground time) g_j (time needed to serve flight j at a gate). Arrival times, ground times and waiting times are assumed to be discrete with interval δ. Hence, the maximal waiting time is $T = \alpha\delta$ for some $\alpha > 0$ and time horizon is $\Theta = \beta\delta$ for some $\beta > \alpha$.

Let $x_{j,i}^t$ be a binary variable which is equal to 1 if flight j is assigned to gate i at time t and equal to 0 ~~if~~ otherwise. No assignment to a gate can be made before flight j arrives, hence for $t < a_j$ the variables $x_{j,i}^t$ are undefined. Similarly, no assignment to a gate can be made after a flight has waited α time intervals for a gate assignment (after that time this flight is served on the apron), hence for $t > a_j + \alpha$ the variables $x_{j,i}^t$ are undefined.

With m gates there are $m\sum_{j=1}^{n}(\alpha + 1) = mn(\alpha + 1)$ variables $x_{j,i}^t$.

A flight j can be assigned at most once to at most one gate, so

$$\sum_{i=1}^{m}\sum_{t=a_j}^{a_j+\alpha} x_{j,i}^t \leq 1, \text{ for } j = 1, \dots, n. \quad (1)$$

There are n constraints of type (1).

Let $y_{j,i}^{t}$ be a binary variable which is equal to 1 if gate i is serving flight j at time t, and equal to 0 ~~if~~ otherwise. No assignment to a gate can be made before flight j arrives, hence for $t < a_j$ the variables $y_{j,i}^{t}$ are undefined. Similarly, gate i cannot serve flight j after $t > a_j + \alpha + g_j$ since after $t > a_j + \alpha$ flight j is served on the apron. Hence, for $t > a_j + \alpha + g_j$ the variables $y_{j,i}^{t}$ are undefined.

The number of variables $y_{j,i}^{t}$ is equal to $m \sum_{j=1}^{n} (\alpha + g_j + 1)$.

If flight j is assigned to gate $i,\ i = 1, \dots, m$, at time t (i.e. $x_{j,i}^{t} = 1$) then this gate is not available for another flight assignment for $1 + g_j$ consecutive time intervals starting from interval t, i.e. for interval $t,\ t+1, \dots, t + g_j$. This condition is equivalent to:

$$\begin{aligned} & x_{j,i}^{t} \le y_{j,i}^{t} , \\ & x_{j,i}^{t} \le y_{j,i}^{t+1} , \\ & \vdots \\ & x_{j,i}^{t} \le y_{j,i}^{t+g_j} , \end{aligned} \tag{2}$$

where $a_j \le t \le a_j + \alpha$.

There are $m \sum_{j=1}^{n} \alpha (g_j + 1)$ constraints of type (2).

As constraints of that type are the most numerous in the model, we propose a more concise formulation resulting from replacing constraints (2) by their logical equivalent:

$$g_j x_{j,i}^{t} \le y_{j,i}^{t} + y_{j,i}^{t+1} + \cdots + y_{j,i}^{t+g_j}, \tag{2'}$$

where $a_j \le t \le a_j + \alpha$.

The number of constraints of type (2') is equal to the number of variables $x_{j,i}^{t}$, hence it is equal to $m \sum_{j=1}^{n} (\alpha + 1) = mn(\alpha + 1)$.

No more than one flight can be assigned to a gate at a time, so:

$$\sum_{j=1}^{n} x_{j,i}^{t} \le 1,\ t = a_{*}, \dots, \beta,\ i = 1, \dots, m , \tag{3}$$

where * is the index of the flight scheduled as second. There are at most $m(\beta - a_{*} + 1)$ constraints of type (3); the exact number of these constraints depends on the flight arrival time structure.

Gate i at time t can serve at most one flight, so:

$$\sum_{j=1}^{n} y_{j,i}^{t} \le 1,\ t = a_{*}, \dots, \beta ,\ i = 1, \dots, m . \tag{4}$$

There are at most $m(\beta - a_{*} + 1)$ constraints of type (4); the exact number of these constraints depends on the flight arrival time structure.

If flight j is assigned to a gate at its arrival time a_j then there is no waiting time. Otherwise, the waiting time for flight j equals:

$$\delta\left(\sum_{i=1}^{m} x_{j,i}^{a_j+1} + 2 \sum_{i=1}^{m} x_{j,i}^{a_j+2} + \cdots + \alpha \sum_{i=1}^{m} x_{j,i}^{\alpha}\right)$$

and the total waiting time over all flights is:

$$f_1(x) = \delta \sum_{j=1}^{n} (\sum_{i=1}^{m} x_{j,i}^{a_j+1} + 2\sum_{i=1}^{m} x_{j,i}^{a_j+2} + \cdots + \alpha \sum_{i=1}^{m} x_{j,i}^{\alpha}). \quad (5)$$

If flight j is not assigned to a gate then it is assigned to the apron, so:

$$\sum_{i=1}^{m} \sum_{t=a_j}^{t=a_j+\alpha} x_{j,i}^{t} = 1 - y_j, \text{ for } j = 1, \dots, n, \quad (6)$$

where y_j is a binary variable equal to 0 if flight j is assigned to a gate and equal to 1 otherwise.
There are n variables y_j and n constraints of type (6).

With $f_2(y)$,

$$f_2(y) = \sum_{j=1}^{n} y_j , \quad (7)$$

as the objective function to be minimised, at optimality with respect to $f_1(x)$ (i.e. when variables $x_{j,i}^t$, $y_{j,i}^t$ and y_j are optimal with respect to $f_1(x)$ first and then with respect to $f_2(x)$, in that order) or at efficiency (i.e. when variables $x_{j,i}^t$, $y_{j,i}^t$ and y_j are efficient with respect to $f_1(x)$ and $f_2(x)$, for the definition of efficiency see the next section), the number of variables y_j taking value 1 will be minimal but not less than the value dictated by constraints (6).

Objective functions (5) and (7) together with constraints (1), (2'), (3), (4) and (6) constitute a bicriteria model for AGAP at a small airport. Values of objective functions at efficient assignment represent rational compromises between waiting time and the number of apron operations.

The model can accommodate also other objective functions because the multiple criteria decision making methodology we present in the next section can deal with any number of criteria. In Kaliszewski, Miroforidis 2012a, we considered a similar model where instead of the total waiting time the maximal waiting time over all flights was minimised. For that purpose in our earlier paper we used the following form of the first objective function:

$$f_1(x) = \delta\, max_j (\sum_{i=1}^{m} x_{j,i}^{a_j+1} + 2\sum_{i=1}^{m} x_{j,i}^{a_j+2} + \cdots + \alpha \sum_{i=1}^{m} x_{j,i}^{a_j+\alpha}). \quad (5')$$

However, this function works in the context of the AGAP problem correctly only if a solving method (solver) assigns flights to gates as soon as a gate is idle. However, with a general method (solver) this is not always guaranteed and this is the case with the general purpose solver CPLEX. In consequence, when working with objective function (5'), assignments can be derived in which for the minimal value of maximal (over flights) waiting time the minimal individual flight waiting time is not minimal. In other words, assignments can be derived in which, for the minimal value of maximal (over flights) waiting time, flights are not assigned to gates as early as possible (*aeap* assignments). Such assignments cannot be accepted in practice. This situation can be avoided either by adding to the model a significant number of constraints modelling precedence-type relations, which in consequence may hamper the scalability of the model, or by adding to objective function (5') a penalty term in the form of a small fraction of objective function (5) to eliminate non-*aesp* assignments.

In this paper we have decided to apply a symmetric approach, namely to work primarily with objective function (5). This guarantees that all assignments will be *aeap* but raises, in turn, the question of individual flight waiting times. One can expect, however, that the maximal flight waiting time limited to α and a decomposition of the AGAP problem to a series of time windows (cf. the next section) will moderate the maximal waiting times of AGAP assignments produced. Eventually, to ensure that for a given value of the total waiting time the maximal waiting time is minimal (there can be multiple solutions with the same values of the total waiting time), one can add to objective function (5) a penalty term in the form of a small fraction of objective function (5').

2.2 Time windows

The model presented is all-integer and linear and if a penalty term in the form of a small fraction of objective function (5') is added to objective function (5), the model can be again made linear by a transformation of that term (a transformation analogous to that used in the next subsection to linearize in optimisation problem (9) the *maximum* function).

The problems (instances of the model) to be solved are of considerable size even for modest values of n, m and β (cf. the previous subsection). Although we have no influence on the magnitude of n and m, we can decrease the magnitude of β significantly by employing the concept of *time windows*.

Observe that in the model an apron is a buffer which absorbs all flights which cannot wait sufficiently long for an assignment to a gate. In the previous subsection we have set that threshold to $\alpha\delta$. Hence, in the model any flight is assigned to the apron at the latest at its arrival time plus $\alpha\delta$. Suppose that the time horizon $\Theta = \delta\beta$ of AGAP is divided into time windows of equal size such that window width is not greater than $\alpha\delta$ (see Appendix 1 for an example). Then, in the model a flight whose arrival time is in time window q will never compete for a gate assignment with a flight whose arrival time is in time window $q + 2$. Hence, the model for AGAP can be solved separately in each time window for all flights with arrival times in that window. Gate assignments in a given time window which overlap with the next time window can be represented in time window $t + 1$ by fixing the corresponding variables $y_{j,i}^{t}$ to 1 but this requires that AGAP have to be solved sequentially in separate time windows, starting from the first time window.

Solving AGAP in separate time windows does not guarantee optimality with respect to the whole time horizon Θ but makes the entire problem much more manageable from the computational point of view.

3 The Multiobjective Methodology

Let x denote a (decision) *variant* (solution), X a space of variants, X_0 a set of *feasible variants*, $X_0 \subseteq X$. Then the multiobjective optimisation problem is:

$$"\text{vmax}"\ f(x)$$
$$x \in X_0\ , \tag{8}$$

where $f: X \to R^k$, $f = (f_1, \dots f_k)$, $f_i: X \to R$, $i = 1, \dots, k$, $k \geq 2$, are *objective functions* (*criteria*); "vmax" denotes the operator of deriving all *efficient* (as defined below) *variants* in X_0 .

A variant $\bar{x}$ of X_0 is *efficient* if $f_i\,(x) \geq f_i(\bar{x})$, $i = 1, \dots, k,\ x \in X$, implies $f(x) = f(\bar{x})$.

It is a well-established result (cf. Kaliszewski, 2006; Ehrgott, 2005; Miettinen, 1999) that variant $\bar{x}$ is efficient[1] if and only if it solves the optimisation problem:

$$min_{x \in X_0} max_i [\lambda_i \big(y_i^* - f_i(x)\big) + \ \rho e^k (y^* - f(x))]\ , \tag{9}$$

where $\lambda_i > 0,\ i = 1, \dots, k,\ e^k = (1, \dots, 1)$, y^* is such that $y_i^* > \ f_i(x),\ i = 1, \dots, k,\ x \in X_0$, and ρ is a positive "sufficiently small" number[2].

By the "only if" part of this result no efficient variant is a priori excluded from being derived by solving an instance of optimisation problem (9). In contrast to that, the maximisation of a weighted sum of objective functions over X_0 does not have, in general (and especially in the case of problems with discrete variables), this property.

Let $\hat{y}_i = max_{x \in X_0} f_i(x),\ i = 1, \dots, k$. We can restate the above result saying that a variant $\bar{x}$ such that $f_i(\bar{x}) \neq \ \hat{y}_i$ for all $i = 1, \dots, k$, is efficient (cf. footnote 4) if and only if it solves the optimisation problem:

$$min_{x \in X_0} max_i [\lambda_i \big(\hat{y}_i - f_i(x)\big) + \ \rho e^k (\hat{y} - f(x))]\ , \tag{9'}$$

This is an immediate consequence of accounting in the proof of the only if part of Theorem 4.3 in Kaliszewski (1994) for the condition $f_i(\bar{x}) \neq \ \hat{y}_i$ for all $i = 1, \dots, k$. Efficient solutions with $f_i(\bar{x}) = \ \hat{y}_i$ for some $i \in \widetilde{K} \subseteq \{1, \dots, k\}$, can be derived with the optimisation problem:

$$min_{x \in X_0} max_{i \notin \widetilde{K}} [\lambda_i \big(\hat{y}_i - f_i(x)\big) + \ \rho e^k (\hat{y} - f(x))]\ , \tag{9''}$$
$$f_i(x) = \ \hat{y}_i\ ,\ i \ \in \widetilde{K}\ .$$

At the first glance, the objective function in (9') ($max_i [\lambda_i \big(\hat{y}_i - f_i(x)\big) + \rho e^k (\hat{y}_i - f(x)]$) seems difficult to handle. However, observe that optimisation problem (9') is equivalent to:

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[1] Actually, variant $\bar{x}$ is $\rho -$*properly efficient*, for a formal treatment of this issue cf. e.g. Kaliszewski (2006), Ehrgott (2005), Miettinen (1999).

[2] Kaliszewski (2006), Ehrgott (2005), Miettinen (1999).

$$
\begin{aligned}
& min\ s\ , \\
& s \geq \lambda_i\big(\hat{y}_i - f_i(x)\big) + \rho e^k\big(\hat{y}_i - f(x)\big),\ i = 1, \dots, k\ , \\
& x \in X_0\ .
\end{aligned}
$$

An analogous observation applies to optimisation problem (9'').

Besides the potential ability to derive each efficient variant, optimisation problem (9') provides an easy and intuitive capture of decision maker's preferences. Observe that an element $\hat{y}$ (recall that $\hat{y}_i = max_{x \in X_0} f_i(x),\ i = 1, \dots, k$) represents maximal values of objective functions which can be attained if they are maximised separately.

To assist the decision maker in the search for *the most preferred variant* one can employ the optimisation problem (9') or (9''). By this we assume a modicum of rationality on the part of the decision maker, namely we assume that the decision maker prefers an efficient variant to any variant dominated by it (variant $\bar{x}$ dominates variant x if $f_i(\bar{x}) \geq f_i(x),\ i = 1, \dots, k,$ and $f_i(\bar{x}) > f_i(x)$ for at least one i.

Suppose that an element $x \in X_0$ such that $\hat{y} = f(x)$ does not exist which is rather a standard situation of conflicting criteria (otherwise, x is clearly the most preferred variant). Then, the decision maker knows that to derive an efficient variant he (or she) has to compromise on values of objective functions f_i with respect to values $\hat{y}_i$, $i = 1, \dots, k$. He can define his acceptable compromises on values $\hat{y}_i$, $i = 1, \dots, k$, and by this direct of search for an efficient variant which corresponds to these compromises in three ways:

1) providing a *vector of concessions* τ ,
2) providing a reference point y^{ref},
3) providing weights λ_i , $i = 1, \dots, k$.

Way 1. The components of a vector of concessions τ specify concessions the decision maker is willing to make with respect to $\hat{y}_i$, $i = 1, \dots, k$. The components of the vector τ can be defined in absolute values ("the decision maker is willing to make a concession of τ_i units on the value $\hat{y}_i$, $i = 1, \dots, k$") or in relative values ("the decision maker is willing to make a concession of τ_i per cent on the value $\hat{y}_i$, $i = 1, \dots, k$").

Way 2. A reference point y^{ref} ($y^{ref} \in R^k$, $y_i^{ref} \leq \hat{y}_i$, $i = 1, \dots, k$), (it is irrelevant whether there exists an element $x \in X_0$ such that $y^{ref} = f(x)$ or not) specifies explicitly a compromise between values of objective functions f_i with respect to values $\hat{y}_i$, $i = 1, \dots, k$, which the decision maker regards as agreeable (Wierzbicki, 1999). A reference point specifies indirectly a vector of concessions:

$$\tau_i = \hat{y}_i - y_i^{ref},\ i = 1, \dots, k\ . \tag{10}$$

Way 3. An experienced decision maker can define a vector of concessions τ in terms of weights $\lambda_i > 0,\ i = 1, \dots, k$, in optimisation problem (9') or (9").

For $\tau_i > 0,\ i = 1, \dots, k,$ the vector of concessions τ and the vector of weights λ are related by the formula (Kaliszewski, 2006):

$$\lambda_i = (\tau_i)^{-1},\ i = 1, \dots, k, \tag{11}$$

If $\tau_i = 0$ for some i then this means that the decision maker is not willing to make any concessions on $\hat{y}_i$ and he (she) is interested only in efficient solutions for which $f_i(x) = \hat{y}_i$ for that i. Then optimisation problem (9') should be replaced by optimisation problem (9") with $\widetilde{K}$ composed of all indices $i = \{1, \dots, k\}$ such that $\tau_i = 0$. Hence, from now on we shall assume without loss of generality that $\tau_i > 0,\ i = 1, \dots, k$.

The optimisation problem (9'), if solved with λ_i, $i = 1, \dots, k$, given by formula (11), has the following property:

- it finds an efficient variant x such that $f(x)$ is on the half line $y = \hat{y} - t\tau,\ t \geq 0$, whenever such a variant exists,
- otherwise, it finds an efficient variant x such that $max_i \lambda_i(\hat{y}_i - f_i(x)) + \rho e^k(\hat{y} - f(x)) = max_i \lambda_i(\hat{y}_i - \tilde{y}_i) + \rho e^k(\hat{y} - \tilde{y})$, where $\tilde{y}$ is on the half line $y = \hat{y} - t\tau,\ t \geq 0$.

4 Solving an AGAP Instance

Consider the following instance of AGAP. In the time horizon of 2 hours there are 5 flights scheduled as in Table 1. These flights can be served at two gates or on apron. The discretisation interval is $\delta = 5$ minutes. All ground times are equal to 50 minutes. The upper bound on waiting time α is 6 discretisation intervals, i.e. 30 minutes.

Table 1

TIME WINDOW *i*

FLIGHT	ARRIVAL TIME
1	0:05
2	0:15
3	0:30
4	0:40
5	0:45

We illustrate on this example the MCDM methodology for decision maker's preference capture presented in the previous section. The methodology has been outlined for problems where all objective functions are maximised whereas in our example objective functions, waiting time and the number of apron operations, are to be minimised. Thus, to have in our example both objective functions in the "max" form, we simply change signs of values of objective functions. For example, if we maximise in the *adapted* problem "minus" the number of apron operations then -1 apron operation is better than -3 apron operations, which is a purely technical convention.

4.1 The AGAP Instance

The instance of our AGAP model to be solved has the following structure.

There are 70 binary variables $x_{j,i}^t$, 170 binary variables $y_{j,i}^t$ and 5 binary variables y_j , 245 binary variables in total.

There are 70 constraints of type (2'), 44 constraints of type (3), 23 constraints of type (4) and 5 constraints of type (6), 142 constraints in total.

In addition to structural variables $x_{j,i}^t$, $y_{j,i}^t$, y_j , to simplify manipulations with objective functions (5) and (7), we add (in this case nonnegative) variables f_1, f_2:

$$f_1 = f_1(x) \text{ and } f_2 = f_2(x).$$

This increases the size of the model to 247 variables, 245 binary and 2 continuous and nonnegative, and 144 constraints among which 137 are of "less than or equal to" type and 7 are of "equal to" type.

For optimisation problem (9) one additional (in this case nonnegative) variable and two additional "greater than or equal to" type constraints are needed. ~~I~~In this case the model has 248 variables, 245 binary and 3 continuous and nonnegative, and 146 constraints among which 137 are of "less than or equal to" type, 2 are of "greater than or equal to" type and 7 are of "equal to" type.

In CPLEX nonnegativity of variables is assumed by default, so there is no need to add such constraints explicitly.

4.2 Solving the AGAP Instance with CPLEX

We used the above model to simulate an AGAP decision making process with calculations performed by CPLEX solver.

To derive $\hat{y}$, first we have calculated the maximal value of the function $-f_1(x)$ (formula (5)). The optimal value of this function is 0. It is worth observing that CPLEX produced this value with 0 gate and 5 apron operations, while clearly for the same value of $-f_1(x)$ only 3 apron operations are feasible (any 2 out of 5 flights can be served by two gates without a delay, so there are $\binom{5}{2} = 20$ solutions with this optimal value), thus the assignment produced is not efficient (it is weakly efficient). The efficiency of assignments is not essential when deriving just $\hat{y}_i$ but to work with efficient assignments only we maximized $-f_1(x) - \varepsilon f_2(x)$ with $\varepsilon = 0.00001$ (cf. a comment in Section 2 on the value of the function $f_2(x)$ at the optimality of the function $f_1(x)$). This modified objective function produced an efficient assignment #1 (Table 2; in tables below WT denotes waiting time and #APRON denotes the number of apron operations).

Next we have calculated the maximal value of the function $-f_2(x)$ (formula (7)). The optimal value of this function is -1. For the same reasons as above we maximized $-\varepsilon f_1(x) - f_2(x)$ with the same value of ε. This modified objective function produced an efficient assignment #2 (Table 2).

Table 2

FLIGHT-GATE/APRON ASSIGNMENTS BY CPLEX

No	Scenario	GATE 1	GATE 2	APRON	- WT	- #APRON
#1	y^{ref} = (-15.00,-1.00) λ = (1.00,1.00)	1	4	2,3,5	0	-3
#1*a*	τ = (5.00,1.00)	2	1	3,4,5	0	-3
#2	-	1,4	2,5	3	-45	-1
#3	τ = (10.00,1.00)	1,5	3	2,4	-15	-2
#3*a*	y^{ref} = (-25.00,-2.00)	3	1,5	2,4	-15	-2
#3b	λ = (1.00,23.00)	4	1,5	2,4	-15	-2

So in the *adapted* problem we have $\hat{y} = (0, -1)$.

The clearly best combination: 0 waiting time and -1 apron operation is not possible in that problem (if it were, the maximization of $-f_1(x) - \varepsilon f_2(x)$ or $-f_1(x) - \varepsilon f_2(x)$ would produce it) so the decision maker has to compromise on $\hat{y}$, i.e. accept assignments which are worse than this combination with respect to at least one objective function.

As presented in the previous section, the decision maker can define his favourable compromises in three ways. Here we show how he can act along each of these ways. Below, in all computations we have used $\rho = 0.00001$ (cf. formulae (9), (9'), (9")).

1. Suppose that the decision maker is willing to make concessions on the (impossible) best combination $\hat{y}$ and he defines such concessions by (the vector of) favourable concessions: (10 minutes waiting time, 1 apron operation). Hence $\tau = (10.00, 1.00)$. By formula (11) $\lambda = (0.10, 1.0)$. With these weights the objective function of optimisation problem (9') has the smallest value for assignment #3 (15 minutes waiting time, 2 apron operations) (see Figure 1).

Suppose now that the decision maker is willing to make concessions on the (impossible) best combination $\hat{y}$ but this time he defines such concessions by (the vector of) favourable concessions: (5 minutes waiting time, 1 apron operation). Hence $\tau = (5.00, 1.00)$. By formula (11) $\lambda = (0.20, 1.00)$. With these weights the objective function of optimisation problem (9') has the smallest value for assignment #1a (0 minutes waiting time, 3 apron operations). Here #*Nx* denotes an assignment which has the same values of objective functions as assignment #*N* but differs from assignment #*N* and possibly other assignments #*Nx* by flight gate/apron assignments.

2. Suppose that the decision maker specifies explicitly a compromise between the number of apron operations and waiting time he would like to achieve or at least to approximate as closely as possible: 15 minutes waiting time, 1 apron operation. Observe that in our problem there is no such assignment, nevertheless the reference point $y^{ref} = (-15.00, -1.00)$ (signs have to be reverted for the *adapted* problem) captures the decision maker's preferences for that point as described in the previous section. By formula (10) and formula (11) $\tau = (15.00, 0.00)$. As $\tau_2 = 0$, this means that the decision maker is not willing to make any concessions on $\hat{y}_2 = -1$. Hence the problem reduces to maximization of $-f_2(x)$ which has been already done when calculating $\hat{y}_1$ and this yielded assignment #1.

Suppose now that the decision maker specifies explicitly another compromise between the number of apron operations and waiting time he would like to achieve or at least to approximate as closely as possible: 25 minutes waiting time, 2 apron operations. Hence, $y^{ref} = (-25.00, -2.00)$ and by formula (10) $\tau = (25.00, 1.00)$ and by formula (11) $\lambda = (0.04, 1.00)$. With these weights the objective function of optimisation problem (9) has the smallest value for assignment #3a.

Suppose that the decision maker specifies directly two vectors of weights λ where from his experience with the problem (e.g. the problem was solved many times in the past) he knows that the first vector leads to assignments with a small number of apron operations whereas the second leads to assignments with low waiting times. Let those vectors be: $\lambda^1 = (1.00, 23.00)$ and $\lambda^2 = (1.00, 1.00)$.

In the first case the objective function of optimisation problem (9') has the smallest value for assignment #3b (15 minutes waiting time, 2 apron operations).

In the second case the objective function of optimisation problem (9') has the smallest value for assignment #1 (0 minutes waiting time, 3 apron operations).

All scenarios for the problem have been solved with CPLEX on a UNIX platform in less than 0.1 seconds.

4.3 Solving the AGAP Instance with Evolutionary Optimisation

We have mirrored all calculations listed in the previous section with a custom-made evolutionary optimisation algorithm (E−AGAP). A brief description of this algorithm is given in Appendix 2.

As expected, for all the decision making scenarios considered in Section 4.2, E−AGAP has produced the same results as CPLEX with respect to values of objective functions, as shown in Table 3. However, for the same values of objective functions, E−AGAP and CPLEX produced different solutions (assignments).

All scenarios for the problem have been solved with algorithm E−AGAP on a PC computer with Linux operating system in less than 1 second.

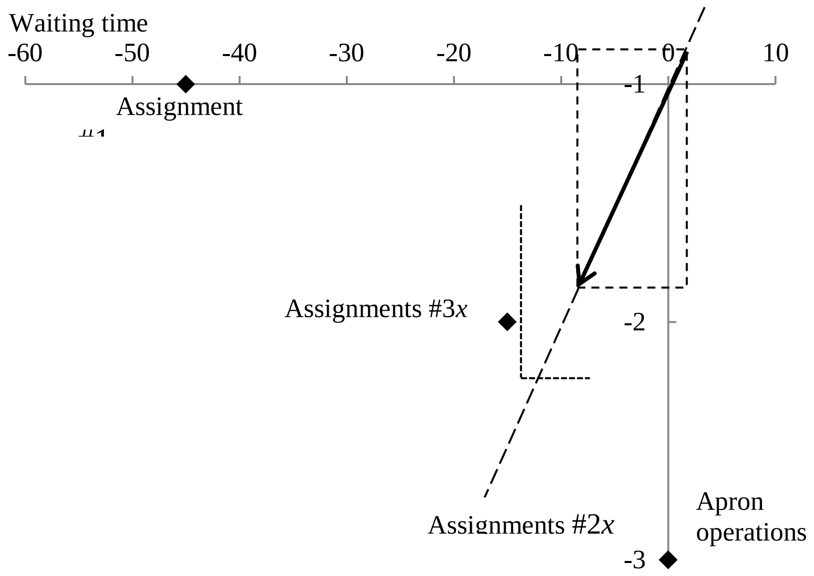

Figure 1. A graphical interpretation of searching for optimal solution in the example problem with the vector of concessions τ = (10.00, 1.00). Here the smallest value of the objective function in optimisation problem (9') is attained for assignment #3

Table 3

FLIGHT-GATE/APRON ASSIGNMENTS BY E−AGAP

No	Scenario	GATE 1	GATE 2	APRON	- WT	- #APRON
#1*b*	τ = (5.00, 1.00)	4	5	1, 2, 3	0	-3
	y^{ref} = (-15.00,-1.00)	4	5	1,2,3	0	-3
	λ = (1.00, 1.00)	4	5	1,2,3	0	-3
#3c	τ = (10.00, 1.00)	1,5	2	3,4	-15	-2
	y^{ref} = (-25.00, -2.00)	1,5	2	3,4	-15	-2
	λ = (1.00, 23.00)	1,5	2	3,4	-15	-2

5 Concluding Remarks

In principle there is no limit for applicability of the methodology we have developed and we recommend to problems of higher dimensions than those solved in the paper as long as a solver can handle problem (9).

In the paper we have solved a small instance of a multiple criteria decision problem using a commercial optimisation package. Here by solving multiple criteria decision problems we mean the ability to derive any efficient solution (decision variant) which the decision maker implicitly points to by his/her preferences.

We have shown that the methodology for decision makers' preference capture seamlessly works with CPLEX with no need for any adaptation of the package, so in fact it would work in that manner with any commercial solver. As CPLEX is reported to be able to solve problems with thousands of variables and constraints this opens the door for applying MCDM to practical problems of considerable sizes as required in some industries.

We have also shown that heuristics, such as Evolutionary Multiobjective Optimisation, can be easily fitted to our methodology as a potential viable alternative to CPLEX. The lack of guarantee of optimality in such methods is outweighed by their flexibility, adaptability and low cost.

This poses the question of how those two computing paradigms relate to each other in the function of growing size and complexity of decision making problems such as AGAP. In more general terms, it would be of utmost interest and importance for MCDM community to to which extent heuristics such as Evolutionary Multiobjective Optimisation (cf. e.g. Kaliszewski et al., 2012), can be competitive with the exact optimisation. Investigations of that question are the intended topic for our future research.

Appendix 1

Consider a flight schedule as in Table 1, Section 4. As said in Section 4.2, the upper bound for the width of time windows allowing sequential solving of the AGAP is $\alpha\delta$, which in this case is 30 minutes. It is shown in Figure 1 that to this aim time windows of $3\delta = 15$ minutes are not wide enough and in Figure 2 it is shown that time windows of $4\delta = 20$ minutes of the minimal required size.

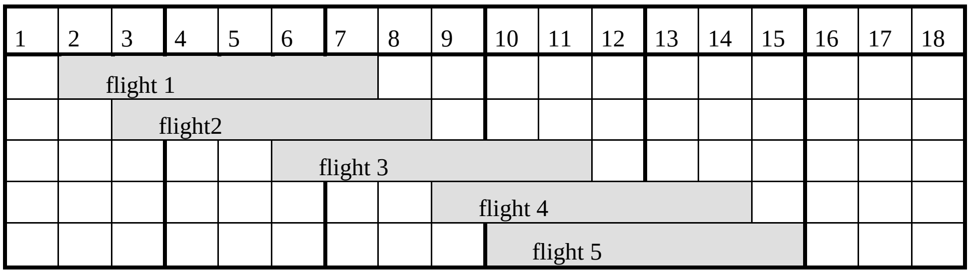

Figure 1. For time windows of minutes some flights extend over more than two windows

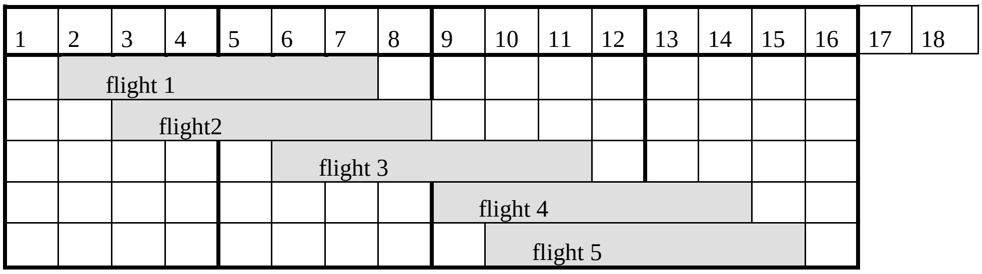

Figure 2. For time windows of minutes no flight extends over more than two windows

Appendix 2

The purpose of solving AGAP with evolutionary optimisation was to get some comparative experience in solving MCDM problems with a commercial software (such as CPLEX) and a sort of heuristics. To that aim we have implemented a heuristic algorithm – E−AGAP − which mimics principles of evolutionary computations. At the current stage, the algorithm solves just the instance of AGAP used in this paper as an illustrative numerical example, but it can be easily parameterised to handle any AGAP instances.

In the algorithm each individual is composed of three queues, two for the gates and one for the apron. In other words, each individual represents a variant of gate/apron assignment. In the initialisation step for each individual flights are randomly scheduled to queues. Then, in consecutive iterations, flights are rescheduled by genetic-like operators. Three operators are used: exchange operator, relocation operator and selection operator.

The exchange operator chooses randomly two flights assigned to different queues and exchanges them. The relocation operator selects randomly a flight from a queue and places it in a randomly selected position in a gate queue (not apron queue). If possible with respect to the fitness function, both operators can be used in the same iteration several times on the same individual.

Following the idea exposed in Stańczak (2003), in the course of the algorithm a record of actions of these two operators is kept for each individual. An operator with more historical success in improving values of objective functions for a given individual has a better chance to be called to act on that individual.

In contrast to the exchange operator and the relocation operator, which act if called to, the selection operator is called in each iteration to select individuals from the next iteration population.

The fitness function used in our paper is in fact a mechanism which besides computing values of objective functions has a built-in functionality of correcting infeasible solutions, namely: if a flight waits too long for a service, the fitness function assigns it to the apron queue.

To account for multiple criteria interplay the selection operator employed promotes nondominated solutions. If the population of nondominated solutions becomes too small some dominated best-fit solutions also become candidates for selection.

The evolutionary optimisation algorithm composed in this manner has been effective in solving the instance of AGAP problem considered in this paper. For more challenging instances of AGAP the algorithm can be endowed with some other (as many as appropriate) genetic-like operators (cf. Stańczak, 2003).

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M U L T I P L E   C R I T E R I A   D E C I S I O N   M A K I N G
Vol. 8 2013

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MULTICRITERIA APPROACH TO ENVIRONMENTAL DECISION-MAKING IN A COMPANY – A PATH GOAL PROGRAMMING APPLICATION

In the paper we propose a practical application of the path goal programming approach, by which we understand a goal programming approach with goals being not numbers, but paths in networks. This method, with goal paths corresponding to desired schedules of environmental investment implementation, is used for the environmental decision-making in a selected company. The company has to introduce some investments aimed at natural resources reduction. However, not all the required investments can be realized in one budgeting period, because of budget constraints. The desired compromise schedule of investment realization has to be worked out. This schedule becomes the goal. The path goal programming method helps to reduce the undesired deviations from the goal schedule.

Keywords: environmental decisions, environmental awareness, path goal programming

1 Introduction

The dynamic development of industry and population explosion have been observed since the twentieth century. It has been leading to an excessive use of natural resources and environmental devastation. Thus, there is a need for a change of companies' attitude toward nature. This need was noticed during the Conference of the United Nations in Rio de Janeiro in 1992. The delegates assembled at the conference stated clearly that environment, economy and society were closely linked. The concept of sustainable development was

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created, meaning a development where the balance between economic growth, environmental protection and human development is preserved.

That is why the environment plays a more and more important role in decision making in companies. Apart from the obvious goals such as profit increase or cost minimization, the companies more and more often have to take into account the negative influence of their activity on the environment and to minimize it. They also have to make investments minimizing this negative influence.

In this paper we deal with decisions about such investments. They have to be made on the basis of several criteria: cost (the investments should be as cheap as possible), time (the negative influence on the environment should be eliminated as soon as possible), reduction value of the negative influence of various factors of the company activity on the environment (this value should be as high as possible) and influence of the environmental decisions on company image and its compliance with law (of course this factor has to be maximized too). All the above listed criteria were taken into account in various stages of the decision process in the studied company and three actions were selected for implementation. In the final stage the budget criterion remained to be considered. The budget criteria made it necessary to distribute the implementation of the selected actions in time, thus to choose a schedule of their implementation. We proposed to use the so called path goal programming in this stage.

Goal programming is a very well known tool for multicriteria decision making. However, in all goal programming versions the goals are numbers – sometimes "generalized" (interval, probabilistic or fuzzy), but always numbers (surveys of all the existing goal programming approaches can be found e.g. in Chang et al. (2012); Ghahtarani and Najafi (2013); Nha et al. (2013)). As mentioned above, in the decisions considered in this paper we had to refer to schedules: the desired schedules played the role of goals. That is why the path goal programming was used in this case, whose idea was proposed for the first time by the present authors (Kuchta and Urbańska, 2012). The path goal programming takes into account deviations from a desired schedule, thus it helps to find an optimal schedule of investments. The original idea of the method is extended here and applied to the company in question.

The outline of the paper is as follows: In Section 2 we present briefly the concept and state of art of the multicriteria approach in environmental management and, more generally, in sustainable decision making. In Section 3 we analyse one instance of environmental decision making in the company studied. In Section 4 we apply the path goal programming to the scheduling of selected investments in the company studied.

2 Multicriteria analysis and environmental management

Environmental management in companies is one of the latest trends in the theory of organization and management. It has been dynamically developing since the 1990s (Graczyk, 2008; O'Brien 2000). One of the reasons for this ongoing development is the increasing environmental awareness among human beings, who often have the double role: that of the private persons, wanting to live in healthy and good conditions, and that of workers or employees in companies, who exercise pressure on the management to take into account the environment. Environmental awareness can be described as: ideas, values and opinions about the environment which for human beings is a place of life, personal development and social life (Papuziński, 2006).

Environmental decisions are taken on four basic levels: on the operational level of companies, on the tactic level of companies, on the strategic level of companies and on the level of legal regulations of a state (Graczyk, 2008; Merad et al., 2013). In this paper the company strategic level is the most important one. Companies should integrate environmental and economic objectives. If this is so, environmental objectives become usual components of business management. The environmental decisions include decisions concerning prevention, compensation, reduction, regulation, innovation, vitalization and substitution (Nahotko, 2002). In our case, we are dealing with investment decisions concerning substitution of a heat source with another one, with the objective of energy saving, as well as investment decisions concerning reduction of water and gas usage.

By implementing the concept of environmental management a company can achieve two kinds of benefits: direct and indirect. The reduction of operational cost, e.g. thanks to the reduction of the use of natural resources and of waste management costs, is a direct benefit. Another direct benefit is the reduction of environmental fines. The reduction of social costs, such as environmental pollution and natural resources depletion, and the creation of an eco-friendly image, is an indirect benefit.

The environmental management is a part of what we call sustainable management (Daub and Ergenzinger, 2005). Environmental or sustainable decision making in a company is always a multicriteria decision making process, where the criteria have to be taken from at least two groups: that of economic criteria and that of environmental ones (there are many definitions of sustainability – see e.g. http://sustainability.about.com/od/Sustainability/a/What-Is-Sustainability.htm – the specific criteria may change, but usually some so-called social criteria are taken too). There is a vast literature on multicriteria decision making in environmental or sustainable management. The authors apply all the most popular and verified in practice multicriteria decision making methods, such as ELECTRE, PROMETHEE, distance from ideal solution methods (a review can be found e.g. in Merad et al., (2013); Doukas et al., (2007); Khalili-Damghani et al., (2013); Macharis et al., (2012)), in order to select projects (actions) which should be implemented. Also the goal programming (crisp, interval, fuzzy or stochastic) is used in order to select projects within the framework of sustainable (environmental) management (Bilbao-Terol et al., 2012; Khalili-Damghani et al., 2013), but never to schedule them. In general, we are not aware of any literature where the question of seeking the optimal schedule for environmental (sustainable) actions is considered. In our opinion, this is the main novelty of the present paper, along with the application of goal programming to the question of optimal schedule search.

3 Environmental decision in the company under study

The company which is the object of the present case study is a large Polish company, not willing, however, to reveal its identity. The company has been following an environmental policy following all the goals mentioned in the introduction. The case described here is a decision making process within the company's consequent implementation of its environmental policy. Unfortunately, not all the details of the decision making process were given to the present authors, hence our presentation has to be rather superficial.

At first the company environmental policy was analysed by managers and experts. Then the company managers prepared a workshop. The aim of the workshop was to analyse all the activity areas of the company in terms of environmental management. Employees and managers of the company took part in the workshop. The result was the identification of a handful of possible solutions which were propositions of environmental decisions. 38 possible solutions (we will call them also projects, actions or investments) were identified. However, not each of them had a chance for realization. The criteria for selecting solutions were set by the managers. In this way, a list of criteria was made. They are, in order of importance:

a) the economic criteria,
b) the legal criteria,
c) the ecological criteria.

Managers and experts have analysed all the possible solutions and checked their compatibility with the company environmental policy, taking into account (entirely informally, during workshops and expert meetings) the goals and their hierarchy. However, some of the legal requirements were fixed, i.e. they had to be fulfilled under all circumstances. Three solutions listed below (Tables 1, 2, 3) were selected. Other top-priority actions included the exchange of traditional light switches for photoelectric cells in all the buildings and modernising the drainage ditch in order to use rain water to fill up the fire-fighting water tank. Both were excluded for the moment: the former because of a complicated legal procedure necessary to accomplish before starting the project, the latter because it seemed to bring fewer financial advantages than the three solutions selected eventually.

For each selected action a goal was identified as well as an indicator measuring the achievement of the goal. As it was known from the beginning that it might not be possible to achieve all the goals fully, at least not immediately, the steps of gradual goal achievements were elaborated, together with the corresponding cost. The results are shown below:

Table1

Solution I and its characteristics

Description	Construction of a pipeline installation for heat distribution
General goal	Energy saving
Specific goal	Replacement of the heat obtained from electric heaters with heat obtained from gas
Total cost	$10 000
Stages	Stage I: pipeline installation in the store house (50% of the cost), Stage II: pipeline installation in the factory building

Source: company internal documents

Table 2

Solution II and its characteristics

Description	Heat cast on freon systems
General goal	Water usage and effluents reduction
Specific goal	Reducing the amount of water used for cooling systems
Total cost	$13 000
Stages	Stage I: Heat cast on half of the existing production lines (50% of the cost) Stage II: Heat cast on the other half of the existing production lines

Source: company internal documents

Table 3

Solution III and its characteristics

Description	Automatic system of water outmeasuring
General goal	Gas usage reduction
Specific goal	Reducing the amount of water consumed for steam preparation
Total cost	$20 000
Stages	Stage I: the system on half of the existing production lines (50% of the cost) Stage II: the system on the other half of the existing production lines

Source: company internal documents

The three investments selected were consistent with the company environmental policy. However, the company had a limited budget. As the budget was too small to realize all three investments in one budgeting period, the decision had to be made how to schedule them. The path goal programming approach, proposed in Kuchta and Urbańska (2012) and applied to the discussed case in the next section, supported the management in the decision making about the investments schedule. The company management decided to limit the decision manoeuvre to two budgeting periods.

4 Path goal programming approach applied in the company under study

The idea of path goal programming is to treat network paths, which may represent schedules, as goals and to minimize negative deviations from the desired schedules. The idea of path goal programming was presented in Kuchta and Urbańska (2012). The formulation there, however, concerns just one investment and combines one schedule goal with a "classic" numerical goal. Here we apply this approach in a situation with three investments and three desired schedules treated as goals, with the numerical goals taken into account in an informal way during workshops and expert evaluations in the company. The decision to be made refers to two budgeting periods.

As it is always the case in any goal programming application, first the company management were asked about their goals – here, the desired schedules for the implementation of each of the selected solutions. They were asked to be moderate, i.e. not to choose the quickest schedules for all three investments, which, clearly, could not be achieved because of budgetary limits. The managers were asked to reveal not the ideal schedules, but only those which would make them fairly satisfied.

First, the managers said they wanted to achieve, if possible, three things:

a) after the first budgetary period they wanted to be able to announce to the public that they had already introduced a pro-environmental solution, even if it had to be just one solution and even if it was not implemented fully,
b) after the second budgetary period they wanted to be able to announce to the public that they had implemented all three solutions, again, not necessarily fully,
c) after the second budgetary period they wanted to have at least one of the solutions implemented fully, preferably the second solution.

Having these statements in mind, the authors presented to the managers three goal schedules, shown in Figure 1. They were accepted as goal schedules, although of course other schedules would also satisfy the management requirements a), b), c).

Figure 1 presents the network of the decision considered here. Each of the three parts of the network refers to one of the investments (the solutions from Table 1, 2 and 3). The arcs starting in nodes 1, 2 and 3 stand for the decisions concerning each investment in the first budgeting period. The arc from node 1 to node 110 denotes the decision not to do anything regarding solution I in the first budgeting period, while the arc from node 1 to node 111, the decision to implement 50% of Solution I in the first budgeting period, and the arc from node 1 to node 112, the decision to implement the whole solution I in the first budgeting period. The arc from node 112 to 122 denotes the only possible decision in the second budgeting period in case the whole Solution I is implemented in the first period: not to do anything about this solution. Both arcs leaving node 111 correspond to the two decision possibilities about Solution I in the second period in case half of it was implemented in the first decision period: to leave it as it is (the arc leading to node 121) or implement the other half of it (the arc leading to node 122). The arcs leaving node 110 can be interpreted analogously (this node stands for the situation when nothing was done about Solution I in the first budgeting period, thus in the second period it can be left undone (120), implemented in 50% (121) and implemented fully (122)). The other arcs have the same meaning, but refer to the other two solutions. The three arcs leaving node 0 are auxiliary arcs, whose task is to link the decisions about the three solutions into one decision process. The arcs entering nodes 13, 23 and 33 have a similar role.

The patterns of the arcs represent the schedule goals proposed to the managers on the basis of their opinion. The dotted arcs show the desired schedules. We can see that if the proposed schedules were achieved, Solution I would be implemented only in 50%, but already in the first budgeting period, Solution II would be implemented fully, but it would be started possibly only in the second budgeting period. As far as Solution III is concerned, it would be implemented in 50%, and this would happen only in the second budgeting period.

The hatched arcs represent positive deviations from the desired schedules: quicker and fuller than desired implementations of the three solutions. The company would be even more satisfied if some of the hatched arcs were used. The continuous arcs represent the negative, undesired deviations. Each use of a continuous arc means a behind-schedule implementation of one of the solutions.

The budget for each budgeting period was set to \$12 000. The implementation cost is given in Tables 1, 2 and 3.

Now we can formulate the following dynamic goal programming model:

Objective function: $\delta_1^+ + \delta_2^+ + \delta_3^+ \rightarrow \min$ (1)

where $\delta_1^+, \delta_2^+, \delta_3^+$ are, the negative deviations for the schedule for Solution I, II and III, respectively.

The deviations are defined in the following way:

$$\begin{aligned}&p_{1-112}x_{1-112} + p_{1-111}x_{1-111} + p_{1-110}x_{1-110}+\\&+p_{112-122}x_{112-122} + p_{111-122}x_{111-122} + p_{111-121}x_{111-121}+\\&+p_{110-122}x_{110-122} + p_{110-121}x_{110-121} + p_{110-120}x_{111-120} +\\&+p_{122-13}x_{122-13}+p_{121-13}x_{121-13} + p_{120-13}x_{120-13}+\delta_1^- - \delta_1^+=0\end{aligned} \tag{2}$$

$$
\begin{aligned}
&p_{2-212}x_{2-212} + p_{2-211}x_{2-211} + p_{2-210}x_{2-210}+\\
&+p_{212-222}x_{212-222} + p_{211-222}x_{211-222} + p_{211-221}x_{211-221}+\\
&+p_{210-222}x_{210-222} + p_{210-221}x_{210-221} + p_{210-220}x_{211-220} +\\
&+p_{222-23}x_{222-23}+p_{221-23}x_{221-23} + p_{220-23}x_{220-23}+\delta_2^- - \delta_2^+=0
\end{aligned} \tag{3}
$$

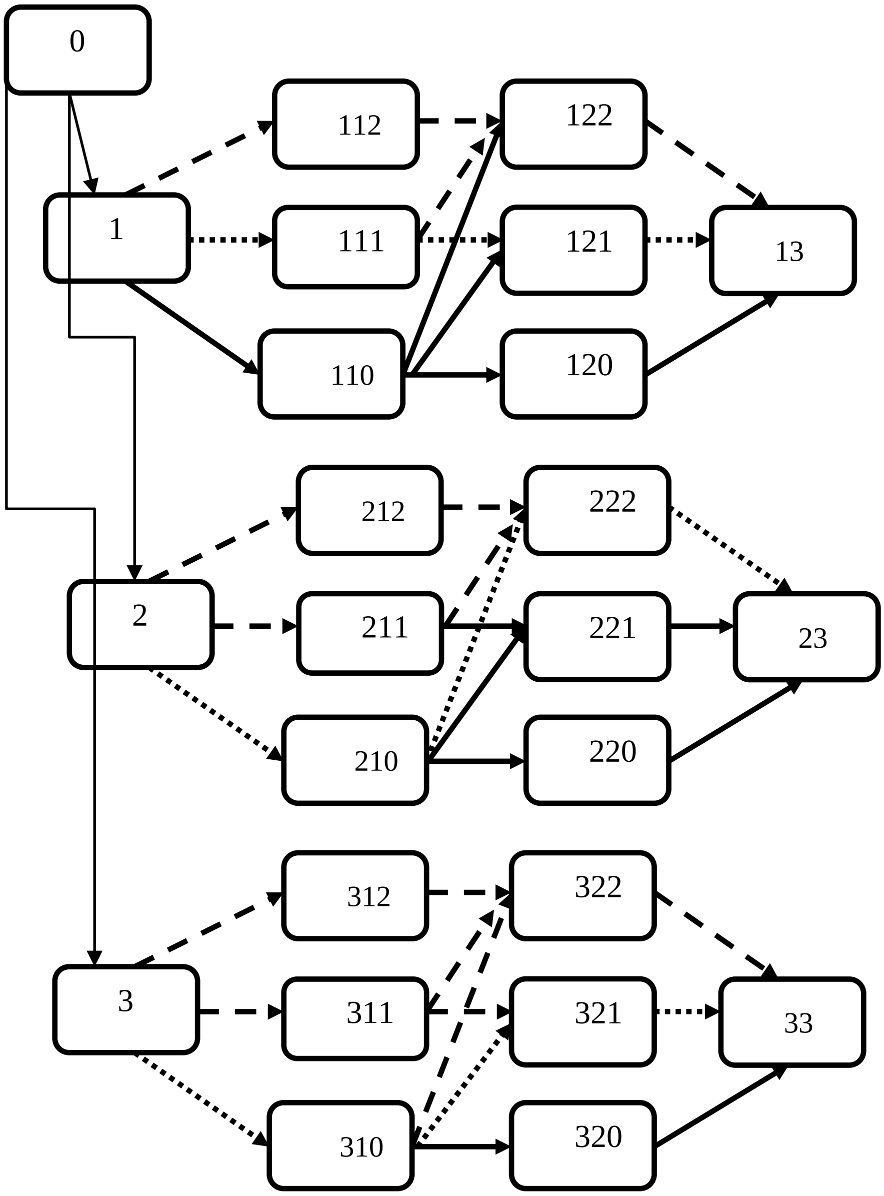

Figure 1. Decision network

$$\begin{aligned}&p_{3-312}x_{3-312}+p_{3-311}x_{3-311}+p_{3-310}x_{3-310}+\\&+p_{312-322}x_{312-322}+p_{311-322}x_{311-322}+p_{311-321}x_{311-321}+\\&+p_{310-322}x_{310-322}+p_{310-321}x_{310-321}+p_{310-320}x_{310-320}+\\&+p_{322-13}x_{322-13}+p_{321-33}x_{321-33}+p_{320-33}x_{320-33}+\delta_3^- -\delta_3^+=0\end{aligned} \tag{4}$$

where the x_{i-j} are binary decision variables, taking on value 1 if the arc leading from node *i* to node *j* is used and value 0 otherwise. p_{i-j} are coefficients equal to 0 if the corresponding arc belongs to the goal schedule, to 1 if the arc causes an undesired deviation from the goal schedule and to -1 if it causes a positive deviation from the goal schedule.

In our case (Figure 1) we have thus:

$$\begin{aligned}&p_{1-112}=-1,\ p_{1-111}=0, p_{1-110}=1, p_{112-122}=-1,\\&p_{111-122}=-1, p_{111-121}=0,\ p_{110-122}=1\\&p_{110-121}=1,\ p_{110-120}=1,\\&p_{122-13}=-1, p_{121-13}=0, p_{120-13}=1;\end{aligned} \tag{5}$$

$$\begin{aligned}&p_{2-212}=-1,\ p_{2-211}=-1, p_{2-210}=0,\ p_{212-222}=-1,\\&p_{211-222}=-1, p_{211-221}=1,\ p_{210-222}=0\\&p_{210-221}=1,\ p_{210-220}=1,\\&p_{222-23}=0, p_{221-23}=1, p_{220-23}=1;\end{aligned} \tag{6}$$

$$\begin{aligned}&p_{3-312}=-1,\ p_{3-311}=-1, p_{3-310}=0,\ p_{312-322}=-1,\\&p_{311-322}=-1, p_{311-321}=-1,\ p_{310-322}=1\\&p_{310-321}=0,\ p_{310-320}=1,\\&p_{322-33}=-1, p_{321-33}=0, p_{320-33}=1;\end{aligned} \tag{7}$$

Then we have the network constraints, assuring that exactly three investments are considered and that the decision in case of each of the investments is unequivocal and actually taken:

$$x_{0-1}+x_{0-2}+x_{0-3}=3 \tag{8}$$

$$\begin{aligned}&x_{1-110}+x_{1-111}+x_{1-112}=x_{0-1}\\&x_{2-210}+x_{2-211}+x_{2-212}=x_{0-2}\\&x_{3-310}+x_{3-311}+x_{3-312}=x_{0-3}\end{aligned} \tag{9}$$

$$\begin{aligned}&x_{1-112}=x_{112-122}\\&x_{2-212}=x_{212-222}\\&x_{3-312}=x_{312-322}\end{aligned} \tag{10}$$

$$\begin{aligned} &x_{1-111} = x_{111-122} + x_{111-121} \\ &x_{2-211} = x_{211-222} + x_{211-221} \\ &x_{3-311} = x_{311-322} + x_{311-321} \end{aligned} \tag{11}$$

$$\begin{aligned} &x_{1-110} = x_{110-122} + x_{110-121} + x_{110-120} \\ &x_{2-210} = x_{210-222} + x_{210-221} + x_{210-220} \\ &x_{3-310} = x_{310-322} + x_{310-321} + x_{310-320} \end{aligned} \tag{12}$$

$$\begin{aligned} &x_{110-122} + x_{111-122} + x_{112-122} = x_{122-13} \\ &x_{210-222} + x_{211-222} + x_{212-222} = x_{222-23} \\ &x_{310-322} + x_{311-322} + x_{312-322} = x_{322-33} \end{aligned} \tag{13}$$

$$\begin{aligned} &x_{110-121} + x_{111-121} = x_{121-13} \\ &x_{210-221} + x_{211-221} = x_{221-23} \\ &x_{310-321} + x_{311-321} = x_{321-33} \end{aligned} \tag{14}$$

$$\begin{aligned} &x_{110-120} = x_{120-13} \\ &x_{210-220} = x_{220-23} \\ &x_{310-320} = x_{320-33} \end{aligned} \tag{15}$$

where (8) assure that exactly three solutions will be considered, and (9)–(15) are balance constrains for the individual nodes from Figure 1.

Finally, we have budget constraints for the two budgeting periods:

$$\begin{aligned} &k_{1-112}x_{1-112} + k_{1-111}x_{1-111} + k_{1-110}x_{1-110} + k_{2-212}x_{2-212} + \\ &k_{2-211}x_{2-211} + k_{2-210}x_{2-210} + k_{3-312}x_{3-312} + k_{3-311}x_{3-311} + \\ &k_{3-310}x_{3-310} \leq B1 \end{aligned} \tag{16}$$

$$\begin{aligned} &k_{112-122}x_{112-122} + k_{111-122}x_{111-122} + k_{111-121}x_{111-121} \\ &+k_{110-122}x_{110-122} + k_{110-121}x_{110-121} + k_{110-120}x_{111-120} + \\ &+k_{212-222}x_{212-222} + k_{211-222}x_{211-222} + k_{211-221}x_{211-221} \\ &+k_{210-222}x_{210-222} + k_{210-221}x_{211-221} + k_{210-220}x_{211-220} \\ &+k_{312-322}x_{312-322} + k_{311-322}x_{311-322} + k_{311-321}x_{311-321} \\ &+k_{310-322}x_{310-322} + k_{310-321}x_{311-321} + k_{310-320}x_{311-320} \\ &\leq B2 \end{aligned} \tag{17}$$

where $B1$ and $B2$ are, the budgets for each period (in our case both are equal to \$12 000) and k_{i-j} are the costs linked to each decision. In our case, according to Tables 1, 2 and 3, we have:

$$\mathrm{k}_{1-112} = 10\,000, \mathrm{k}_{1-111} = 5\,000, \mathrm{k}_{1-110} = 0$$
$$\mathrm{k}_{112-122} = 0, \mathrm{k}_{111-122} = 5\,000, \mathrm{k}_{111-121} = 0,$$
$$\mathrm{k}_{110-122} = 10\,000, \mathrm{k}_{110-121} = 5\,000, \mathrm{k}_{110-120} = 0$$

$$\mathrm{k}_{2-212} = 13\,000, \mathrm{k}_{2-211} = 6\,500, \mathrm{k}_{2-210} = 0$$
$$\mathrm{k}_{212-222} = 0, \mathrm{k}_{211-222} = 6\,500, \mathrm{k}_{211-221} = 0,$$
$$\mathrm{k}_{210-222} = 13\,000, \mathrm{k}_{210-221} = 6\,500, \mathrm{k}_{210-220} = 0$$

$$\mathrm{k}_{3-312} = 20\,000, \mathrm{k}_{3-311} = 10\,000, \mathrm{k}_{3-310} = 0$$
$$\mathrm{k}_{312-322} = 0, \mathrm{k}_{311-322} = 10\,000, \mathrm{k}_{311-321} = 0,$$
$$\mathrm{k}_{310-322} = 20\,000, \mathrm{k}_{310-321} = 10\,000, \mathrm{k}_{310-320} = 0$$

The solution of problem (1)-(17) is:

$$\mathrm{x}_{1-111} = \mathrm{x}_{111-121} = \mathrm{x}_{121-13} = 1$$
$$\mathrm{x}_{2-211} = \mathrm{x}_{211-221} = \mathrm{x}_{221-23} = 1$$
$$\mathrm{x}_{3-310} = \mathrm{x}_{310-321} = \mathrm{x}_{321-33} = 1$$

the other x_{i-j} being zero. This gives us $\delta_1^+ + \delta_2^+ + \delta_3^+ = 1$, and $\delta_1^+ = 0, \delta_2^+ = 1, \delta_3^+ = 0$.

The value $\delta_1^+ + \delta_2^+ + \delta_3^+$ is a measure of the number of periods the investments will be behind schedule. In our case Solution II will be behind schedule in the second budgeting period (however, in the first budgeting period it will be ahead of schedule – we would have then a positive deviation from the schedule). Solution I and III will be exactly on schedule in both periods. Thus, postulate c) would not be satisfied. However, the management of the company accepted this, as the first two postulates would be held.

In fact, it might always be useful to consider alternative solutions. In our case we might seek other solutions with the objective function $\delta_1^+ + \delta_2^+ + \delta_3^+ = 1$, but $\delta_2^+ = 0$. The alternative solutions are obtained, if they exist, by adding to the model the additional constraint $\delta_2^+ = 0$. However, the managers preferred to have $\delta_2^+ = 1$ (their third postulate not fulfilled) rather than $\delta_1^+ = 1$ or $\delta_3^+ = 1$ (the other postulates not fulfilled), thus the current solution was finally accepted. With this solution all three actions would be implemented in 50% after the second budgetary period and action 1 would be implemented in 50% already after the first budgetary period, which satisfied the management.

Conclusions

In this paper we described and applied the path goal programming approach, which allows to consider schedules as goals. The application concerned a big company implementing an environmental policy which had to choose between various desired environment-influencing investments, as the budget did not allow to implement all of them at the same time. Not only the investments had to be chosen, but also a compromise implementation schedule. Three investments have been chosen using an informal multicriteria decision making process, and the authors helped to determine satisfactory schedules of their implementation. We formulated a path goal programming model allowing to find a solution which best suited the desired schedule. Solving the model (which turned out to be, in the final stage, a linear programming model with 18 decision variables) allowed to minimize negative deviations from the desired schedule.

Our future research will go in two directions. The first one is a further development of the path goal programming approach. The negative deviations from the schedule can be weighted according to their distance from the desired schedule and their significance for the environment or the internal objectives of the company. This is a problem still to be considered – in this paper the deviations do not have any weights. The other research direction is the consideration of more complex environmental decisions. This will be accomplished, among other things, in cooperation with the company considered in this paper. The first step will be to persuade the management to use formal methods also in the initial investment selection process. For the moment this phase is accomplished in a totally informal way, but many authors propose fairly simple and verified in practice formal sustainable multicriteria decision making methods (e.g. Merad et a. 2013) which might facilitate the process and make it more effective.

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M U L T I P L E C R I T E R I A D E C I S I O N M A K I N G
Vol. 8 2013

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INCOMPLETE PAIRWISE COMPARISON MATRIX AND ITS APPLICATION TO RANKING OF ALTERNATIVES

Abstract

A fuzzy preference matrix is the result of pairwise comparison - a powerful method in multi-criteria optimization. When comparing two elements, the decision maker assigns a value between 0 and 1 to any pair of alternatives representing the element of the fuzzy preference matrix. Here, we investigate relations between transitivity and consistency of fuzzy preference matrices and multiplicative preference ones. The results obtained are applied to decision situations where some elements of the fuzzy preference matrix are missing. We propose a new method for completing the fuzzy preference matrix with missing elements called the extension of the fuzzy preference matrix and investigate an important particular case of the fuzzy preference matrix with missing elements. Next, using the eigenvector of the transformed matrix we obtain the corresponding priority vector. Illustrative numerical examples are supplied.

Keywords: pairwise comparison matrix, fuzzy preference matrix, reciprocity, consistency, transitivity, fuzzy preference matrix with missing elements.

1 Introduction

In various fields of evaluation, selection, and prioritization processes the decision makers (DM) try to find the best alternative(s) from a feasible set of alternatives. In many cases, the comparison of different alternatives according to their desirability in decision problems cannot be done by one person or using only a single criterion. In many DM problems, procedures have been established to combine opinions about alternatives related to different points of view. These procedures are often based on pairwise comparisons, in the sense that processes

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are linked to some degree of preference of one alternative over another. According to the nature of the information expressed by every DM, for every pair of alternatives different representation formats can be used to express preferences, e.g. multiplicative preference relations (see Alonso et al., 2008; Saaty, 1980; Saaty, 1991), fuzzy preference relations (see Fodor, Roubens, 1994; Herrera-Viedma et al., 2004; Ramík, 2011), interval-valued preference relations and also linguistic preference relations (see Alonso et al., 2008).

Usually, experts are characterized by their own personal background and knowledge of the problem to be solved. Expert opinions may differ substantially, some of them cannot efficiently express a preference degree between two or more of the available options. This may be true due to an imprecise or insufficient level of knowledge of the problem on the part of an expert, or because the expert is unable to determine the degree to which some options are better than others. In such situations the expert will provide an incomplete fuzzy preference relation (see Alonso et al., 2008; Herrera-Viedma et al., 2007).

In this paper, we present a general method to estimate the missing information in the form of incomplete fuzzy preference relations- multiplicative or fuzzy. Our proposal is different to the approach described in Herrera-Viedma, et al., (2007) and Ma et al., (2006), where special averages of expert evaluations and consistency/transitivity properties are applied. In the literature (Xu and Da, 2005), the problem is solved by the least deviation method to obtain a priority vector of a fuzzy preference relation. Here, we propose the classical result of Perron-Frobenius theory to obtain the priority vector for a transformed fuzzy preference matrix. Moreover, our approach enables us to obtain a priority vector for additive-transitive preference relations and also for additive-consistent ones, i.e. additive-reciprocal and multiplicative-transitive ones. It also allows for completing a pairwise comparison matrix with missing elements and for finding out the closest consistent/transitive matrix to the inconsistent/intransitive one, i.e. by repairing the inconsistency of fuzzy preference relations.

2 Multiplicative and additive preferences

The DM problem can be formulated as follows. Let $X=\{x_1, x_2,...,x_n\}$ be a finite set of alternatives. These alternatives have to be ordered from best to worst, using the information given by a DM in the form of pairwise comparison matrix.

The preferences over the set of alternatives, X, may be represented in two ways: multiplicative and additive (also called fuzzy preference relations). Let us assume that the preferences on X are described by a preference relation on X given by a positive $n\times n$ matrix $\boldsymbol{A}=\{a_{ij}\}$, where $a_{ij} > 0$ for all i,j indicates the preference intensity for the alternative x_i to that of x_j. The elements of $\boldsymbol{A}=\{a_{ij}\}$ satisfy the following reciprocity condition.

A positive n×n matrix $\mathbf{A}=\{a_{ij}\}$ is multiplicative-reciprocal (m-reciprocal), if:

$$a_{ij}.\ a_{ji} =1 \text{ for all } i,j\in\{1,2,\dots,n\}. \quad (1)$$

A positive n×n matrix $\mathbf{A}=\{a_{ij}\}$ is multiplicative-consistent (or, m-consistent), if:

$$a_{ij} = a_{ik}.\ a_{kj} \quad \text{for all } i,j,k\in\{1,2,\dots,n\}. \tag{2}$$

Note that $a_{ii} = 1$ for all i, and that an m-consistent matrix is m-reciprocal (however, not vice versa). Here, $a_{ij} > 0$ and m-consistency is not restricted to the Saaty scale. In particular, we extend this scale to the closed interval $[1/\sigma; \sigma]$, where $\sigma>1$.

Sometimes it is more natural, when comparing x_i to x_j, that the decision maker (DM) assigns nonnegative values b_{ij} to x_i and b_{ji} to x_j, such that $b_{ij} + b_{ji} = 1$. With this interpretation, the preferences on X can be understood as a fuzzy preference relation, with membership function $\mu_R : X\times X \rightarrow [0;1]$, where $\mu_R (x_i, x_j) = b_{ij}$ denotes the preference of the alternative x_i over x_j. The most important properties of the above mentioned matrix $\boldsymbol{B} =\{b_{ij}\}$, called here the *fuzzy preference matrix*, can be summarized as follows.

An $n\times n$ matrix $\boldsymbol{B} =\{b_{ij}\}$ with $0 \le b_{ij} \le 1$ for all i and j is *additive-reciprocal* (*a-reciprocal*), if:

$$b_{ij} + b_{ji} = 1 \text{ for all } i,j\in\{1,2,\dots,n\}. \tag{3}$$

Evidently, if (3) holds, then $b_{ii} = 0.5$ for all $i\in\{1,2,\dots,n\}$.

To make a coherent choice of evaluations b_{ij} (when assuming fuzzy preference matrix $\boldsymbol{B} =\{b_{ij}\}$), a set of properties to be satisfied by such relations has been suggested in the literature, the terminology of properties of relations is, however, not established yet, compare e.g. Alonso et al. (2008); Fodor, Roubens (1994); Tanino (1984). Here, we use the usual terminology which is as close as possible to the one used in the literature.

Transitivity is one of the most important properties of preferences, and it represents the idea that the preference intensity obtained by comparing two alternatives directly should be equal to or greater than the preference intensity between those two alternatives obtained using an indirect chain of alternatives.

Let $\boldsymbol{B} =\{b_{ij}\}$ be an $n\times n$ a-reciprocal matrix with $0 \le b_{ij} \le 1$ for all i and j.

We say that $\mathbf{B} =\{b_{ij}\}$ is multiplicative-transitive (m-transitive), if:

$$\frac{b_{ij}}{b_{ji}} = \frac{b_{ik}}{b_{ki}}\frac{b_{kj}}{b_{jk}} \qquad \text{for all } i,j,k\in\{1,2,\dots,n\}. \tag{4}$$

Note that if $\boldsymbol{B}$ is m-consistent then $\boldsymbol{B}$ is m-transitive. Moreover, if $\boldsymbol{B}$ is m-reciprocal, then $\boldsymbol{B}$ is m-transitive iff $\boldsymbol{B}$ is m-consistent.

We say that $\mathbf{B} =\{b_{ij}\}$ is additive-transitive (a-transitive), if:

$$(b_{ij} - 0{,}5) = (b_{ik} - 0{,}5) + (b_{kj} - 0{,}5) \text{ for all } i,j,k\in\{1,2,\dots,n\}. \tag{5}$$

This property is also called additive consistency; here, we reserve, however, this name for a different notion, see below.

Now, we shall investigate some relationships between a-reciprocal and m-reciprocal pairwise comparison matrices. We start with an extension of the result published in Herrera-Viedma et al. (2004). For this purpose, given $\sigma > 1$, we define the following function φ_σ as:

$$\varphi_\sigma(t) = \sigma^{2t-1} \text{ for } t \in [0;1]. \tag{6}$$

We obtain the following result, characterizing a-transitive and m-consistent matrices; for the proof see Ramik, Vlach (2013).

Proposition 1. Let $\sigma > 1$, $\boldsymbol{B} = \{b_{ij}\}$ be an $n \times n$ matrix with $0 \le b_{ij} \le 1$ for all i, $j \in \{1,2,\ldots,n\}$. If $\boldsymbol{B}$ is a-transitive then $\boldsymbol{A} = \{\varphi_\sigma(b_{ij})\}$ is m-consistent.

Now, let us define the function Φ as follows:

$$\Phi(t) = \frac{t}{1-t} \text{ for } 0 < t < 1. \tag{7}$$

We obtain the following result (see Ramik, Vlach, 2013).

Proposition 2. Let $\boldsymbol{B} = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 \le b_{ij} \le 1$ for all $i,j \in \{1,2,\ldots,n\}$. If $\boldsymbol{B}$ is m-transitive then $\boldsymbol{A} = \{a_{ij}\} = \{\Phi(b_{ij})\}$ is m- consistent.

From Proposition 2 it is clear that the notion of m-transitivity plays a similar role for a-reciprocal fuzzy preference matrices as the notion of m-consistency does for m-reciprocal matrices. That is why it is reasonable to introduce the following *definition*:

Any a-reciprocal m-transitive $n \times n$ matrix $\boldsymbol{B} = \{b_{ij}\}$ is called *additively consistent* (*a-consistent*).

According to this definition Proposition 2 can be reformulated as follows:

Proposition 2*. Let $\boldsymbol{B} = \{b_{ij}\}$ be an $n \times n$ matrix with $0 \le b_{ij} \le 1$ for all $i,j \in \{1,2,\ldots,n\}$. If $\boldsymbol{B}$ is a-consistent then $\boldsymbol{A} = \{a_{ij}\} = \{\Phi(b_{ij})\}$ is m-consistent.

By Proposition 1, resp. Proposition 2* we can transform a-transitive, resp. a-consistent matrices into m-consistent ones by an appropriate transformation functions φ_σ, resp. Φ.

In practice, perfect consistency/transitivity is difficult to obtain, particularly when measuring preferences on a set with a large number of alternatives.

3 Inconsistency of pairwise comparison matrices, priority vectors

If for some positive $n \times n$ matrix $\boldsymbol{A} = \{a_{ij}\}$ and for some $i,j,k \in \{1,2,...,n\}$, the multiplicative consistency condition (2) does not hold, then $\boldsymbol{A}$ is said to be *multiplicative-inconsistent* (or *m-inconsistent*). If for some $n \times n$ fuzzy preference matrix $\boldsymbol{B} = \{b_{ij}\}$ with $0 \le b_{ij} \le 1$ for all i and j, and for some indices $i,j,k \in \{1,2,...,n\}$, (4) does not hold, then $\boldsymbol{B}$ is said to be *additive-inconsistent* (or, *a-inconsistent*). Finally, if for some $n \times n$ fuzzy matrix $\boldsymbol{B} = \{b_{ij}\}$ with $0 \le b_{ij} \le 1$ for all i and j, and for some indices $i,j,k \in \{1,2,...,n\}$, (5) does not hold, then $\boldsymbol{B}$ is said to be *additive-intransitive* (*a-intransitive*). In order to measure the degree of inconsistency/intransitivity of a given matrix several measurement methods have been proposed in the literature (see e.g. Alonso et al., 2008). In AHP, multiplicative reciprocal matrices have been considered (Saaty, 1991).

As far as additive-reciprocal matrices are concerned, some methods for measuring a-inconsistency/a-intransitivity are proposed here. We start, however, with measuring the inconsistency of positive matrices which is based on Perron-Frobenius theory (see e.g. Fiedler, Nedoma, Ramík, Rohn, 2006). Later on, we shall deal with measuring a-inconsistent and a-intransitive matrices.

The Perron-Frobenius theorem describes some of the remarkable properties enjoyed by the eigenvalues and eigenvectors of irreducible nonnegative matrices (e.g. positive matrices).

Theorem (Perron-Frobenius). Let $\boldsymbol{A}$ be an irreducible nonnegative $n\times n$ matrix. Then the spectral radius, $\rho(\boldsymbol{A})$, is a positive (real) eigenvalue, with a positive (real) eigenvector $\boldsymbol{w}$ such that $\boldsymbol{Aw}= \rho(\boldsymbol{A})\boldsymbol{w}$.

In the decision making context the above mentioned eigenvalue $\rho(\boldsymbol{A})$ is called the *principal eigenvalue of* $\boldsymbol{A}$. It is a simple eigenvalue (i.e. it is not a multiple root of the characteristic equation), and its eigenvector, called the *priority vector*, is unique up to a multiplicative constant.

Now, let $\boldsymbol{A}$ be a nonnegative m-reciprocal $n\times n$ matrix. The m-consistency of $\boldsymbol{A}$ is characterized by the *m-consistency index* $I_{mc}(\boldsymbol{A})$ defined in (Saaty, 1980) as:

$$I_{mc}(\boldsymbol{A}) = \frac{\rho(\boldsymbol{A})-n}{n-1}, \tag{8}$$

where $\rho(\boldsymbol{A})$ is the spectral radius of $\boldsymbol{A}$ (in particular, the principal eigenvalue of $\boldsymbol{A}$).

Moreover, we suppose that $\boldsymbol{A}=\{a_{ij}\}$ is a pairwise comparison matrix with elements a_{ij} based on evaluation of alternatives x_i and x_j, for all i and j. For the purpose of decision making, the rank of the alternatives in $X=\{x_1, x_2,...,x_n\}$ is determined by the vector of weights $\boldsymbol{w} = (w_1,w_2,...,w_n)$, where $w_i > 0$, for all $i\in\{1,2,...,n\}$, such that $\sum_{i=1}^{n} w_i = 1$, satisfying the characteristic equation $\boldsymbol{Aw} = \rho(\boldsymbol{A})\boldsymbol{w}$. This vector $\boldsymbol{w}$ is the (normalized) priority vector of $\boldsymbol{A}$. Since the element of the priority vector w_i is interpreted as the relative importance of the alternative x_i, the alternatives $x_1, x_2,...,x_n$ in X are ranked by their relative importance. The following result has been derived in Saaty (1980).

Proposition 3. If $\boldsymbol{A} =\{a_{ij}\}$ is an $n\times n$ positive m-reciprocal matrix, then $I_{mc}(\boldsymbol{A}) \geq 0$. Moreover, $\boldsymbol{A}$ is m-consistent if and only if $I_{mc}(\boldsymbol{A}) = 0$.

To provide a consistency measure independently of the dimension n of the matrix $\boldsymbol{A}$, T. Saaty (1980) proposed the consistency ratio. To distinguish it here from the other consistency measures, we shall call it *m-consistency ratio*. This is obtained by taking the ratio of I_{mc} to its mean value R_{mc}, estimated by an arithmetic average over a large number of positive m-reciprocal matrices of dimension n, whose entries are randomly and uniformly generated (see Saaty, 1980), i.e.:

$$CR_{mc} = I_{mc}/R_{mc}. \tag{9}$$

It was proposed that a pairwise comparison matrix could be accepted (in a DM process) if its m-consistency ratio does not exceed 0.1 (see Saaty, T.L., 1980). The m-consistency index I_{mc} has been defined for m-reciprocal matrices; here, we investigate the inconsistency/intransitivity of a-reciprocal matrices. For this purpose we use relations between m-consistent and a-transitive/a-consistent matrices derived in Propositions 1 and 2*.

Let $\boldsymbol{B} = \{b_{ij}\}$ be an m-reciprocal matrix with $0 \leq b_{ij} \leq 1$ for all i and j. We define the a-consistency index $I_{ac}(\boldsymbol{B})$ of $\boldsymbol{B} = \{b_{ij}\}$ as:

$$I_{ac}(\boldsymbol{B}) = I_{mc}(\boldsymbol{A}), \text{ where } \boldsymbol{A} = \{\Phi(b_{ij})\}. \tag{10}$$

From (10) we easily obtain the following result, which is parallel to Proposition 3.

Proposition 4. If $\boldsymbol{B} = \{b_{ij}\}$ is an a-reciprocal $n\times n$ fuzzy matrix with $0 \leq b_{ij} \leq 1$ for all i and j, then $I_{ac}(\boldsymbol{B}) \geq 0$. Moreover, $\boldsymbol{B}$ is a-consistent if and only if $I_{ac}(\boldsymbol{B}) = 0$.

Now, we shall deal with measuring a-intransitivity of a-reciprocal matrices. Let $\sigma > 1$ be a given value characterizing the scale. Let $\boldsymbol{B} = \{b_{ij}\}$ be an a-reciprocal $n\times n$ fuzzy matrix with $0 \leq b_{ij} \leq 1$ for all i and j. We define the *a-transitivity index* $I_{at}^{\sigma}(\boldsymbol{B})$ of $\boldsymbol{B} = \{b_{ij}\}$ as:

$$I_{at}^{\sigma}(\boldsymbol{B}) = I_{mc}(\boldsymbol{A}^{\sigma}), \tag{11}$$

where:

$$\boldsymbol{A}^{\sigma} = \{\varphi_{\sigma}(b_{ij})\}. \tag{12}$$

By applying (8) and (12) we obtain the following results corresponding to Propositions 3 and 4.

Proposition 5. If $\boldsymbol{B} = \{b_{ij}\}$ is an a-reciprocal $n\times n$ matrix with $0 \leq b_{ij} \leq 1$ for all i and j, then $I_{at}^{\sigma}(\boldsymbol{B}) \geq 0$. Moreover, $\boldsymbol{B}$ is a-transitive if and only if $I_{at}^{\sigma}(\boldsymbol{B}) = 0$.

Let $\boldsymbol{A} = \{a_{ij}\}$ be an a-reciprocal $n\times n$ matrix. In (12), the m-consistency ratio of $\boldsymbol{A}$ denoted by $CR_{mc}(\boldsymbol{A})$ is obtained by taking the ratio of $I_{mc}(\boldsymbol{A})$ to its mean value $R_{mc}(n)$, i.e.:

$CR_{mc}(\boldsymbol{A}) = I_{mc}(\boldsymbol{A})/R_{mc}(n)$.

The values of $R_{mc}(n)$ for n=3,4,…, can be found in Saaty (1980). Similarly, we define the a-consistency ratio $CR_{ac}(\boldsymbol{A})$ and the a-transitivity ratio $CR_{at}^{\sigma}(\boldsymbol{A})$.

Denote $\Phi(B) = \{\Phi(b_{ij})\}$, then the corresponding priority vector $\boldsymbol{w}^{ac}$ is given by the characteristic equation $\Phi(B)\,\boldsymbol{w}^{ac} = \rho(\Phi(B))\boldsymbol{w}^{ac}$.

Given $\sigma > 1$, let us denote $\varphi_{\sigma}(B) = \{\varphi_{\sigma}(b_{ij})\}$, then the priority vector $\boldsymbol{w}^{at}$ is defined by the characteristic equation $\varphi_{\sigma}(B)\,\boldsymbol{w}^{at} = \rho(\varphi_{\sigma}(B))\boldsymbol{w}^{at}$.

In practice, a-inconsistency of a positive a-reciprocal fuzzy priority matrix $\boldsymbol{B}$ is “acceptable” if $CR_{ac}(\boldsymbol{B}) < 0.1$. Also, a-intransitivity of a positive a-reciprocal pairwise comparison matrix $\boldsymbol{B}$ is "acceptable" if $CR^{\sigma}_{at}(\boldsymbol{B}) < 0.1$. The final ranking of alternatives is given by the corresponding priority vector, see Example 1.

The following two results give a characterization of a m-consistent matrix as well as an a-consistent matrix by vectors of weights, i.e. positive vectors with the sum of their elements equal to one. The proofs are straightforward and can be found in Ramik, Vlach (2013).

Proposition 6. Let $\boldsymbol{A} = \{a_{ij}\}$ be a positive $n\times n$ matrix. $\boldsymbol{A}$ is m-consistent if and only if there exists a vector $\boldsymbol{w} = (w_1, w_2,...,w_n)$ with $w_i > 0$ for all $i\in\{1,2,...,n\}$, and $\sum_{i=1}^{n} w_i = 1$ such that:

$$a_{ij} = \frac{w_i}{w_j} \quad \text{for all i,j} \in \{1,2,...,n\}. \tag{13}$$

Proposition 7. Let $\boldsymbol{A} = \{a_{ij}\}$ be an a-reciprocal $n\times n$ matrix with $0 < a_{ij} < 1$ for all $i,j\in\{1,2,...,n\}$. $\boldsymbol{A}$ is a-consistent if and only if there exists a vector $\boldsymbol{v} = (v_1, v_2,...,v_n)$ with $v_i > 0$ for all $i\in\{1,2,...,n\}$, and $\sum_{i=1}^{n} v_i = 1$ such that:

$$a_{ij} = \frac{v_i}{v_i + v_j} \quad \text{for all i,j} \in \{1,2,...,n\}. \tag{14}$$

An associated result can be derived also for a-transitive matrices.

Proposition 8. Let $\boldsymbol{A} = \{a_{ij}\}$ be an a-reciprocal $n\times n$ matrix with $0 < a_{ij} < 1$ for all $i,j \in\{1,2,...,n\}$. $\boldsymbol{A}$ is a-transitive if and only if there exists a vector $\boldsymbol{u}$ $= (u_1, u_2,...,u_n)$ with $u_i > 0$ for all $i \in\{1,2,...,n\}$, and $\sum_{i=1}^{n} u_i = 1$ such that:

$$a_{ij} = \tfrac{1}{2}(1+nu_i - nu_j) \quad \text{for all } i,j\in\{1,2,...,n\}. \tag{15}$$

The proof of this proposition is based on the observation that for a-transitive matrix $\boldsymbol{A} = \{a_{ij}\}$ we have:

Setting for all $i \in\{1,2,...,n\}$, we obtain the required result.

Example 1

Let $X=\{x_1, x_2, x_3, x_4\}$ be a set of 4 alternatives. The preferences on X are described by a positive matrix $\boldsymbol{B} = \{b_{ij}\}$:

$$\boldsymbol{B} = \begin{pmatrix} 0.5 & 0.6 & 0.6 & 0.9 \\ 0.4 & 0.5 & 0.6 & 0.7 \\ 0.4 & 0.4 & 0.5 & 0.5 \\ 0.1 & 0.3 & 0.5 & 0.5 \end{pmatrix}. \tag{16}$$

Here, $\boldsymbol{B}$ is a-reciprocal and a-inconsistent, which may be directly verified by (7), e.g. $b_{12}.b_{23}.b_{31} \neq b_{21}.b_{32}.b_{13}$. At the same time, $\boldsymbol{B}$ is a-intransitive as $b_{12}+b_{23}+b_{31}=1.9 \neq 1.5$. We consider $\sigma = 9$ and calculate:

$$\boldsymbol{E} = \Phi(B) = \begin{pmatrix} 1 & 1.50 & 1.50 & 9.0 \\ 0.67 & 1 & 1.50 & 2.33 \\ 0.67 & 0.67 & 1 & 1 \\ 0.11 & 0.43 & 1 & 1 \end{pmatrix},$$

$$\boldsymbol{F} = \varphi_{\sigma}(B) = \begin{pmatrix} 1 & 1.55 & 1.55 & 5.80 \\ 0.64 & 1 & 1.55 & 2.41 \\ 0.64 & 0.64 & 1 & 1 \\ 0.17 & 0.42 & 1 & 1 \end{pmatrix}.$$

We calculate $\rho(\boldsymbol{E}) = 4.29$, $\rho(\boldsymbol{F}) = 4.15$, then we obtain $CR_{ac}(\boldsymbol{B}) = 0.11>0.1$ with the priority vector $\boldsymbol{w}^{ac} = (0.47;\ 0.25;\ 0.18;\ 0.10)$, which gives the ranking of alternatives: $x_1 > x_2 > x_3 > x_4$. Similarly, $CR_{at}^{9}(\boldsymbol{B}) = 0.056 < 0.1$ with the priority vector $\boldsymbol{w}^{at} =(0.44;0.27;0.18;0.12)$, with the same ranking of alternatives: $x_1 > x_2 > x_3 > x_4$.

As it is evident, a-consistency ratio $CR_{ac}(\boldsymbol{B})$ is too high for the matrix $\boldsymbol{B}$ to be considered a-consistent. On the other hand, a-transitivity ratio $CR_{at}^{9}(\boldsymbol{B})$ is sufficiently low for the matrix $\boldsymbol{B}$ to be considered a-transitive. The ranking of the alternatives given by both methods remains, however, the same.

In this example we can see that the values of consistency ratio and transitivity ratio can be different for an a-reciprocal matrix. In order to investigate a possible relationship between the inconsistency/in-transitivity indices, we performed a simulation experiment with randomly generated 1000 a-reciprocal matrices, (n=4 and n=15). Then we calculated the corresponding consistency and transitivity indexes. Numerical experiments show that there is no strong relationship between a-consistency and a-transitivity.

4 Fuzzy preference matrix with missing elements

In many decision-making procedures we assume that experts are capable of providing preference degrees between any pair of possible alternatives. However, this may not always be true, which creates a missing information problem. A missing value in a fuzzy preference matrix is not equivalent to a lack of preference of one alternative over another. A missing value can be the result of the incapacity of an expert to quantify the degree of preference of one alternative over another. In this case he/she may decide not to guess the preference degree between some pairs of alternatives. It must be clear that when an expert is not able to express a particular value a_{ij}, because he/she does not have a clear idea of how the alternative x_i is better than the alternative x_j, this does not mean that he/she prefers both options with the same intensity. The DM could be also bored by evaluating too many pairs of alternatives. To model these situations, in the following we introduce the incomplete preference relation matrix. Here, we use a different approach and different notation than Alonso (2008).

Let $n > 2$, $I = \{1,2,...,n\}$ be a set of indices, $I^2 = I \times I$ the Cartesian product of I, i.e. $I^2 = \{(i,j) | i,j \in I\}$. Here, we assume that the reciprocity condition is satisfied. Therefore, we shall consider only a-reciprocal fuzzy preference matrices.

Let $L \subseteq I^2$, $L = \{(i_1,j_1),(i_2,j_2),...,(i_q,j_q)\}$ be the set of pairs (i,j) of indices such that there exists a pairwise comparison value a_{ij}, $0 \le a_{ij} \le 1$. By L' we denote the symmetric subset to L, i.e. $L' = \{(j_1,i_1),(j_2,i_2),...,(j_q,i_q)\}$. By reciprocity, each subset $K \subseteq I^2$ of the given elements can be expressed as follows

$$K = L \cup L' \cup D, \tag{17}$$

where L is a set of pairs of indices (i,j) of the evaluated elements a_{ij} and D is the diagonal of the fuzzy preference matrix, $D = \{(1,1),(2,2),...,(n,n)\}$, here a_{ii} = 0.5 for all i. The elements a_{ij} with $(i,j) \in I^2 - K$ are called *missing elements*.

Now we define the fuzzy preference matrix $\mathbf{B}(\mathrm{K}) = \{\mathrm{b}_{\mathrm{ij}}\}_{\mathrm{K}}$ with missing elements by:

$$b_{ij} = \begin{cases} a_{ij} & \text{if} \quad (i,j) \in K, \\ - & \text{if} \quad (i,j) \notin K. \end{cases}$$

Here, the missing elements of the matrix $\boldsymbol{B}(K)$ are denoted by a dash "-". On the other hand, the elements evaluated by the experts are denoted by a_{ij} where $(i,j) \in K$. By a-reciprocity, it is sufficient that the expert quantifies only elements a_{ij} where $(i,j) \in L$, such that $K = L \cup L' \cup D$; the other elements are calculated automatically by (3). In what follows we shall investigate a particular important case of L, namely, $L = \{(1,2);(2,3);...,(n-1,n)\}$.

5 Extension of fuzzy preference matrix with missing elements and its consistency/transitivity

In this section we shall deal with the problem of finding the values of missing elements of a given fuzzy preference matrix so that the extended matrix is as much a-consistent/a-transitive as possible. In the ideal case the extended matrix will become a-consistent/a-transitive. We start with the a-consistency property.

Let $K \subseteq I^2$, let $\boldsymbol{B}(K) = \{b_{ij}\}_K$ be a fuzzy preference matrix with missing elements. The matrix $\boldsymbol{B}^{ac}(K) = \{ b_{ij}^{ac} \}_K$ called an *ac-extension of* $\boldsymbol{B}(K)$ is defined as follows:

$$b_{ij}^{ac} = \begin{cases} b_{ij} & \text{if} \quad (i,j) \in K, \\ \dfrac{v_i^*}{v_i^* + v_j^*} & \text{if} \quad (i,j) \notin K. \end{cases} \tag{18}$$

Here, $v^* = (v_1^*, v_2^*, ..., v_n^*)$, called the *ac-priority vector with respect to K*, is the optimal solution of the following optimization problem:

$$(\mathbf{P_{ac}}) \qquad d_{ac}(\boldsymbol{v},K) = \sum_{(i,j)\in K} (b_{ij} - \frac{v_i}{v_i + v_j})^2 \to \min; \tag{19}$$

subject to:

$\sum_{j=1}^{n} v_j = 1$, $v_i \geq \varepsilon > 0$ for all i=1,2,...,n

($\varepsilon > 0$ is a given sufficiently small number).

Note that the a-consistency index of the matrix $\boldsymbol{B}^{ac}(K) = \{ b_{ij}^{ac} \}_K$ is defined by (15) as $I_{ac}(\boldsymbol{B}^{ac}(K))$. The following proposition follows directly from Proposition 7.

Proposition 9. $\boldsymbol{B}^{ac}(K) = \{ b_{ij}^{ac} \}_K$ is a-consistent, i.e. $I_{ac}(\boldsymbol{B}^{ac}(K)) = 0$ if and only if $d_{ac}(\boldsymbol{v},K) = 0$.

Now, we look for the values of missing elements of a given fuzzy preference matrix so that the extended matrix is as much a-transitive as possible. In the ideal case the extended matrix will become a-transitive.

Again, let $K \subseteq I^2$, $\boldsymbol{B}(K) = \{b_{ij}\}_K$ be a fuzzy preference matrix with missing elements.

The matrix $\mathbf{B}^{\mathbf{at}}(\mathrm{K}) = \{ b_{ij}^{at} \}_{\mathrm{K}}$ called an at-extension of $\mathbf{B}(\mathrm{K})$ is defined as follows:

$$b_{ij}^{at} = \begin{cases} b_{ij} & \text{if} \quad (i,j) \in K, \\ \max\{0, \min\{1, \tfrac{1}{2}(1 + nu_i^* - nu_j^*)\}\} & \text{if} \quad (i,j) \notin K. \end{cases} \tag{20}$$

Here, $\boldsymbol{u}^* = (u_1^*, u_2^*, \ldots, u_n^*)$ called the *at-priority vector with respect to K* is the optimal solution of the following optimization problem:

$$(\mathbf{P_{at}}) \qquad d_{at}(\boldsymbol{u},K) = \sum_{(i,j)\in K} (b_{ij} - \tfrac{1}{2}(1 + nu_i - nu_j))^2 \to \min; \tag{21}$$

subject to:

$\sum_{j=1}^{n} u_j = 1$, $u_i \geq \varepsilon > 0$ for all i=1,2,...,n

($\varepsilon > 0$ is a given sufficiently small number).

In general, the optimal solution $\boldsymbol{u}^* = (u_1^*, u_2^*, \dots, u_n^*)$ of (P_{at}) does not satisfy the following condition:

$$0 \le \tfrac{1}{2}(1 + nu_i^* - nu_j^*) \le 1, \tag{22}$$

i.e. $\boldsymbol{B} = \{b_{ij}\} = \{\tfrac{1}{2}(1 + nu_i - nu_j)\}$ fails to be a fuzzy preference matrix. That is why in the definition of the at-extension of $\boldsymbol{B}(K)$ we use formula (20) ensuring that all the elements b_{ij} belong to the unit interval [0;1]. In the next section we shall derive the necessary and sufficient conditions for (22) to be satisfied.

Note that the a-transitivity index (for a given $\sigma > 1$) of the matrix $\boldsymbol{B}^{at}(K) = \{ b_{ij}^{at} \}_K$ is defined by (11) as $I_{at}^{\sigma}(\boldsymbol{B}^{at}(K))$. The next proposition follows directly from Proposition 8.

Proposition 10. Let $\sigma > 1$. If $\boldsymbol{B}^{at}(K) = \{ b_{ij}^{at} \}_K$ is a-transitive, i.e. $I_{at}^{\sigma}(\boldsymbol{B}^{ac}(K)) = 0$, then $d_{at}(\boldsymbol{v},K) = 0$.

6 A particular case of fuzzy preference matrix with missing elements

For a complete definition of a reciprocal fuzzy preference $n\times n$ matrix we need $N = \dfrac{n(n-1)}{2}$ pairs of elements to be evaluated by an expert. For example, if n=10, then N=45, which is a considerable number of pairwise comparisons. In practice we ask that the expert evaluates only around n pairwise comparisons of alternatives which seems a reasonable number. In this section we shall deal with an important particular case of fuzzy preference matrix with missing elements where the expert should evaluate only n-1 pairwise comparisons of elements.

Let $K \subseteq I^2$ be a set of indexes given by an expert, $\boldsymbol{B}(K) = \{b_{ij}\}_K$ be a fuzzy preference matrix with missing elements. Moreover, let $K = L \cup L' \cup D$. In fact, it is sufficient that the expert evaluates matrix elements only from L.

Here, we assume that the expert evaluates the following n-1 elements of the fuzzy preference matrix $\boldsymbol{B}(K)$: $b_{12}, b_{23}, \dots, b_{n-1,n}$.

First, we investigate the ac-extension of $\boldsymbol{B}(K)$. We obtain the following result.

Proposition 11. Let $L=\{(1,2);(2,3);\dots;(n-1,n)\}$, $0 < b_{ij} < 1$ with $b_{ij} + b_{ji} = 1$ for all $(i,j) \in L$, let $K = L \cup L' \cup D$, and $L' = \{(2,1);(3,2);\dots;(n, n-1)\}$, $D = \{(1,1),\dots,(n,n)\}$. Then the ac-priority vector $v^* = (v_1^*, v_2^*, \dots, v_n^*)$ with respect to K is given as:

$$v_1^* = \frac{1}{S}, \tag{23}$$

$$v_{i+1}^* = a_{i,i+1} v_i^*, \text{ for } i=1,2,..,n-1, \tag{24}$$

where:

$$a_{ij} = \frac{1-b_{ij}}{b_{ij}} \text{ for all } (i,j) \in L \text{ and:} \tag{25}$$

$$S = 1 + \sum_{i=1}^{n-1} a_{i,i+1} a_{i+1,i+2} \cdot \ldots \cdot a_{n-1,n} \,. \tag{26}$$

Remark. The proof of Proposition 11 is straightforward by using (25), (26) and the optimal solution of (19). By Proposition 7 it follows that the ac-extension of $\boldsymbol{B}(K)$, i.e. the matrix $\boldsymbol{B^{ac}}(K)=\{ b_{ij}^{ac} \}_K$ is a-consistent.

Now, we investigate the at-extension $\boldsymbol{B^{at}}(K)$ of $\boldsymbol{B}(K)$. We obtain the following result.

Proposition 12. Let $L=\{(1,2);(2,3);...,(n-1,n)\}$, $0 < b_{ij} < 1$ with $b_{ij} + b_{ji} = 1$ for all $(i,j) \in L$, let $K = L \cup L' \cup D$. Then the at-priority vector $\boldsymbol{u}^* = (u_1^*, u_2^*, \ldots, u_n^*)$ with respect to K is given as:

$$u_i^* = \frac{2}{n^2} \sum_{j=1}^{n-1} \alpha_j - \frac{2}{n} \alpha_{i-1} - \frac{n-i-1}{n} \quad \text{for } i=1,2,...,n, \tag{27}$$

where:

$$\alpha_0 = 0,\ \alpha_j = \sum_{i=1}^{j} b_{i,i+1} \quad \text{for } j=1,2,...,n-1. \tag{28}$$

Remark. The proof of Proposition 12 is straightforward by using (27), (28) and the optimal solution of (21). In general, the optimal solution $\boldsymbol{u}^* = (u_1^*, u_2^*, \ldots, u_n^*)$ of (P_{at}) does not satisfy the condition:

$$0 \le \tfrac{1}{2}(1 + nu_i^* - nu_j^*) \le 1, \text{ for all } i,j=1,2,...,n, \tag{29}$$

i.e. $\boldsymbol{B} = \{b_{ij}\} = \{\tfrac{1}{2}(1 + nu_i - nu_j)\}$ is not a fuzzy preference matrix. We can easily prove the necessary and sufficient condition for satisfying (29) based on evaluations $b_{i,i+1}$.

Proposition 13. Let $L=\{(1,2);(2,3);...,(n-1,n)\}$, $0 \le b_{ij} \le 1$ with $b_{ij} + b_{ji} = 1$ for all $(i,j) \in L$, let $K = L \cup L' \cup D$. Then the at-extension $\boldsymbol{B^{at}}(K)=\{ b_{ij}^{at} \}_K$ is a-transitive if and only if:

$$\left| \sum_{k=i}^{j-1} b_{k,k+1} - \frac{j-i}{2} \right| \le \frac{1}{2} \quad \text{for } i=1,2,...,n-1,\ j=i+1,...,n. \tag{30}$$

The proof of Proposition 13 follows directly from Proposition 8 and Proposition 12.

Example 2

Let L={(1,2);(2,3);(3,4)}, let expert evaluations be b_{12}=0.9, b_{23}=0.8, b_{34}=0.6, with $b_{ij}+b_{ji}=1$ for all $(i,j) \in L$, $K = L \cup L' \cup D$. Then $\boldsymbol{B}(K) = \{b_{ij}\}_K$ is a fuzzy preference matrix with missing elements as follows:

$$\boldsymbol{B}(K) = \begin{pmatrix} 0.5 & 0.9 & - & - \\ 0.1 & 0.5 & 0.8 & - \\ - & 0.2 & 0.5 & 0.6 \\ - & - & 0.4 & 0.5 \end{pmatrix}.$$

Solving (P_{ac}) we obtain the ac-priority vector $\boldsymbol{v}$* with respect to K, in particular, $v^* = (0{,}864; 0{,}096; 0{,}024; 0{,}016)$. By (20) we obtain $\boldsymbol{B}^{ac}(K)$ - the ac-extension of $\boldsymbol{B}(K)$ as follows:

$$\boldsymbol{B}^{ac}(K) = \begin{pmatrix} 0.5 & 0.9 & 0.97 & 0.98 \\ 0.1 & 0.5 & 0.8 & 0.86 \\ 0.03 & 0.2 & 0.5 & 0.6 \\ 0.02 & 0.14 & 0.4 & 0.5 \end{pmatrix}.$$

By Proposition 9, $\boldsymbol{B}^{ac}(K)$ is a-consistent, hence $I_{ac}(\boldsymbol{B}^{ac}(K)) = 0$. Solving ($P_{at}$) we obtain the at-priority vector $\boldsymbol{u}$* with respect to K as follows:

$u^* = (0{,}487; 0{,}287; 0{,}137; 0{,}088)$. By (27) we obtain $\boldsymbol{B}^{at}(K)$ – the at-extension of $\boldsymbol{B}(K)$ as follows:

$$\boldsymbol{B}^{at}(K) = \begin{pmatrix} 0.5 & 0.9 & 1.0 & 1.0 \\ 0.1 & 0.5 & 0.8 & 0.9 \\ 0.0 & 0.2 & 0.5 & 0.6 \\ 0.0 & 0.1 & 0.4 & 0.5 \end{pmatrix},$$

where, by Proposition 10, $\boldsymbol{B}^{at}(K)$ is a-intransitive, as $d_{ac}(\boldsymbol{v},K) > 0$. In particular, $I^{9}_{at}(\boldsymbol{B}^{ac}(K))$= 0.057.

7 Conclusions

In this paper we have dealt with some properties of fuzzy preference relations, in particular with reciprocity, consistency and transitivity of relations given in the form of square nonnegative matrices. We have shown how to measure the degree of consistency and/or transitivity, and also how to extend crisp comparisons to fuzzy ones, i.e. how to evaluate pairs of elements by fuzzy values. Also, we have proposed a new method for measuring inconsistency based on Saaty´s principal eigenvector method. Moreover, we have dealt with the problem of the incomplete fuzzy preference matrix, where some elements of pairwise comparison are missing. We have proposed a special method for dealing with that case. Some illustrating examples have been presented to clarify the theory proposed.

Acknowledgements

The research work was supported by the Czech Science Foundation (GACR), grant No. 402/09/0405.

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MULTIPLE CRITERIA DECISION MAKING IN THE VALUATION OF REAL OPTIONS

Abstract

Traditional project evaluation is based on discounted cash flow method (DCF) with Net Present Value (NPV) as the main measure. This approach sometimes leads to the abandonment of profitable projects, because the DCF method does not take into account the role of managerial flexibility. The Real Options Valuation (ROV) method takes into account future situations in the valuation, assuming that the project is properly managed. The Project Manager shall have the right to take action as appropriate.

A widely used method for the valuation of real options is the binomial tree method (CRR), proposed by Cox, Ross and Rubinstein. It takes into account one state variable. In many real problems, however, many factors should be considered. This leads to a multi-criteria decision-making problem. This paper presents an extension of the CRR method for several state variables.

Keywords: project management, real options, dynamic programming

1 Introduction

The term real option was proposed by S. Myers (1974, p. 1-25), who noted similarities between financial options and opportunities that arise in project management. An option can be defined as a right, but not an obligation, which means that the holder of that right can determine when to exercise it, depending on the current market situation. This approach was then developed by A.K. Dixit and R.S. Pindyck (1994), and was later discussed by L. Trigeorgis (1993, p. 202-224). The most important element in the Real Options Analysis (ROA) is the valuation (ROV – Real Option Valuation). In ROV methods known from

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financial market were used first, such as the Black-Scholes model (Black, Scholes, 1973, p. 637-654) or the Cox-Ross-Rubinstein model (CRR)(Cox, Ross, Rubinstein, 1979, p. 229-263). Also used was an approach based on Monte Carlo simulations (Boyle, 1977, p. 323-338). The CRR model is based on the binomial tree. This approach was also adopted in the book by Guthrie (2009) on which this study is based.

Traditionally, project evaluation is based on the discounted cash flow method (DCF) with Net Present Value (NPV) as the main measure of effectiveness. When this value is positive, the project is approved, when it is negative, the project is rejected. This approach sometimes leads to the abandonment of profitable projects. The reason for this is that the DCF method does not take into account the role of managerial flexibility. The Project Manager shall have the right to take action as appropriate. This situation is called a real option. Using ROV, we can provide a quantitative measurement of this situation.

The traditional approach in the valuation of Real Options is based on a single factor called the state variable. There are also attempts to take into account many state variables. The first attempt, based on financial options, was made by Boyle (1988, p.1 - 12), who took into account two assets. Mun (2010) described a commercial solution with such possibilities. Guthrie (2009, p. 403) also described problems for which it is necessary to take into consideration several of variables.

This paper presents problems in Real Options valuation with many state variables, which lead to issues considered in multiobjective analysis. This paper presents such multi-criteria problems. The first section presents the Defer Options that may arise in project management. The next section describes a multi-criteria approach in Real Options valuation. The last section is a numerical example.

2 Problem formulation

Many project management methodologies recommend the division of the project into stages. This raises the problem of decision-making, consisting in the choice of the start time of the next steps. For example, in the PRINCE2 (Office of Government Commerce, 2009) methodology, each project must have at least two stages: initialization and actual implementation. We will consider a project consisting of these two phases, each of which takes a period of time. After the initialization of a project we have to decide when to begin its implementation. We can delay the execution by one period. This is a classic example of the defer options, considered by Ingersoll and Ross (1992, p.1-29).

If planned project is static, the decision maker is not able to react to changes in the environment and in the project itself. If only the duration of the project can be extended and the decision maker is allowed to decide freely about the start times of the consecutive stages, a completely new situation arises, which is presented in Figure 1. The decision maker may start the project (decision ***A***), then move from the current state (***initial*** of project) to the last state (***end*** of project). The decision maker may wait (decision ***W***), but then the project will remain in the starting state.

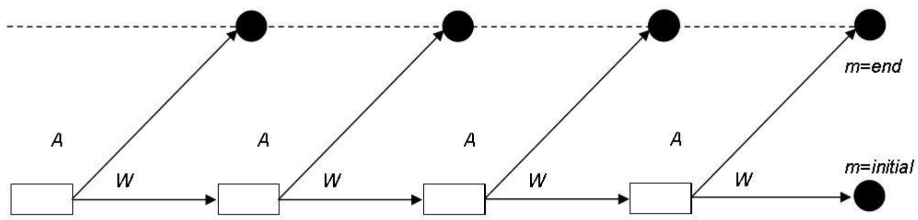

Figure 1. The decision tree

The results of the project, as well as its value, depend on certain factors. If we consider more than one factor that lead to usability design considerations in many areas, the problem is converted from a simple valuation to a multi-criteria evaluation problem.

Decisions are made based on the observation of the change of the factors. These factors vary stochastically according to a certain random process. The idea behind the CRR method is to cover a possible future state variable binomial tree as shown in Figure 2. It meets a role of scenario possible changes in the value of the state variable. At each stage, we consider only the possibility of an increase or decrease in value. This procedure simplifies the decision-making process.

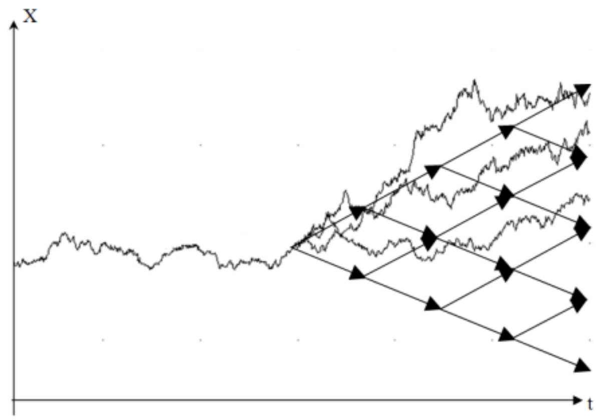

Figure 2. The binomial tree covering a stochastic process

Guthrie (2009, p. 168) considers one-factor models. This paper expands these discussion, by binding results of the project with two factors. There are used methods of multiobjective dynamic programming.

3 Method of evaluation

Each of the two factors called state variables, may be increased *u*-times and fall *d*-times in each period. This assumption leads to the tree of possible state variable values, which consists of nodes marked with indices (*i*, *n*) where *i* is the number of falls and *n* is the period number.

With each node a state variable and cash flow are connected. We denote it by:

- $X_k(i, n)$ – *k-th* state variable in period *n*
- $Y_m(i, j, n)$ – cash flow in period *n* (where *m* is state of project).

Given are: the number of periods *N*, the present value of each state variable $X_k(0,0)$, and also coefficients *u*, *d*. The value of *u* can be obtained from historical data by the calibration procedure (Guthrie, 2009, p. 263).

The proposed procedure consists of the following steps:

Step 1 – Build the decision tree (D-Tree)

We identify the possible states of the project, which may be different phases or specific stages. We also identify the possible decisions that we consider. Taking such a decision leads to a transition from one state to another. Finally, we identify all possible transitions. The result of this step is a D-Tree, shown in Figure 1.

Step 2 – Build the lattice of state variables (X-Tree)

We identify quantifiable economic magnitudes, on which the result of the project may depend (state variables). The method currently proposed does not include the correlation between these variables (we assume that such correlation does not exist).

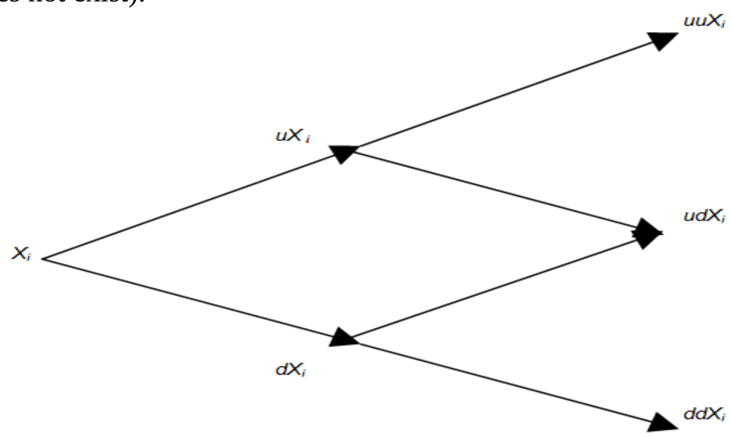

Figure 3. The binomial tree of state variables

The tree starts from a known present value of the state variables. Based on the history of changes of this magnitude, the values *u* and *d* can be determined (this is done in the calibration process – Guthrie, 2009, p. 324). A tree of possible changes of state variable is built, as a possible scenario of the situation; it is presented in Figure 3. Calibration it is an appropriate selection of the number of steps and the choice of parameters (*d*, *u*), so as to best meet the future value of the variable state.

Step 3 – Build the tree of the project values (V-Tree)

The calculation of the V-Tree is based on the principle of optimality, formulated in Bellman's paper (Bellman, 1957). In our case, where decisions are made based on more than one factor, we use the multiobjective dynamic programmming principle of optimality, where we want to find the set of noninferior (efficient) solutions. In our case this principle can be formulated as follows (Li, Haimes, 1989, p. 471-483):

"Each noninferior control sequence has the property that, whatever the initial state, the existing control subsequence must constitute noninferior policy with respect to this initial state".

The application of this principle leads to backward induction in which we consider the sets of efficient (noninferior or nondominated) solutions, in this case the values at the *k*-1 stage:

$$\{V(k-1)\} = \sup_{d_k}\left\{e^{-r\Delta t_m} E\left[V\ (k, d_k)\right]\right\} \quad k = n,...,1 \tag{1}$$

In our case we consider a project evaluation based on many state variables, which is therefore presented as a vector of values:

$$\mathbf{V}_m(i^1, i^2, ..., i^s, n) = \begin{bmatrix} f_1(X_1(i^1, i^2, ..., i^s, n), V_1(i^1, i^2, ..., i^s, n+1)) \\ f_2(X_2(i^1, i^2, ..., i^s, n), V_2(i^1, i^2, ..., i^s, n+1) \\ ... \\ f_s(X_s(i^1, i^2, ..., i^s, n), V_s(i^1, i^2, ..., i^s, n+1) \end{bmatrix} \tag{2}$$

Since we consider two state variables, the resulting V-Tree grows in two dimensions. We denote the present value of the project, which is dependent on two state variables, by:

$V_{(s)}(i, j, n)$ – utility value of project in period n,

where:

i – number of falls of first state variable,

j – number of falls of second state variable.

The calculation of the V-Tree starts from the project results in a final. We assume that the final value of the project is a function of state variables:

$$\mathbf{V}^m(i,j,n)=\begin{bmatrix} f_1^e(X_1(i,j,n)\ ) \\ f_2^e(X_2(i,j,n)\ ) \end{bmatrix} \tag{3}$$

On their basis, moving from the final set, we calculate the value of the project in previous nodes. Trees are constructed for each state project. The calculation of the value is done by backward induction. Knowing the value of the project after its completion (which is usually equal to the state variable or can be calculated using the correct formula for this variable), we calculate the values of the project in the preceding nodes.

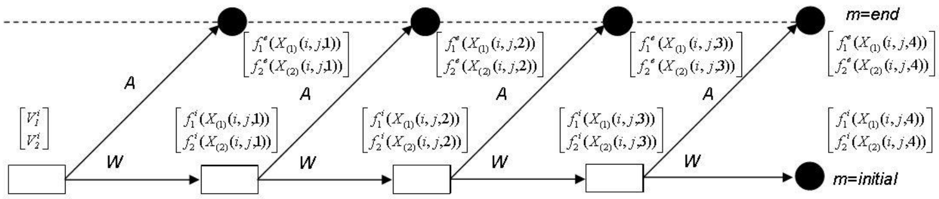

Figure 4. The V-Tree

We will consider two possibilities:

- the case of financial factors, when the present value is the discounted expected value of subsequent values:

$$V_1\ (i,j,n)=\left(\pi_u^1\cdot V_1(i,j,n+1)+\pi_d^1\cdot V_1(i+1,j,n+1)\ \right)e^{-r\Delta t} \tag{4}$$

$$V_2\ (i,j,n)=\left(\pi_u^2\cdot V_2(i,j,n+1)+\pi_d^2\cdot V_2(i,j+1,n+1)\ \right)e^{-r\Delta t} \tag{5}$$

- the case of other factors, when the present value is the expected value of subsequent values:

$$V_1\ (i,j,n)=\pi_u^1\cdot V_1(i,j,n+1)+\pi_d^1\cdot V_1(i+1,j,n+1) \tag{6}$$

$$V_2\ (i,j,n)=\pi_u^2\cdot V_2(i,j,n+1)+\pi_d^2\cdot V_2(i,j+1,n+1) \tag{7}$$

Subsequent values are weighted by the probability of achieving those values. If we denote by *r* the risk free interest rate, we can calculate them from the formulas (Seydel, 2009, p. 15):

$$\pi_u^l=\frac{e^{r\Delta t}-d^l}{u^l-d^l} \tag{8}$$

for the growth of the l-th state variable,

$$\pi_d^l = \frac{u^l - e^{r\Delta t}}{u^l - d^l} \tag{9}$$

for the fall of the l-th state variable in the case of using Geometric Brownian Motion (GBM) and

$$\pi_u^l = \frac{e^{r\Delta t} E\big(X_l(-, n+1)\big) - X_l(i+1, n+1)}{X_l(i, n+1) - X_l(i+1, n+1)} \tag{10}$$

for the growth of the l-th state variable,

$$\pi_d^l = 1 - \pi_u^l \tag{11}$$

for the fall of the l-th state variable in the case of using Brownian Motion (BM) (G. Guthrie, 2009, p. 280).

Step 4 – Determine effective transitions (decisions)

The presented procedure allows not only to determine the possible cash flow, but also to identify the best decisions. As we are in the area of multi-criteria decision analysis, these will not be the optimal decisions but only the effective ones. The best decision is one for which we obtain the 'supremum' of the discounted expected value:

$$d_k^* = \arg\sup_{d_k} \left\{ e^{-r\Delta t_m} E\left[V\,(k, d_k)\right] \right\} \quad k = n, \ldots, 1 \tag{12}$$

In the multicriteria case under consideration, we can obtain a set of efficient decisions for which there is no worse decision at any stage:

$$\neg\exists d_k \quad e^{-r\Delta t_m} E\left[V\,(k, d_k)\right] \succ e^{-r\Delta t_m} E\left[V\,(k, d_k^*)\right] \quad k = n, \ldots, 1 \tag{13}$$

In our case, we have at each stage two decisions as shown in Figure 1:

$$\mathrm{m} = \{\mathrm{A}, \mathrm{W}\} \tag{14}$$

and also two criteria values:

- V_1 – dependent on first state value (X_1),
- V_2 – dependent on second state value (X_2).

Using the scalarization approach (Trzaskalik, 1988, p. 64), we can simplify the calculation to a simple comparison of the two values obtained as the sum of the weighted components of the vector **V**.

The values determined by (4)-(7) must be calculated for each decision, so we introduce a superscript denoting the relevant decision:

$$\mathbf{V}^A(i,j,n) = \begin{bmatrix} V_1^A(i,j,n) \\ V_2^A(i,j,n) \end{bmatrix} \tag{15}$$

for the decision **<u>A</u>ct** and

$$\mathbf{V}^W(i,j,n) = \begin{bmatrix} V_1^W(i,j,n) \\ V_2^W(i,j,n) \end{bmatrix} \tag{16}$$

for the decision **<u>W</u>ait**.

Because we assume that the project is properly managed, a favorable decision will be chosen.

$$\begin{aligned} &\text{IF } \mathbf{V}^W(i,j,n) \succ \mathbf{V}^A(i,j,n) \quad \text{THEN } \mathit{Wait}; \mathbf{V}^m = \mathbf{V}^W \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{ELSE } \mathit{Act}; \mathbf{V}^m = \mathbf{V}^A \end{aligned} \tag{17}$$

The only problem is to determine the preferred vector evaluation. Using the scalarization approach (Trzaskalik, 1988, p. 64), we denote:

$$Q^W(i,j,n) = \sum_l w_l V_l^W(i,j,n) \tag{18}$$

$$Q^A(i,j,n) = \sum_l w_l V_l^A(i,j,n) \tag{19}$$

where: $w_l \geq 0$ are weights of assessments, hence the calculations are simplified to:

$$\begin{aligned} &\text{IF } Q^W(i,j,n) > Q^A(i,j,n) \quad \text{THEN } \mathit{Wait} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{ELSE } \mathit{Attempt} \end{aligned} \tag{20}$$

which gives not only effective decisions but also allows to calculate the value of the cash flow associated with the project.

4 Numerical example

We will consider a three-month project of social nature. The project may begin in any quarter of 2013. After its implementation financing in the amount of 10 million euros will be obtained. Costs were estimated at 41 million PLN. In addition, if the project proves to be purposeful further co-operation of the financing institution will be possible. The purposefulness the project depends on the development of the level of unemployment. If it remains high, the project will be deemed purposeful. If the unemployment rate drops, its implementation will be useless.

X-Trees determined on the basis of the observations of variables in 2012 are presented in Table 1 for the exchange rate EUR/PLN and Table 2 for the level of unemployment. For the first state variable we use GBM, for the second, BM.

Table 1

X-Tree for EUR/PLN exchange rate

X_1	n				
i	0	1	2	3	4
0	4,0946	4,2727	4,4585	4,6525	4,8548
1		3,9239	4,0946	4,2727	4,4585
2			3,7604	3,9239	4,0946
3				3,6036	3,7604
4					3,4534

Table 2

X-Tree for unemployment rate

X_2	n				
i	0	1	2	3	4
0	10,4	10,7	11,0	11,3	11,6
1		10,1	10,4	10,7	11,0
2			9,8	10,1	10,4
3				9,5	9,8
4					9,2

In the following tables the final values obtained by the project are shown. The final value for the first state value is calculated as project profit :

$$f_1^e(X_1(i,j,n)\ ) = M \cdot X_1(i,j,n) - K, \tag{21}$$

where:

M = 10 M EUR
K = 41 M PLN

The final value of the second state value is calculated as project utility. If unemployment is greater than 10%, this value is 100, otherwise it is 0:

$$f_2^e(X_2(i,j,n)\ ) = \begin{cases} 100 & X_2(i,j,n) > 10{,}0\% \\ 0 & X_2(i,j,n) <= 10{,}0\% \end{cases} \tag{22}$$

The calculated final values are presented in Tables 3 to 6.

Table 3

Final values for n = 4

$(f_1^e; f_2^e)$	n = 4				
i, j	0	1	2	3	4
0	(7,5; 100)	(7,5; 100)	(7,5; 100)	(7,5; 0)	(7,5; 0)
1	(3,6; 100)	(3,6; 100)	(3,6; 100)	(3,6; 0)	(3,6; 0)
2	(-0,1; 100)	(-0,1; 100)	(-0,1; 100)	(-0,1; 0)	(-0,1; 0)
3	(-3,4; 100)	(-3,4; 100)	(-3,4; 100)	(-3,4; 0)	(-3,4; 0)
4	(-6,5; 100)	(-6,5; 100)	(-6,5; 100)	(-6,5; 0)	(-6,5; 0)

Table 4

Final values for n = 3

$(f_1^e; f_2^e)$	n = 3			
i, j	0	1	2	3
0	(5,5;100)	(5,5; 100)	(5,5; 100)	(5,5; 0)
1	(1,7; 100)	(1,7; 100)	(1,7; 100)	(1,7; 0)
2	(-1,8; 100)	(-1,8; 100)	(-1,8; 100)	(-1,8; 0)
3	(-5,0; 100)	(-5,0; 100)	(-5,0; 100)	(-5,0; 0)

Table 5

Final values for n = 2

$(f_1^e; f_2^e)$	n = 2		
i, j	0	1	2
0	(3,6; 100)	(3,6; 100)	(3,6; 0)
1	(-0,1; 100)	(-0,1; 100)	(-0,1; 0)
2	(-3,4; 100)	(-3,4; 100)	(-3,4; 0)

Table 6

Final values for n = 1

$(f_1^e; f_2^e)$	n = 1	
i, j	0	1
0	(1,7; 100)	(1,7; 100)
1	(-1,8; 100)	(-1,8; 100)

Using backward induction, from the equations (4)-(7), we calculate the vectors of values for each decision stage. The calculations used the values r = 4%, π_u^1 = 0,3725, π_d^1 = 0,6275, π_u^2 = 0,5 , π_d^2 = 0,5. The results are presented in Tables 7 to 8.

Table 7

Decision values for n = 3

$\left(\mathbf{V}^A(i,j,n)\right)^T$ $\left(\mathbf{V}^W(i,j,n)\right)^T$	n = 3			
i, j	0	1	2	3
0	(5,0; 100)* (0; 100)	(5,0; 100)* (0; 100)	(5,0; 50)* (0; 50)	(5,0; 0)* (0; 0)
1	(1,3; 100)* (0; 100)	(1,3; 100)* (0; 100)	(1,3;50)* (0;50)	(1,3; 0)* (0; 0)
2	(-2,1;100) (0; 100)*	(-2,1; 100) (0; 100)*	(-2,1; 50) (0; 50)*	(-2,1; 0) (0; 0)*
3	(-5,3; 100) (0; 100)*	(-5,3; 100) (0; 100)*	(-5,3; 50) (0; 50)*	(-5,3; 0) (0; 0)*

Table 8

Decision values for n = 2

$\left(\mathbf{V}^A(i,j,n)\right)^T$ $\left(\mathbf{V}^W(i,j,n)\right)^T$	n = 2		
i, j	0	1	2
0	(3,1; 100)* (2,7; 100)	(3,1; 100)* (2,7; 75)	(3,1; 50)* (2,7; 25)
1	(-0,5; 100) (0,5; 100)*	(-0,5; 100)? (0,5; 75)?	(-0,5; 50)? (0,5; 25)?
2	(-3,8; 100) (0; 100)*	(-3,8; 100)? (0; 75)?	(-3,8; 50)? (0; 25)?

The dominant elements are marked with an asterisk. There are no such elements in nodes (1,1), (1,2), (2,1) and (2,2). The definitive decision can be calculated using the preference structure obtained by weights in the scalarization approach. Assume that current revenues are more important than the possibility of implementing a similar project in the future, so we may take w_1 = 0,9 and w_2 = 0,1. Then we have:

Q^A (1,1,2) = 0,9 ·(- 0,5) + 0,1 · 100 = 9,05!

Q^W (1,1,2) = 0,9 · 0,5 + 0,1 · 75 = 7,95

Q^A (1,2,2) = 0,9 ·(- 0,5) + 0,1 · 50 = 4,55!

Q^W (1,2,2) = 0,9 · 0,5 + 0,1 · 25 = 2,95

Q^A (2,1,2) = 0,9 ·(- 3,8) + 0,1 · 100 = 6,58

Q^W (2,1,2) = 0,9 · 0,0 + 0,1 · 75 = 7,5!

Q^A (2,2,2) = 0,9 ·(- 3,8) + 0,1 · 50 = 1,58

Q^W (2,2,2) = 0,9 · 0,0 + 0,1 · 25 = 2,5!

By comparing the calculated values we get the preferred choice. This time the preferred element is marked with an exclamation mark.

Table 9

Decision values for n = 1

$\left(\mathbf{V}^A(i,j,n)\right)^T$ $\left(\mathbf{V}^W(i,j,n)\right)^T$	n = 1	
i, j	0	1
0	(1,39; 100)* (1,14; 100)	(1,39; 50)? (0,83; 75)?
1	(-2,15; 100)? (0;91, 24)?	(-2,15;50) (-0,18; 58,7)*

Once again, this time for the nodes (0,1) and (1,0), we calculate the preferred decisions:

Q^A (0,1,1) = 0,9 ·1,39 + 0,1 · 50 = 6,25

Q^W (0,1,1) = 0,9 · 0,83 + 0,1 · 75 = 8,25!

Q^A (1,2,2) = 0,9 ·(- 2,15) + 0,1 · 100 = 8,07

Q^W (1,2,2) = 0,9 · 0 + 0,1 · 91,24 = 9,12!

And finally for n = 0 we have $\left(\mathbf{V}^{A}(i,j,n)\right)^{T}$ = (-0,49; 100) and $\left(\mathbf{V}^{W}(i,j,n)\right)^{T}$ = (0,35; 78,85) which gives us:

Q^{A} (0,0,0) = 0,9 ·(-0,49) + 0,1 · 100 = 9,56

Q^{W} (0,0,0) = 0,9 · 0,35 + 0,1 · 78,85 = 8,2

The best decision is to start the project at the beginning of 2013. Although this approach brings a small loss in the implemented project, it also raises hopes for future profitable projects.

5 Conclusions

The present paper outlines the valuation method of development projects in which real option situations occur. The method proposed takes into account the dependence of the project on two independent random factors, which are called state variables. Our procedure is based on binomial trees and uses a multicriteria dynamic programming method. The numerical example shows the need of computer implementation of the method. The calculations performed are straightforward but tedious.

The method discussed here, as shown in the example, allows not only to make the right decisions about the beginning the project, but also to support decision making during the project’s implementation. It does this by determining the appropriate start times of the subsequent phases.

Acknowledgements

This work is supported by the grant from National Science Centre (NCN), project number NN 111 477740.

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ESTIMATING PRIORITIES IN GROUP AHP USING INTERVAL COMPARISON MATRICES

Abstract

In this paper analytic hierarchy process (AHP), a well-known approach for handling multi-criteria decision making problems, is discussed. It is based on pairwise comparisons. The methods for deriving the priority vectors from comparison matrices are examined. The existing methods for aggregating the individual comparison matrices into a group comparison matrix are revised. A method for aggregation, called WGMDEA, is proposed for application in the case study.

Because exact (crisp) values cannot always express the subjectivity and the lack of information on the part of a decision maker, the interval judgments are more suitable in such cases. Two main methodological problems emerge when dealing with interval comparison matrices in group AHP:

a) to aggregate individual crisp preferences into the joint interval matrix,
b) to calculate the weights from the joint interval comparison matrix.

In the paper we first discuss the already proposed approaches to the aggregation of individual matrices, and the derivation of weights from interval comparison matrices, pertained to AHP group decision making methodology. Then, a new method, ADEXTREME, for generating the interval group judgments from individual judgments is proposed. A numerical example based on Rural Development Program of the Republic of Slovenia in 2007-2013 is presented to illustrate the new methodology for deriving the weights from interval comparison matrices. The results obtained by WGMDEA, MEDINT and ADEXTREME methods are compared.

Keywords: multiple criteria decision making, group decision making, analytic hierarchy process, aggregating individual comparison matrices, interval judgments, deriving the weights from interval comparison matrices, management of natural resources.

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1 Introduction

The analytic hierarchy process (AHP) (Saaty, 1980) is a well-known method for solving multiple criteria problems; it has already been applied to several problems from different domains (Vaidya and Kumar, 2006). AHP is very flexible since it allows for combining empirical data and subjective judgments, and also intangible and non-measurable criteria. Thus, AHP is suitable for evaluating and ranking alternatives, and supporting the selection of the best alternative. It is based on a hierarchical structure of criteria, subcriteria, and alternatives. The pairwise comparisons of objects (criteria, subcriteria, alternatives) on the same level with respect to the object on the next higher level are gathered in a comparison matrix. The 1-9 ratio scale is used to express the strength of preference between the compared objects. The priority vectors are derived from a pairwise comparison matrix by one of the known methods and synthesized in the final priority vector (Saaty, 2006).

As a number of stakeholders (decision makers) who have different goals, or common interests are getting gradually more important when solving the multi-criteria problems, group (participatory) decision making methods replace the single decision maker multi-criteria methods. Among multi-criteria group methods, group AHP gained wide acceptance (Peniwati, 2007). The main problem in group AHP is to aggregate the individual judgments, i.e. the individual comparison matrices, into a group matrix, or individual priority vectors into a group priority vector. Although the group AHP has already been extensively used in practice, the problem of choosing an appropriate aggregation method to aggregate individual judgments (priorities) is still not fully solved. In the literature there are many approaches to aggregation (Bryson and Joseph, 1999; Cho and Cho, 2008; Escobar and Moreno-Jimenez, 2007; Forman and Peniwati, 1998; Grošelj et al., 2011; Hosseinian et al., 2009; Huang et al., 2009; Mikhailov, 2004; Moreno-Jimenez et al., 2008; Ramanathan and Ganesh, 1994; Regan et al., 2006; Sun and Greenberg, 2006), but it is not clear how good they are. In the literature, we seldom find studies comparing several group aggregation methods.

The complexity and uncertainty of the decision problems, the subjectivity, and the lack of information on the part of decision makers can sometimes be hard to express with exact values. Interval judgments can be more suitable in such cases. Another necessity for interval judgments occurs when the group is not satisfied with the aggregated judgment and expresses it with an interval (Arbel and Vargas, 2007; Chandran et al., 2005). If every decision maker in the group provides an interval comparison matrix, their aggregation is even more complicated than in the case of crisp matrices (Entani and Inuiguchi, 2010; Yang et al., 2010). Thus, two main methodological issues emerge when dealing with interval comparison matrices in group AHP:

a) aggregation of individual crisp judgments into the group interval matrix,

b) determination the weights from the interval comparison matrix, what was already studied in many ways (Arbel and Vargas, 2007; Arbel and Vargas, 1993; Conde and de la Paz Rivera Pérez, 2010; Cox, 2007; Lan et al., 2009; Wang et al., 2005a; Wang et al., 2005b).

In the paper we first discuss the already proposed approaches to the aggregation of individual crisp judgments or priorities with respect to AHP group decision making methodology. Aggregation into the group interval matrix is also discussed. Then, a new way of aggregation of individual crisp matrices into the group interval matrix is proposed. A new approach using a modified minimum and maximum method is suggested. Further, the method of generating and ranking weights from the interval comparison matrix is discussed. Finally, a numerical example based on Rural Development Program of the Republic of Slovenia in 2007-2013 is presented (PRP, 2007) to illustrate the new methodology for deriving the weights from interval comparison matrices.

2 AHP and group AHP

2.1 Pairwise comparison matrix and priority vector in AHP for one decision maker

In AHP the decision maker pairwise compares all elements on the same level of the hierarchy (alternatives, subcriteria, criteria) with respect to the element to which they are connected on the next higher level. For comparing two elements a 1-9 Saaty's scale is used (Saaty, 2006). All comparisons are gathered in a pairwise comparison matrix *A*. The matrix $A=(a_{ij})_{n\times n}$ is reciprocal:

$$a_{ij}=\frac{1}{a_{ji}} \quad \text{for } i,j=1,\dots,n \tag{1}$$

The matrix A is acceptably consistent if the consistency ratio CR < 0.1 (Saaty, 2006):

$$CR=\frac{CI}{RI_n}, CI=\frac{\lambda_{max}-n}{n-1} \tag{2}$$

Here n is the dimension of the matrix A resulting from the comparison of n elements and $\lambda_{\max}$ is the maximal eigenvalue of the matrix A. RI_n is the average consistency index (Saaty, 2006) which depends on the size of the matrix A.

In order to calculate the priority (preference) vector ω from the matrix A, several methods can be used (Bryson, 1995; Chandran et al., 2005; Chu et al., 1979; Cook and Kress, 1988; Crawford and Williams, 1985; Gass and Rapcsák, 2004; Mikhailov, 2000; Saaty, 2006; Wang and Chin, 2009; Zahedi, 1986; Zu, 2000). In applications, the eigenvector method (EV) (Saaty, 2006):

$$A\omega=\lambda_{\max}\omega \tag{3}$$

or the logarithmic least-squares method (LLSM) (Crawford and Williams, 1985):

$$\min \sum_{i=1}^{n} \sum_{j>i}^{n} \left(\ln a_{ij} - \left(\ln \omega_i - \ln \omega_j \right) \right)^2 \tag{4}$$

the solution of which is the geometric mean of the row elements of the matrix A:

$$\omega_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}}, \ i = 1, \ldots, n \tag{5}$$

are most often applied.

2.2 Aggregation of individual matrices in group AHP and WGMDEA method

The main problem in group AHP is the aggregation of a set of individual judgments or preferences into the group judgment or preference. This problem is formulated as follows: let m be the number of decision makers and let n be the number of elements (criteria or alternatives) compared with respect to the element on the next higher level; further let $A_k = \left(a_{ij}^{(k)} \right)_{n \times n}$, $k=1,\ldots,m$ be their pairwise comparison matrices, and let $w^{(k)} = \left(w_1^{(k)}, \ldots, w_n^{(k)} \right)$, $k=1,\ldots,m$ be priority vectors derived from A_k; let α_k, $k=1,\ldots,m$, $\alpha_k > 0$, $\sum_{k=1}^{m} \alpha_k = 1$ be the weights of decision makers' importance. A group matrix A^{group} is finally derived from A_k, $k=1,\ldots,m$ with one of the group methods. A group priority vector $w = \left(w_1, \ldots, w_n \right)$ is then derived from A^{group} or directly from A_k, $k=1,\ldots,m$.

For aggregation of individual priorities, weighted arithmetic mean method (WAMM) is normally used (Ramanathan and Ganesh, 1994) but also weighted geometric mean can be used (Forman and Peniwati, 1998).

In WAMM the individual priority vectors are synthesized into the group priority vector $\mathrm{w} = \left(w_1, \ldots, w_n \right)$ using the weighted arithmetic mean:

$$w_i = \sum_{k=1}^{m} \alpha_k w_i^{(k)}, \ \mathrm{i}=1,\ldots,\mathrm{n} \tag{6}$$

The Lehrer-Wagner model, adopted for AHP (Regan et al., 2006), is likewise the WAMM used for the aggregation of individual priorities.

In aggregating individual judgments the weighted geometric mean method (WGMM) is the only method that satisfies several required axiomatic conditions, such as separability, unanimity homogeneity and power conditions (Saaty and Peniwati, 2008).

Using the WGMM the group matrix $A^{group} = A^{WGMM}$ is calculated as:

$$a_{ij}^{(WGMM)} = \prod_{k=1}^{m} \left(a_{ij}^{(k)} \right)^{\alpha_k} \tag{7}$$

To derive the priority vector from A^{WGMM} EV (3) is usually used.

Some other group AHP methods are:

1. Weighted group least-squares method for deriving group priorities which minimizes the weighted Minkowski distance (Sun and Greenberg, 2006).
2. Group method with aggregation on preferential differences and rankings which considers the differences of preference among criteria (or alternatives) and the ranks of the criteria (or alternatives) for each decision maker (Huang et al., 2009).
3. A method of aggregation of individual preference structures (Escobar and Moreno-Jimenez, 2007).
4. The weighted geometric mean DEA method (WGMDEA) (Grošelj et al., 2011) is a group method, based on data envelopment analysis, which uses weighted geometric mean for aggregation of individual judgments and linear programming for deriving the group priority vector. The solution of the linear program (8) for all w_i, $i = 1,...,n$ gives the group priority vector.

$$\begin{aligned} &\max w_0 = \sum_{j=1}^{n} \left(\prod_{k=1}^{m} \left(a_{0j}^{(k)} \right)^{\alpha_k} \right) x_j \\ &subject\ to\ \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \prod_{k=1}^{m} \left(a_{ij}^{(k)} \right)^{\alpha_k} \right) x_j = 1 \\ &\sum_{j=1}^{n} \left(\prod_{k=1}^{m} \left(a_{ij}^{(k)} \right)^{\alpha_k} \right) x_j \geq n x_i,\ i=1,...,n \\ &x_j \geq 0,\ j=1,...,n \end{aligned} \tag{8}$$

In the case study we applied WGMDEA method, since it is easily solved and provides good results as compared to some other group AHP methods (Grošelj et al., 2011).

3 Interval judgments in group AHP

3.1 Generating group interval matrix from individual crisp comparison matrices

Here we present the problem of combining individual judgments into a group interval judgment. Let $A^{group} = \left(\left[l_{ij}, u_{ij} \right] \right)_{n \times n}$ be a group interval matrix, composed of intervals with lower bounds l_{ij} and upper bounds u_{ij}, derived from the individual crisp comparison matrices. Intervals can be constructed using minimum and maximum judgments for the bounds of the intervals (Chandran et al., 2005; Wang et al., 2005b). If there are many intermediate judgments they do not influence the bounds of the intervals. One possibility to overcome this drawback is the MEDINT method (Grošelj and Zadnik Stirn, 2011) which uses values that are lower than the median for constructing the lower bound of the interval and values that are greater than the median for constructing the upper bound of the interval.

3.1.1 Method of aggregation of individual judgments into a group interval judgment using MEDINT method

Let m be the number of decision makers included in the process of evaluating n criteria (or alternatives) in respect to the element on the next higher level. Let $A^{(k)} = \left(a_{ij}^{(k)}\right)_{n\times n}$, $k=1,\dots,n$ be their comparison matrices.

An Ordered Weighted Geometric (OWG) operator (Chiclana et al., 2000) F of dimension m generates a weighting vector $W = \left(w_1,\dots,w_m\right)$ with the properties: $w_i \in [0,1], \sum_{i=1}^{m} w_i = 1$, and such that: $F\left(a_1,\dots,a_m\right) = \prod_{i=1}^{m} c_i^{w_i}$, where c_i is the i-th largest value from the set $\left\{a_1,\dots,a_m\right\}$.

The OWG operator preserves reciprocity. Different vectors W assign different weights to the values $a_1,\dots,a_m$.

Let all decision makers be equally important. Two vectors $W_L = \left(w_1^L,\dots,w_m^L\right)$ and $W_U = \left(w_1^U,\dots,w_m^U\right)$ for the lower and upper bounds of the intervals, respectively, are generated, depending on the odd or even number of individual judgments. If m is an odd number, then $\frac{m+1}{2}$ is the median of the numbers $1,2,\dots,m$, $s_{\frac{m+1}{2}} = \frac{(m+1)(m+3)}{8}$ is the sum of the numbers from 1 to $\frac{m+1}{2}$ and:

$$W_L^{odd} = \left(\underbrace{0,\dots,0}_{\frac{m-1}{2}}, \frac{1}{s_{\frac{m+1}{2}}}, \frac{2}{s_{\frac{m+1}{2}}},\dots,\frac{\frac{m-1}{2}}{s_{\frac{m+1}{2}}}, \frac{\frac{m+1}{2}}{s_{\frac{m+1}{2}}}\right); W_U^{odd} = \left(\frac{\frac{m+1}{2}}{s_{\frac{m+1}{2}}}, \frac{\frac{m-1}{2}}{s_{\frac{m+1}{2}}},\dots,\frac{2}{s_{\frac{m+1}{2}}}, \frac{1}{s_{\frac{m+1}{2}}}, \underbrace{0,\dots,0}_{\frac{m-1}{2}}\right) \quad (9)$$

If m is an even number, then the median of the numbers $1,2,\dots,m$ is not an integer and $s_{\frac{m}{2}} = \frac{m(m+2)}{8}$ is the sum of the numbers from 1 to $\frac{m}{2}$, which are smaller than the median. Then:

$$W_L^{even} = \left(\underbrace{0,\dots,0}_{\frac{m}{2}}, \frac{1}{s_{\frac{m}{2}}}, \frac{2}{s_{\frac{m}{2}}},\dots,\frac{\frac{m-2}{2}}{s_{\frac{m}{2}}}, \frac{\frac{m}{2}}{s_{\frac{m}{2}}}\right); \; W_U^{even} = \left(\frac{\frac{m}{2}}{s_{\frac{m}{2}}}, \frac{\frac{m-2}{2}}{s_{\frac{m}{2}}},\dots,\frac{2}{s_{\frac{m}{2}}}, \frac{1}{s_{\frac{m}{2}}}, \underbrace{0,\dots,0}_{\frac{m}{2}}\right) \quad (10)$$

The aggregated interval group matrix A^{group} is defined as:

$$A^{group}=\begin{bmatrix} 1 & \left[\prod_{k=1}^{m}\left(c_{12}^{(k)}\right)^{w_k^L},\prod_{k=1}^{m}\left(c_{12}^{(k)}\right)^{w_k^U}\right] & \cdots & \left[\prod_{k=1}^{m}\left(c_{1n}^{(k)}\right)^{w_k^L},\prod_{k=1}^{m}\left(c_{1n}^{(k)}\right)^{w_k^U}\right] \\ \left[\prod_{k=1}^{m}\left(c_{21}^{(k)}\right)^{w_k^L},\prod_{k=1}^{m}\left(c_{21}^{(k)}\right)^{w_k^U}\right] & 1 & \cdots & \left[\prod_{k=1}^{m}\left(c_{2n}^{(k)}\right)^{w_k^L},\prod_{k=1}^{m}\left(c_{2n}^{(k)}\right)^{w_k^U}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \left[\prod_{k=1}^{m}\left(c_{n1}^{(k)}\right)^{w_k^L},\prod_{k=1}^{m}\left(c_{n1}^{(k)}\right)^{w_k^U}\right] & \left[\prod_{k=1}^{m}\left(c_{n2}^{(k)}\right)^{w_k^L},\prod_{k=1}^{m}\left(c_{n2}^{(k)}\right)^{w_k^U}\right] & \cdots & 1 \end{bmatrix} \quad (11)$$

where $c_{ij}^{(k)}$ is the *k*-th largest value from the set $\left\{a_{ij}^{1},...,a_{ij}^{m}\right\}$.

3.1.2 Method of aggregation of individual judgments into a group interval judgment using ADEXTREME method

We suggest a new approach, called adopted extreme values (ADEXTREME) method, for aggregating individual judgments into a group interval judgment, where all individual judgments have an impact on the bound of the group interval but not all have equal power. The highest power belongs to the minimum and the maximal values, respectively.

We assume that all decision makers are equally important. The smallest value has influence equal to one half and all the other values have influence equal to one half on the lower bound l_{ij} of the group interval. The highest value has influence equal to one half and all the other values have influence equal to one half on the upper bound u_{ij} of the group interval. Let *t* and *T* be indexes:

$t,T\in\{1,...,m\}$ and $a_{ij}^{(t)}=\min\limits_{k\in\{1,...,m\}}a_{ij}^{(k)}$ and $a_{ij}^{(T)}=\max\limits_{k\in\{1,...,m\}}a_{ij}^{(k)}$

Then:

$$l_{ij}=\left(a_{ij}^{(t)}\right)^{(1/2)}\prod_{\substack{k=1\\k\neq t}}^{m}\left(a_{ij}^{(k)}\right)^{1/(2m-2)} \text{ and } u_{ij}=\left(a_{ij}^{(T)}\right)^{(1/2)}\prod_{\substack{k=1\\k\neq T}}^{m}\left(a_{ij}^{(k)}\right)^{1/(2m-2)} \quad (12)$$

The following holds: $l_{ij}\le u_{ij}$, $l_{ij}=1/u_{ji}$ and $u_{ij}=1/l_{ji}$ for all *i,j=1,…,n*.

The new ADEXTREME method is easier for calculations than the MEDINT method and it enables all decision makers to influence the interval group judgment.

3.2 Calculating interval weights from interval comparison matrices

For deriving interval weights from the interval comparison matrix A^{group} (11) or (12) we propose the approach of splitting A^{group} into two crisp comparison matrices $A_L^{group}=\left(a_{ij}^{L}\right)$ and $A_U^{group}=\left(a_{ij}^{U}\right)$ (Liu, 2009), where:

$$a_{ij}^L = \begin{cases} l_{ij}, & i < j \\ 1, & i = j \\ u_{ij}, & i > j \end{cases}, \qquad a_{ij}^U = \begin{cases} u_{ij}, & i < j \\ 1, & i = j \\ l_{ij}, & i > j \end{cases}. \tag{13}$$

Matrices A_L^{group} and A_U^{group} are reciprocal comparison matrices.

The weights can be obtained from A_L^{group} and A_U^{group} by the LLSM (4), (5). The interval weights belonging to A^{group} are defined as:

$$\omega_i = \left[\omega_i^L, \omega_i^U\right] = \left[\min\left\{\omega_i^{A_L}, \omega_i^{A_U}\right\}, \max\left\{\omega_i^{A_L}, \omega_i^{A_U}\right\}\right]. \tag{14}$$

For ranking interval weights the matrix of degrees of preference is used:

$$P = \begin{bmatrix} - & p_{12} & \cdots & p_{1n} \\ p_{21} & - & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & - \end{bmatrix} \tag{15}$$

In recent years the possibility-degree formula for p_{ij} has been used several times (Facchinetti et al., 1998; Wang et al., 2005b; Xu and Chen, 2008; Xu and Da, 2002):

$$p_{ij} = P\left(\omega_i > \omega_j\right) = \frac{\max\left\{0, \omega_i^U - \omega_j^L\right\} - \max\left\{0, \omega_i^L - \omega_j^U\right\}}{(\omega_i^U - \omega_i^L) + (\omega_j^U - \omega_j^L)}, \quad i,j = 1, \ldots, n, i \neq j \tag{16}$$

The preference ranking order is obtained using row-column elimination method (Wang et al., 2005b).

4 A case study – evaluation of Natura 2000 development scenarios

Natura 2000 is a European network of ecologically significant natural areas. Natura 2000 sites are managed through sectorial management plans. The agricultural priorities are outlined in the Rural Development Programme of the Republic of Slovenia 2007–2013 (PRP, 2007) and the objectives are divided into four development scenarios (alternatives):

1. Alternative 1 – improving the competitiveness of the agricultural and forestry sector. The activities included in the first alternative should support modernization and innovations and raise the qualification and competitive position. They should contribute to improved employment possibilities, increased productivity, and added value in agriculture and forestry.
2. Alternative 2 – improving the environment and rural areas. The activities included in the second alternative should contribute to environmental and water resource protection, conservation of natural resources, and

implementation of nature friendly technologies in agriculture and forestry. They should provide sustainable development of rural areas and ensure a favorable biodiversity status and the preservation of habitats in the Natura 2000 sites.

3. Alternative 3 – quality of life in the rural areas and diversification of the rural economy. The activities included in the third alternative promote entrepreneurship and raise the quality of life in rural areas through enhanced employment opportunities, rural economic development, and natural and cultural heritage conservation.
4. Alternative 4 – presents the scenario of providing support for rural development through implementing local development strategies. The activities included in the fourth alternative should stimulate the cooperation and connection of local action groups.

To assure the best results for the Natura 2000 sites we should rank these four aims (alternatives) and seek a balance between them. The weighting depends on the approach to the objectives which differs among stakeholders (decision makers). With an objective of incorporating different perspectives, we identified four main stakeholders for the Natura 2000 sites: representatives of environment protection (NGOs), representatives of farmers (owners), the government, and representatives of research/education institutions. Pairwise comparisons of the four objectives are represented by the matrices A, B, C and D for: environmentalist (NGOs representative), farmer (owner), representative of the government, and representative of the research and educational institution, respectively.

$$A=\begin{bmatrix}1 & \frac{1}{4} & \frac{1}{3} & 2\\ 4 & 1 & 2 & 3\\ 3 & \frac{1}{2} & 1 & 2\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & 1\end{bmatrix}\quad B=\begin{bmatrix}1 & 3 & 1 & 1\\ \frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{2}\\ 1 & 3 & 1 & 2\\ 1 & 2 & \frac{1}{2} & 1\end{bmatrix}\quad C=\begin{bmatrix}1 & \frac{1}{2} & 3 & 8\\ 2 & 1 & 4 & 6\\ \frac{1}{3} & \frac{1}{4} & 1 & 3\\ \frac{1}{8} & \frac{1}{6} & \frac{1}{3} & 1\end{bmatrix};$$

$$D=\begin{bmatrix}1 & 2 & 1 & 2\\ \frac{1}{2} & 1 & \frac{1}{4} & 1\\ 1 & 4 & 1 & 1\\ \frac{1}{2} & 1 & 1 & 1\end{bmatrix}$$

All four matrices are acceptably consistent with CR = 0.058 (0.017; 0.039 and 0.070), respectively, calculated by (2). The priorities of the four decision makers, obtained by the EV method (3) are presented in Table 1.

Table 1

The priorities and the ranks of the four alternatives for four decision makers

	A		B		C		D	
	priorities	ranks	priorities	ranks	priorities	ranks	priorities	ranks
alternative 1	0.140	3	0.302	2	0.337	2	0.320	2
alternative 2	0.465	1	0.110	4	0.483	1	0.140	4
alternative 3	0.280	2	0.358	1	0.127	3	0.339	1
alternative 4	0.116	4	0.230	3	0.053	4	0.201	3

The ranking differs between the decision makers. The environmentalist prefers alternative 2, i.e., nature protection. Owners favor rural economic development, which is reflected by alternatives 3 and 1. The government's weights indicate that the government is focused on sharing funds for particular objectives regarding protection and favors the alternative 2, while the institutional (research, education) representative favors the quality of rural life, i.e., alternative 3.

We calculate the group priorities using methods WGMDEA, MEDINT and ADEXTREME method.

WGMDEA:

	WGMDEA	
	priorities	ranks
alternative 1, (ω_1)	0.294	1
alternative 2, (ω_2)	0.274	3
alternative 3, (ω_3)	0.287	2
alternative 4, (ω_4)	0.145	4

The geometric mean matrix is calculated by (7), while the priorities by (8):

$$A^{WGMM} = \begin{bmatrix} 1 & 0.931 & 1 & 2.378 \\ 1.075 & 1 & 0.904 & 1.732 \\ 1 & 1.107 & 1 & 1.861 \\ 0.420 & 0.577 & 0.537 & 1 \end{bmatrix},$$

The ranks of the alternatives obtained by the WGMDEA method are: $\omega_1 > \omega_3 > \omega_2 > \omega_4.$

MEDINT:

The comparison matrices *A*, *B*, *C* and D are aggregated in the A^{group} according to (11). The associated lower and upper weighted vectors (10) are defined as $W_L = \left(0, 0, \frac{1}{3}, \frac{2}{3}\right)$ and $W_U = \left(\frac{2}{3}, \frac{1}{3}, 0, 0\right)$. The intervals in the matrix A^{group} are:

$$A^{group}=\begin{bmatrix} 1 & [0.3150,2.6207] & [0.4807,2.0801] & [1.2599,5.0397] \\ [0.3816,3.1748] & 1 & [0.2752,3.1748] & [0.6300,4.7622] \\ [0.4807,2.0801] & [0.3150,3.6342] & 1 & [1.2599,2.6207] \\ [0.1984,0.7937] & [0.2100,1.5874] & [0.3816,0.7937] & 1 \end{bmatrix}$$

Then A^{group} is split into two crisp matrices A_L^{group} and A_U^{group} (13) and priorities are calculated by the logarithmic least-squares method (4), (5). The interval weights are then given by (14).

	MEDINT
	interval weights
alternative 1, (ω_1)	[0.154, 0.459]
alternative 2, (ω_2)	[0.201, 0.311]
alternative 3, (ω_3)	[0.159, 0.411]
alternative 4, (ω_4)	[0.071, 0.234]

The matrix of degrees of preference P (15), (16):

$$P=\begin{bmatrix} - & 0.623 & 0.539 & 0.830 \\ 0.377 & - & 0.420 & 0.881 \\ 0.461 & 0.580 & - & 0.820 \\ 0.170 & 0.119 & 0.180 & - \end{bmatrix}$$

and the final preference order of alternatives obtained by the MEDINT method: $w_1 \overset{53.9\%}{\succ} w_3 \overset{58\%}{\succ} w_2 \overset{88.1\%}{\succ} w_4$.

ADEXTREME:

The comparison matrices A, B, C and D are aggregated into the A^{group} according to (12). The intervals in the matrix A^{group} are:

$$A^{group}=\begin{bmatrix} 1 & [0.6005,1.3747] & [0.6934,1.4422] & [1.7818,3.5636] \\ [0.7274,1.6654] & 1 & [0.5888,1.4837] & [1.1447,2.6207] \\ [0.6934,1.4422] & [0.6740,1.6984] & 1 & [1.5131,2.1822] \\ [0.2806,0.5612] & [0.3816,0.8736] & [0.4582,0.6609] & 1 \end{bmatrix}$$

Then, A^{group} is split into two crisp matrices A_L^{group} and A_U^{group}. Priorities are calculated by the logarithmic least-squares method (4), (5). The interval weights are then given by (14).

	ADEXTREME
	interval priorities
alternative 1, (ω_1)	[0.226, 0.370]
alternative 2, (ω_2),	[0.251, 0.295]
alternative 3, (ω_3)	[0.228, 0.338]
alternative 4, (ω_4)	[0.107, 0.184]

The matrix of degrees of preference P (15), (16):

$$P = \begin{bmatrix} - & 0.636 & 0.559 & 1 \\ 0.364 & - & 0.431 & 1 \\ 0.441 & 0.569 & - & 1 \\ 0.000 & 0.000 & 0.000 & - \end{bmatrix}$$

and the final preference order of alternatives obtained by the ADEXTREME method $w_1 \overset{55.9\%}{\succ} w_3 \overset{56.9\%}{\succ} w_2 \overset{100,0\%}{\succ} w_4$.

A comparison of WGMDEA, MEDINT and PEINT shows that all three methods, the crisp group method WGMDEA, and the interval methods MEDINT and ADEXTREME give the same ranking of the four alternatives discussed, i.e., $\omega_1 > \omega_3 > \omega_2 > \omega_4$. These results are given in Table 2 and in graphical form in Figure 1. We see that MEDMINT intervals are longer than PEINT, while weights are similar.

Table 2

WGMDEA, MEDINT and ADEXTREME method results

for 4 alternatives and 4 decision makers

	WGMDEA		MEDINT		ADEXTREME	
	priorities	ranks	priorities (intervals)	ranks	priorities (intervals)	ranks
alternative 1, (ω_1)	0,294	1	[0.154, 0.459]	1	[0.226, 0.370]	1
alternative 2, (ω_2)	0,274	3	[0.201, 0.311]	3	[0.251, 0.295]	3
alternative 3, (ω_3)	0,287	2	[0.159, 0.411]	2	[0.228, 0.338]	2
alternative 4, (ω_4)	0,145	4	[0.071, 0.234]	4	[0.107, 0.184]	4

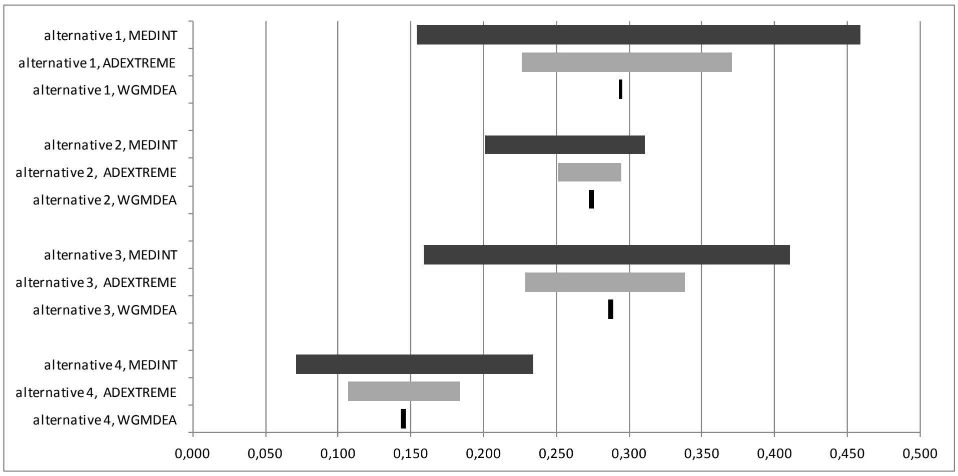

Figure 1. Graphical presentation of WGMDEA, MEDINT and ADEXTREME method results for 4 alternatives and 4 decision makers

5 Concluding remarks

Regarding the theoretical component of operations research, the paper addresses multiple criteria group methods employed in AHP. First, an overview of the methods for aggregating the individual comparison matrices into a group comparison matrix has been presented. A WGMDEA method, which preserves reciprocity, has been proposed for application in a case study. Its advantage is that it uses a linear program, while the most popular method, WGMM, uses an eigenvector procedure (3), which is not linear. Further, interval group matrices in group AHP aggregated from individual comparison matrices were introduced. The method MEDINT was presented and a new method ADEXTREME was suggested.

In the second part of the paper the an application of the proposed group AHP methodology was described. The main goal was the selection of the optimal strategy (alternative) for Natura 2000 site development with group AHP method and interval judgments. The results obtained by WGMDEA, MEDINT and ADEXTREME methods were compared. The results show that all stakeholders (decision makers) support modernization and innovations in agriculture and forestry which should contribute to improved employment opportunities, increased productivity, and added value in agriculture and forestry. The selected alternative can contribute to an enhanced management plan of the area. It can serve as a basis for the establishment of strategic and operational management goals of the area.

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M U L T I P L E C R I T E R I A D E C I S I O N M A K I N G

Vol. 8 2013

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COMPOSITE EVALUATION OF BROADBAND INTERNET ACCESS IN POLAND

Abstract

The level of access to Internet is constantly evaluated and promoted by electronic communications regulators around the world. The issue is especially important in countries, such as Poland, where where Internet access is highly heterogenous among local markets. The objective of this paper is to identify socio-economic factors that influence the level of Internet access in local communities (gminas) in Poland.

The definition of Internet access involves multiple criteria and encompasses in particular its availability, adoption, speed, quality of service and price. In the paper we propose a two-phase approach to a comparison of Internet access in various gminas. First we use Data Envelopment Analysis (DEA) to evaluate Internet broadband access depending on their demographic characteristics based on data from 2010 and 2011 collected by Poland's Office of Electronic Communications (UKE). In the second stage we explain the obtained DEA effectiveness indices using supervised learning techniques with the socio-economic status of the community as explanatory variables. We show that in the period under study rural communities experienced greater Internet access improvement than urban communities, therefore catching up with large cities and reducing technological gap. Moreover, we identify drivers of broadband Internet advancement, including: community type, community education and age structure, computerization level in schools and Herfindahl-Hirschman

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competition index. We show that an effective regulation may foster the advancement of fixed-location broadband Internet access.

1 Introduction

The deployment of broadband technology brings to the society versatile economic and social benefits by its positive influence on (1) Gross Domestic Product (see Czernich et al., 2011; Holt and Jamison, 2009; Koutroumpis, 2009), (2) employment and job creation (see Katz, 2009; Katz et al., 2010), (3) research & development sector (see OECD, 2008), (4) reduction of business costs through e.g. cloud computing (see Zhang et al., 2010), (5) retail, services, manufacturing & industrial sectors (see Fornefeld et al., 2008), (6) education sector through e.g. building human capital and (7) health care sector. Broadband Internet is regarded as a general purpose technology (see Bresnahan and Trajtenberg, 1995), which, similarly to the invention of electricity, engines or railways, fundamentally changed the way an economic activity is now organized. General purpose technology is characterized by its pervasive use in a wide range of sectors bringing about vast productivity gains and enabling new opportunities across an entire economy (see Kelly and Rossotto, 2012). Therefore, fostering the development of broadband technology is regarded as an important policy making objective of many governments around the world. In particular, National Regulatory Authorities (NRAs) are dedicated to support the development of broadband infrastructure and regulation.

Due to the regional diversity of market competition, service accessibility and market penetration, NRAs face the problem of addressing their regulatory objectives and policies taking into account local market heterogeneity. The variety of factors describing the level of market development (service availability, adoption, speed, quality of service and price) makes it difficult to determine which local markets perform better than others. Therefore, we apply Data Envelopment Analysis (DEA) to enable the Polish NRA (UKE – Urząd Komunikacji Elektronicznej) to assign an efficiency score to every region and calculate the distance between them. This can be done without the need of defining the weights of variables a priori which is the disadvantage of many other methods dedicated to the assessment of broadband advancement (see Bayasyan et al., 2011; Commission of EC, 2008; Grubesic, 2010; ITU, 2009; TechNet, 2003).

According to the Polish NRA, the level of competition, as well as the level of demand on the broadband access market, are distributed very unequally throughout the country (see Gaj, 2012). Various market specific characteristics call for different approaches towards market regulation and the allocation of funds. Therefore, UKE concentrates on geographically differentiated regulation scheme and makes it one of its main priorities in its three-year strategic plan. It states that the “identification of areas with unsatisfied demand for fixed line services and broadband internet access is crucial for consumer-oriented policy consisting in improving service accessibility in rural areas and widening the choice of service providers in urbanized areas" (see Gaj, 2012). Applying multi-criteria DEA models enables such an identification and narrows the number of regions used for further, more detailed analysis. Other DEA applications, important from the NRA's point of view, may include setting precise and feasible aims concerning the development of local markets through comparing communities with similar input values. Moreover, operators and third party investors may find the DEA results obtained for local markets valuable and interesting as they reflect the effectiveness of regulatory actions in individual regions and are comparable in time. Therefore, DEA models have many potential applications because of their versatility and simplicity in interpreting the outcomes.

In the literature there are many applications of DEA method to (1) telecommunications markets (see Badasyan et al., 2011; Fernandez-Menendez et al., 2009; Giokas and Pentzaropoulos, 2008; Grubesic, 2010; Kang, 2009; Lam and Lam, 2005; Lam and Shiu, 2008; Sastry, 2009; Tsai et al., 2006; Uri, 2003; Zhu, 2004) and (2) the evaluation of regional economic competitiveness in the UK by creating the omposite competitiveness index (see Huggins, 2003), (3) the evaluation of countries with respect to Gross Domestic Product (see Growiec, 2012) and (4) the performance measurement of banks and public companies (see Brockett et al., 1997; Ho and Zhu, 2004; Zhu, 2000). However, none of these papers tries to explain the DEA score by control variables. Therefore, we propose a second step consisting in modeling DEA scores by control variables with the help of supervised learning techniques. A similar approach is taken with respect to the assessment of local governments by Alfonso and Fernandes (2008). The authors use a Tobit regression to explain DEA indices.

In the paper we propose a two-step procedure for the performance evaluation of regions according to specified criteria and explaining differences in performance among regions by a supervised learning technique. In the first step, we apply Data Envelopment Analysis to assess relative technological advancement of fixed location broadband Internet deployment in individual communities. In the second step, we apply supervised learning techniques (regression trees and random forests) to explain and predict differences in DEA scores depending on socio-economic characteristics of local markets. We show the possibility of effective welfare-improving regulation by influencing some of the explanatory variables that are of partial control of regulator and might foster the deployment of broadband Internet. The novelty of the paper consists in (1) applying the proposed two-step procedure to the data concerning telecommunications local markets in the period of 2010-2011 acquired by Polish telecommunications regulator (UKE) and (2) proposing a new analytic approach based on explaining DEA scores by control variables with the help of supervised learning techniques.

The article is organized as follows. In Section 2 we present sources and structure of data. In Section 3 we describe the proposed two-step procedure consisting of Data Envelopment Analysis part (Section 3.1) and supervised learning approach (Section 3.2). Results of this two-step procedure are presented in Section 4. All figures and computations are performed in the statistical programming environment GNU R (see R Core Team, 2012).

2 Broadband Access Data Characteristic

The employed data are obtained from the following dataset sources:

1. Local Markets Dataset of Poland's Office of Electronic Communications (UKE) from 2010 and 2011.
2. Local Data Bank of Poland’s Central Statistical Office (GUS) from 2010 and 2011.
3. GUS data covering average income in communities from 2010 and 2011.
4. National Census of Population and Housing 2002 provided by GUS.
5. Financial budgets of local governments provided by the Agency of Public Information (BIP) of Poland's Ministry of Finance from 2010.

Local Markets Dataset of UKE allows to obtain data concerning the situation on the market of fixed-location broadband Internet from 2010 and 2011. In particular, the dataset covers the number of subscribers (adoption) and the number of households with ability to subscribe to broadband (availability). The dataset contains also information on fixed-location broadband technology, bundled products and the infrastructure ownership. The UKE data allow to draw conclusions as to the broadband availability, adoption and competition. Based on those data we calculate additionally the Herfindahl-Hirschman Index[1] (HHI) which is a measure of the competition level on the local market. Each data observation describes one local market (gmina). There are 2 462 gminas in the dataset out of 2 479, grouped by three types as follows: 304 urban communities, 599 urban-rural communities and 1 559 rural communities.

The data concerning socio-economic characteristics of communities are derived from the following sources: (1) Local Data Bank of GUS, (2) National Census of Population and Housing 2002 of GUS and (3) GUS data on demand on average income in communities. Local Data Bank provides statistics of population size in communities, as well as age structure of population. There is no available data on current population education structure on the level of communities. However, we find out that the education structure in communities

[1] Herfindahl-Hirschman Index is the sum of squared market shares, i.e. where is the market share of company operating on the market with competitors. The maximum value of HHI is and indicates the monopoly market. The lower HHI (the minimum bound is 0), the more competitive market and higher number of competitors with no significant market power. The usage of Herhindal-Hirschman Index in the evaluation of local markets’ competition performance is presented in Zawisza and Kamiński (2012).

from 2002 is a good proxy for the education structure in 2011. Therefore, we make use of the National Census of Population and Housing 2002 to acquire information about the education structure in 2002. Moreover, GUS Local Data Bank is the source for data concerning computerization level in schools. It enables us to calculate the number of computers with broadband Internet access per pupil in primary and secondary schools in each community. We claim that high level of access to Internet in schools might be an important demand factor influencing the household decisions whether or not to subscribe to broadband. Another potential driver for broadband demand might be the wealth of community citizens, which is measured in our paper by the average income in community. This statistics is calculated on demand by GUS.

The data concerning local government expenditures on telecommunication services and infrastructures is obtained from the BIP of Poland's Ministry of Finance. The data consist of quarterly budgets of communities. In our analysis we calculate the sum of expenditures in 2010.

3 Internet access analysis procedure

The proposed procedure consists of two steps. In the first step, we assign a performance index to each local market with regard to its fixed-location broadband advancement depending on its demographic characteristic. We obtain these indices by using Data Envelopment Analysis (DEA) technique (see Charnes et al., 1978; Cooper et al, 2006; Guzik, 2009; Zhu, 2009). In the second step, DEA indices are explained by socio-economic variables. Understanding the mechanism of the influence of socio-economic variables on broadband advancement might be of great use for a regulator to tailor its policies according to recommendations provided by the model. The modeling part is done with the use of supervised learning techniques, in particular regression trees and random forests (see Hastie et al., 2009; Kamiński and Zawisza, 2012; Koronacki and Ćwik, 2008; Walesiak and Gatnar, 2009).

3.1 Data Envelopment Analysis step

The aim of the Data Envelopment Analysis (DEA) step is to provide a single index describing the level of fixed-location broadband technological advancement for each community. The DEA efficiency index takes into account three criteria[2]:

1. Availability of fixed location broadband refers to the ability of a household to subscribe to broadband via at least one fixed-location broadband technology regardless of whether the household actually subscribes to it or not. It is measured by the number of households with at least one broadband provider.

[2] These criteria are the outputs of DEA method, i.e. variables whose high values are desirable.

2. Adoption of fixed location broadband refers to the actual use of Internet and is measured by the number of households that are subscribed to fixed location broadband services.
3. Competition of fixed location broadband market refers to the competition pressure put by the providers on each other, which has a direct impact on terms and conditions of services offered, e.g. price level, transfer speed, download limits. It is measured by the number of households with at least two broadband service providers.

Since the above three criteria are expressed in absolute terms, it is important to normalize them. Otherwise, larger communities would have higher DEA index values. Therefore, we include the following variables:

- community population size,
- the number of households in community

as DEA inputs[3]. These inputs are objective demographic characteristics of the local market and so they can be treated as exogenous and beyond the control of telecommunications regulator or any government in short or medium term. The aim of the first step of the procedure is to return DEA scores that capture only the objective and technological aspect of fixed location broadband Internet deployment. We do not include any other variables as inputs, since they would unnecessarily justify a poor deployment of broadband Internet in some communities. The explanation of DEA scores is provided in the second step of the procedure proposed.

As the result of DEA method, we assign a single effectiveness index to each local market. The index is normalized between 0% and 100%. The DEA index measures the relative technological performance of fixed location broadband deployment in a given community. The higher DEA index, the higher the evaluation of broadband market performance. The community with DEA efficiency of 100% denotes that it is relatively efficient, e.g. there is no other community in the dataset examined, which is better with respect to all three criteria given the same inputs. On the other hand, communities with DEA efficiency lower than 100% are regarded as not efficient, which means that there is at least one theoretical community that performs better. For instance, DEA efficiency of 25% means that there is a better performing community, which meets all three output criteria having inputs one-fourth as low. For each ineffective community the DEA method returns a benchmarking set, i.e. the set of communities which should be regarded as communities to be followed for a worse performing community.

To assess the performance improvement between 2010 and 2011, we apply DEA to the dataset consisting of communities measured in both years. Hence, the dataset consists of 4 924 observations, i.e. twice the number of communities

[3] These variables are the inputs of DEA method, i.e. variables whose low values are desirable.

in the dataset examined. We apply the input-oriented DEA method with increasing returns to scale. The calculation is performed in GNU R statistical programming environment (see R Core Team, 2012) with the help of the "Benchmarking" package (see Bogetoft and Otto, 2011).

3.2 Supervised learning step

In the second step, we explain DEA indices obtained in the first step. To model DEA indices, we use explanatory variables that are both beyond and of partial control by the national regulatory authority. Therefore, understanding the mechanism of their influence on broadband advancement might be of great use for a regulator to tailor its policies according to recommendations provided by the model. Although some variables are beyond the control of the regulator, it is important to include them in the second step of the procedure, since they may influence variables that are under the regulator's control.

Candidates for variables used in the modeling are the following:

- Computerization level in local schools (SchoolComp),
- Local government expenditures on telecom infrastructure and service (GovExponTelecomInf and GovEspOnTelecomSer),
- Average income in a community (AvgIncome),
- Herfindahl-Hirschman index in a community (HHI),
- Age structure of community population (WorkAge, PreWorkAge),
- Education structure of community population (Prim- & HighEduc).

To model DEA indices with the help of explanatory variables, we can use any of the supervised learning techniques, e.g. linear regression, artificial neural nets or generalized additive model (see Hastie et al., 2009; Kamiński and Zawisza, 2012; Koronacki and Ćwik, 2008; Walesiak and Gatnar, 2009). However, we take the advantage of the regression tree technique, because of its simplicity of interpretation and ease of visualization. In particular, we apply a conditional inference framework in the induction of regression trees (see Hothorn et al., 2006). Additionally, we present the ranking of the importance of variables with the help of random forest technique.

The calculation is performed in GNU R statistical programming environment (see R Core Team, 2012) with the help of the "party" package (see Horthorn et al., 2006) and the "randomForest" package (see Liaw and Wiener, 2002).

4 Results of Internet Access Analysis

4.1 Data Envelopment Analysis – I stage

In the first step, the input-oriented DEA method with increasing returns to scale is employed on the dataset of 4 924 observations (twice the number of communities for 2010 and 2011). A DEA model includes three output criteria: (1) the number of households with at least one broadband provider (availability), (2) the number of households that subscribe to fixed location broadband services (adoption) and (3) the number of households with at least two broadband service providers (competition). DEA inputs are: (1) community population size and (2) the number of households in a community. The result of the DEA method are efficiency indices assigned to each local market in a specified year. The distribution of DEA indices separately for 2010 and 2011 is shown using box-plots and density functions in Figure 1.

Moreover, we consider two approaches: (1) unweighted community observations and (2) community observations weighted by their population size. We show that using the technique of weighting observations by their population sizes significantly influences: (1) calculated summary statistics, (2) their interpretation and (3) implications concerning the performance evaluation of the fixed location broadband market in Poland. The approach of not weighting communities by their population sizes treats communities as units of interest, regardless of their population sizes, whereas the approach of weighting puts emphasis on the citizen, not the community. Summary statistics calculated with the use of weighted observations is a more reliable picture of Poland's fixed-location broadband market performance, since it is taken from the viewpoint of the citizen and not a single community. The unweighted technique assumes that each community is of the same importance and has the same population size. As a result, the unweighted approach puts higher relative importance to rural communities of low population size in comparison with urban communities.

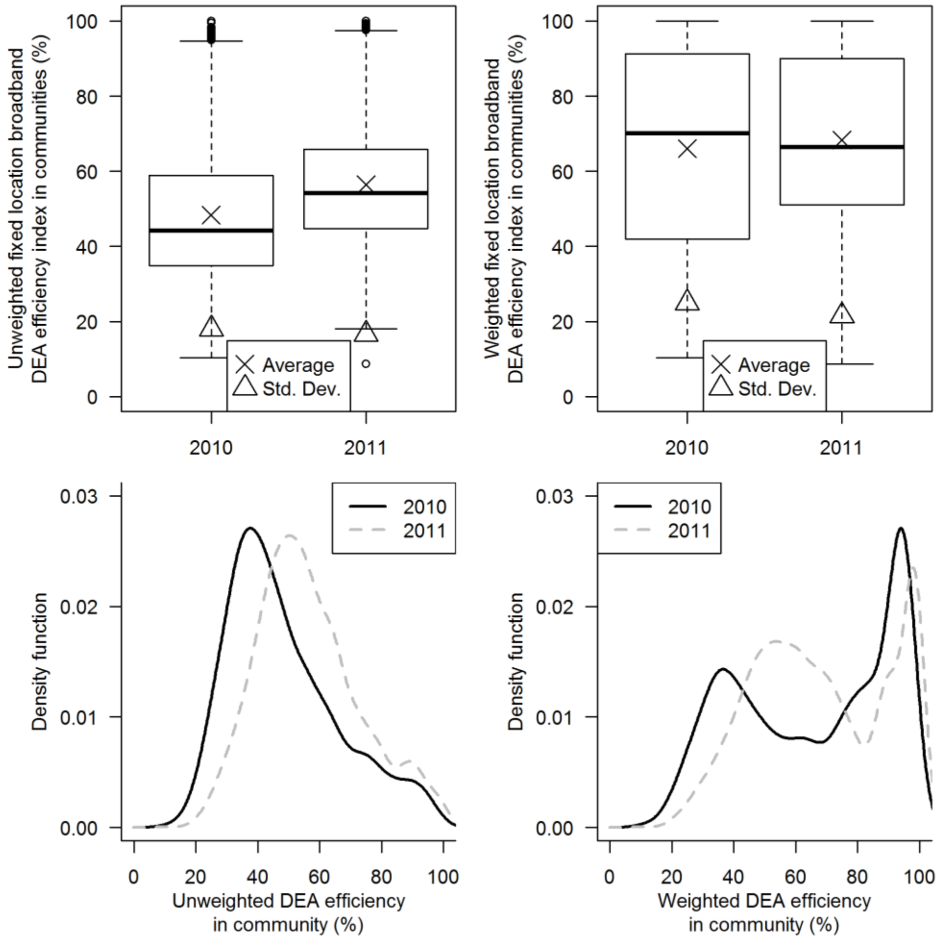

Figure 1. Unweighted (left) and weighted (right) box-plots (top) and density functions (bottom) of Data Envelopment Analysis (DEA) efficiency indices in communities for 2010 and 2011

The upper panels of Figure 1 consist of box-plots of DEA efficiency indices. The upper left panel presents the box-plot of unweighted observations, thus it deals with communities as units of interest. Based on this box-plot we can draw three important conclusions. First, the average of DEA efficiency indices over communities in both years is quite low: 48.3% in 2010 and 56.5% in 2011. This is due to the fact that most communities in our datasets are rural, which perform much worse than urban ones. However, in unweighted approach rural and urban communities have the same weights, left panels tell mainly the story of rural areas. Second, the upper left panel of Figure 1 indicates low dispersion of DEA efficiency indices with interquartile range of 23.9 pp. in 2010 and 21.1 pp. in 2011, i.e. half of communities differed in their DEA efficiency indices by no more than 23.9 pp. in 2010 and 21.1 pp. in 2011. This is again due to the fact that most observations in the dataset are similar to each other, as they are poor performing rural communities. Third, in terms of dynamics, we observe a significant improvement of DEA efficiency indices over the years 2010 and 2011. The median of DEA indices increases by 10.0 pp. from 44.2% to 54.2%. This large improvement is again due to an impressive progress in small, rural communities, which constitute over 60% of all communities.

On the other hand, the upper right panel of Figure 1 tells the story from the viewpoint of a citizen, not a community. As a result, this picture reflects the actual situation more reliably. Based on this panel we can draw the following three conclusions. First, the average of DEA efficiency indices over citizens in both years is quite high 66.0% in 2010 and 68.3% in 2011, especially in comparison to the analogous statistics from the upper left panel. These differences are due to the fact that most citizens live in large cities with well performing broadband market. Second, the upper right panel of Figure 1 indicates a high dispersion of DEA efficiency indices with interquartile range of 49.3 pp. in 2010 and 39.0 pp. in 2011, i.e. half of population differed in their DEA efficiency indices by no more than 49.3 pp. in 2010 and 39.00 pp. in 2011. This indicates large inequalities among Polish citizens from various regions as regards Internet access. Third, in terms of dynamics, we see two interesting facts. The dynamics of DEA efficiency inequality is improving quite remarkably, since the interquartile range decreases by 10.3 pp. from 49.3 pp. in 2010 to 39.0 pp. in 2011. It is reached mainly by the significant upward shift of first quartile by 9.1 pp. from 42.0% to 51.8% and also a minor decrease of third quartile by 1.3 pp. from 91.3% to 90.0%. At the same time the dynamics of level of DEA efficiency stagnates, as the average increases slightly by 2.3 pp. from 66.0% to 68.3% and the median decreases slightly by 3.6 pp. from 70.1% to 66.5%.

The changes observed are due to the fact that poor performing rural communities are catching up, whereas the best performing, mainly urban communities do not progress anymore.

The bottom panels of Figure 1 provide a more comprehensive picture of DEA efficiency indices, as they present density functions. As before, we distinguish the case of unweighted observations (bottom left panel) and weighted observations (bottom right panel). Each panel looks different; they provide two complimentary pictures of the fixed-location broadband market performance in Poland.

The bottom left panel of Figure 1 confirms conclusions drawn from the upper left box-plot panel. On the level of communities we observe: (1) low values of DEA efficiency indices, as the mode[4] of distribution is just ca. 40% in 2010 and 50% in 2011, (2) a remarkable improvement of DEA efficiency over the period

[4] The most probable value of random variable.

2010-2011, as the distribution shifts to the right, (3) small dispersion of DEA efficiency, as most of probability mass is located within an interval of ca. 30%-60%.

The bottom right panel of Figure 1 confirms conclusions drawn from the upper right box-plot panel. On the level of citizens we observe: (1) high values of DEA efficiency indices, as the mode of distribution is just nearly 100% in both years, (2) large dispersion of DEA efficiency, as there is a lot of probability mass assigned to low values of ca. 30% and 100% and (3) the remarkable improvement of inequalities in the DEA efficiency distribution over the period 2010-2011, as the left tail and a significant part of the left distribution mass moved to the right, while the right distribution mass moved slightly toward the center.

The dynamics of DEA efficiency change in the period 2010-2011 is depicted in Figure 2. DEA efficiency indices of both 2010 (the horizontal axis) and 2011 (the vertical axis) are presented on a scatterplot. Each single point represents a community. We distinguish three type of communities: (1) urban communities marked as black circles, (2) urban-rural communities marked as dark grey crosses and (3) rural communities marked as light dark triangles. Moreover, we draw a 45-degree line, which indicates communities with the same DEA efficiency index values in both years. Points lying above this line are communities that experienced an efficiency improvement in the period examined. Analogically, points below this line are communities which regressed into a worse efficiency DEA index. Additionally, for all data points we estimate a spline function presented here with a black thick line to illustrate the average change in DEA efficiency index among communities.

As we can see in Figure 2, rural communities constitute the majority of the dataset and its points are primarily located in lower values of DEA efficiency index (60% and less). However, we also observe a significant improvement among rural communities, since most of them lie above the 45-degree line. The estimated spline function also lies above this line, which indicates that average communities with low and medium DEA index experienced an efficiency improvement of nearly 10 pp.

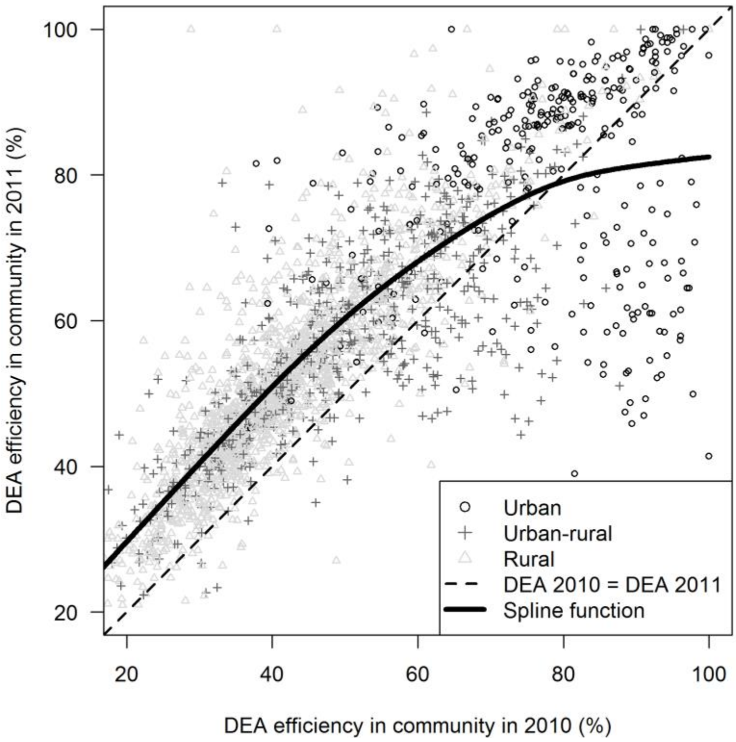

Figure 2. Scatterplot and spline function of relationship between DEA efficiency index values in 2010 and in 2011

On the other hand, as we can see in Figure 2, communities with high DEA index values (80% and more) in 2010 could not have been able to maintain on average a high level of efficiency. Many points with high efficiency in 2010 lie below the 45-degree line, which indicates a regress. These points represent mainly urban and urban-rural communities. This might be due to two reasons. First, some urban communities with saturated broadband market might indeed have experienced some efficiency loss caused by the decrease of the number of subscribers. Second, the phenomena observed might be related to the quality of data. It is possible that data collected for some communities is too optimistic in regard to three criteria in comparison to 2011. The exact source finding would require to analyze each atypical DEA value decrease on its own.

Moreover, communities with DEA efficiency index values of ca. 75%-80% in 2010 experienced on average the same level of efficiency in 2011, as the interval of 75%-80% is the region where the estimated spline function crosses the 45-degree line.

4.2 Supervised learning – II stage

In the second step, we model the DEA efficiency index. In order to do this, we use control variables that are excluded from the first DEA procedure step. We capture the relationship between explanatory variables and the DEA efficiency index by the technique of regression tree. The result of induced regression tree is depicted in Figure 3.

The regression tree provides us with prediction rules concerning the level of DEA efficiency index. A single prediction rule is a path in a graph from the root of the tree to a leaf. For instance, the path following to the leaf number 4 results in the rule that predicts the very high level of DEA index of 85%. The rule has the following form:

IF(CommunityType=urban) AND (HighEduc>11%) → (DEAefficiencyIndex=85%)

This means that an urban community with the share of population with higher education over 11% has on average the DEA efficiency index of 85%. There are 113 communities that fulfill these two conditions. In the case of urban communities with lower than 11% share of higher education in population the average DEA index value is 76%. Hence, the share of population with higher education contributes positively to the level of fixed location broadband advancement. It might be that citizens with a higher educational degree are more aware of Internet advantages and demand for this kind of services. Moreover, higher education population share may not be the direct cause of higher efficiency, but may be correlated with other factors that matter directly.

As we can see to the right of the first node in Figure 3, the remaining cases are concerned with urban-rural and rural communities. All leaves on the right predict lower DEA efficiency index values than those on the left of the first node. Hence, the type of community is crucial for determining the expected level of efficiency. Of course, it is not the administrative decision that matters here, but the fact that community type is strongly correlated with other factors fostering or hampering the development of fixed location broadband infrastructure, e.g. in sparsely populated rural areas the high cost of vast infrastructure may make it unprofitable for entrepreneurs to invest, especially that the demand from less educated population for the broadband is also lower. Hence, regulators should take into account the community type they want to support.

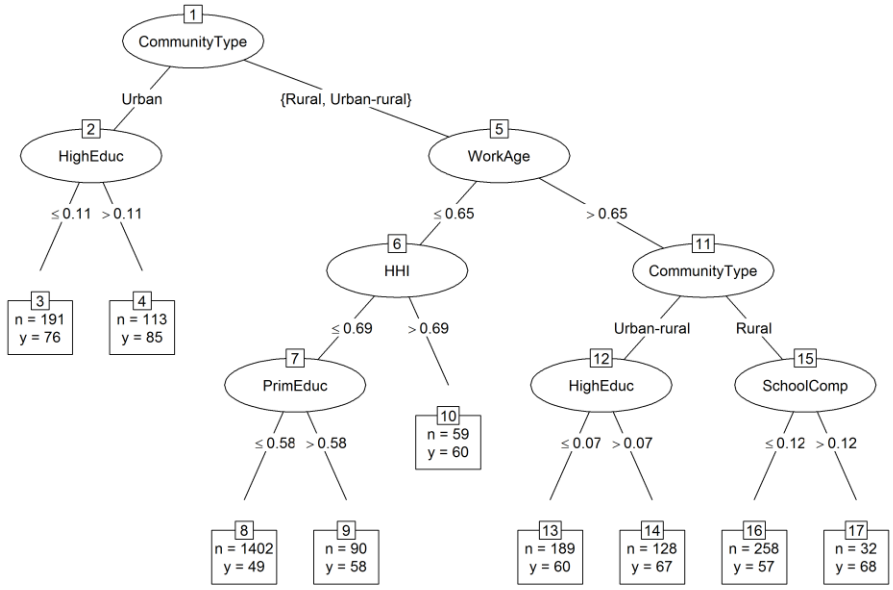

Figure 3. Induced regression tree for the explanation of DEA efficiency index dependent on socio-economic factors and the community type

The next splitting criterion is the share of production age population. The communities with this share larger than 65% have on average higher values of DEA efficiency index, as predictions in leaves on the right side are nearly always higher than those in the two leaves on the left side. For those communities with high production age population share the final splitting criterion depends on the community type. For urban-rural communities, what matters is a higher education population share. If this share is higher than 7%, then an urban-rural community achieves, on average, a DEA efficiency index of 67%, otherwise only 60%. This shows again that the educational structure matters, as it influences the demand for Internet services. On the other hand, for rural communities it is the level of computerization in schools that matters more. If the number of computers with broadband Internet in schools per pupil is higher than 12%, then a rural community can expect on average the DEA efficiency score of 68%, otherwise a score of only 57% is achieved. This shows that in rural areas the impact of demand on the part of children is crucial for household decisions concerning Internet subscription.

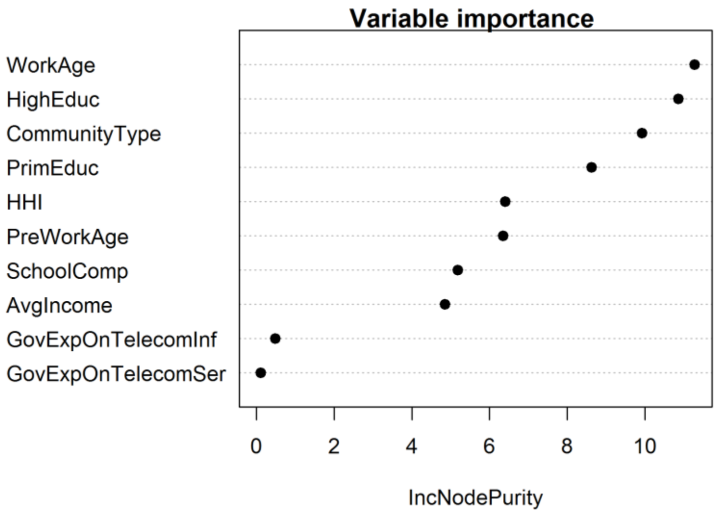

Figure 4. The ranking of socio-economic variable importance in predicting DEA efficiency

As it is shown in Figure 3, the most important determinants of DEA efficiency that occur in a regression tree are: community type, population age structure, population educational structure, school computerization level, and Herfindahl-Hirschman index. The importance of these variables is also confirmed in Figure 4. According to the ranking of variable importance, the most important are variables concerned with population structure with respect to its age and education. The community type is also very important. The Herfindahl-Hirschman Index and average income also play some role. Less significant are local government expenditures on telecommunication infrastructure and services.

The results of the analysis highlight that public policy concentrated on the stimulation of competition on local markets (measured by Herfindahl-Hirschman Index in the analysis) and investment in computerization of community schools influence Internet access more than direct government expenditures.

5 Conclusions

In our paper we propose a two-step procedure for the performance evaluation of regions according to specified criteria. The analysis of communities performed in respect to the advancement of fixed-line broadband provides the Polish telecommunications regulator with valid policy-making implications, which are important for the fulfillment of regulatory requirements and objectives specified in Gaj (2012). The method proposed enables the regulator to compare communities and prepare their ranking. In particular, it allows us to identify communities with both highest and lowest performance of fixed location broadband technological advancement. High performing communities might serve as benchmarks for less successful communities experiencing technological gap, so that UKE can try to replicate good patterns and practices from well performing communities in the poorer ones. The identification of poor performing communities enables the regulator to concentrate its tailor-made activities only in communities of great need. This feature of DEA method fulfills one of UKE's goals set by Gaj (2012), which states that "(…) identification of areas with unsatisfied demand for fixed line services and broadband internet access is crucial for consumer-oriented policy consisting in improving service accessibility in rural areas and widening the choice of service providers in urbanized areas".

Furthermore, regulatory measures aimed at improving poor performing communities may be specified according to the second step of the procedure proposed. We show that the effective regulation may foster the advancement of fixed location broadband Internet access. For instance, the analysis performed reveals that the level of citizens' education influences significantly the broadband advancement in urban and urban-rural communities, whereas in rural communities the level of school computerization plays a vital role. An active regulator might direct its resources into incentivizing local governments to increase their level of computerization in schools as well as introduce innovative multimedia-based classes in schools. All these measures will increase the awareness among pupils which will be propagated and incorporated into their parents' decisions regarding Internet subscription.

The analysis presented enables telecommunication regulators to set precise and attainable, short- and medium-term goals for communities. The results of the analysis highlight that public policy concentrated on stimulation of competition on local markets (measured by the Herfindahl-Hirschman Index in the analysis) and investment in computerization of community schools influence Internet access more than direct government expenditures. Additionally, the method allows us to compare units from various periods so that the progress of performance of communities can be assessed as well. Also the effectiveness of regulatory activities can be evaluated by their impact on a community standing over time. All these advantages of the two-step procedure proposed make it into a versatile and robust tool that enables to assess the communities' advancement with regard to fixed location broadband deployment and the effectiveness of regulatory actions.

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MULTIPLE CRITERIA DECISION MAKING

Vol. 8 2013

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ROBUST ORDINAL REGRESSION APPLIED TO *TOPSIS*

Abstract

This paper proposes a new method for ranking a finite set of alternatives evaluated on multiple criteria. The presented method combines the robust ordinal regression (ROR) approach and the ranking score based on the aggregate distance measure function coming from the $TOPSIS$ method. In our method, the preference model is a set of additive value functions compatible with a non-complete set of pairwise comparisons of some reference alternatives given by the decision maker (DM). Based on this set of compatible value functions, we define an aggregate function representing relative closeness to the reference point (ideal solution) in the value space. The ranking score determined by this distance measure is then used to rank all alternatives. Calculating the distance in the value space permits to avoid normalization used in $TOPSIS$ to transform original evaluations on different criteria scales into a common scale. This normalization is perceived as a weakness of $TOPSIS$ and other methods based on a distance measure, because the ranking of alternatives depends on the normalization technique and the distance measure. Thus, ROR applied to $TOPSIS$ does not only facilitate the preference elicitation but also solves the problem of non-meaningfulness of $TOPSIS$. Finally, an instructive example is given to illustrate the proposed method.

Keywords: Multiple criteria decision aiding, robust ordinal regression, $TOPSIS$ method, UTA^{GMS} and $GRIP$ methods.

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1 Introduction

Multiple criteria decision aiding (MCDA) helps in constructing an aggregation model on the basis of preference information provided by the DM. Such an aggregation model is called preference model. It induces a preference structure in the set of considered alternatives (for a recent state-of-the-art, see Figueira et al. (2005)). The preference information may be either direct or indirect, depending on whether it specifies directly values of some parameters used in the preference model (e.g., trade-off weights, aspiration levels, discrimination thresholds, etc.), or else some examples of holistic judgments from which compatible values of the preference model parameters are induced. Direct preference information is used in the traditional aggregation paradigm, according to which the aggregation model is constructed first and then applied to the whole set of alternatives to get information about the comprehensive preference relation.

Eliciting direct preference information from the DM can be counterproductive in real-world decision making situations because of a high cognitive effort required. Very often this information is not easily definable. Consequently, asking the DM directly to provide values of the parameters makes the DM rather uncomfortable. For example, this is the case of the price or the interest rates in cost-benefit analysis, or the case of the coefficients in objectives and constraints of mathematical programming models, or the case of attribute weights and several thresholds in outranking methods.

Eliciting indirect preferences in the form of holistic pairwise comparisons of some reference or training alternatives is much less demanding of cognitive effort. This kind of preference information is given as decision examples. Such a reverse search of a preference model from decision examples is done by so-called *ordinal regression* (also called disaggregation-aggregation approach). The preference model found by ordinal regression is *compatible* with the given preference information, i.e., it restores the holistic pairwise comparisons made by the DM. Finally, it is used on the whole set of alternatives to recommend the best choice, classification, or ranking. In this paper we will use the preference model to recommend a ranking only.

The ordinal regression paradigm has been known for at least fifty years in the field of multidimensional analysis (see March, 1978). This paradigm has been applied within the two main MCDA approaches: those using a value function as preference model (see Srinivasan and Shocker, 1973; Pekelman and Sen, 1974; Jacquet-Lagrèze and Siskos, 1982), and those using an outranking relation as preference model (see Kiss et al., 1994; Mousseau and Słowiński, 1998; Mousseau et al., 2000). This paradigm has also been used since the mid nineties in MCDA methods involving a new, third family of preference models - a set of dominance decision rules induced from rough approximations of holistic preference relations (see Greco et al., 2001).

Usually, among the many sets of parameters of a preference model representing the preference information, only one specific set is used to give a recommendation on a set of alternatives. For example, among many value functions representing pairwise comparisons of some alternatives made by the DM, only one value function is finally used to recommend the best choice, or sorting, or ranking of alternatives. Since the choice of one among many sets of parameters compatible with the preference information is rather arbitrary, *robust ordinal regression* (ROR) has been recently proposed with the aim of taking into account all the sets of parameters compatible with the preference information given by the DM (see Greco et al., 2008; Figueira et al., 2009; Greco et al., 2010).

The robust ordinal regression builds a set of additive value functions compatible with preference information provided by the DM and results in two rankings: the necessary ranking and the possible ranking. Such rankings answer to robustness concerns, since they provide, in general, "more robust" conclusions than a ranking made by an arbitrarily chosen compatible value function. However, in some decision-making situations, it may be desirable to give a score to different alternatives (solutions), and despite the interest of the rankings provided, some users would like to see, and they indeed need, to know the "most representative" value function among all the compatible ones. This allows assigning a score to each alternative. Recently, a methodology to identify the "most representative" function in ROR without losing the advantage of taking into account all compatible value functions has been proposed in Greco et al. (2011); Kadziński et al. (2012).

In this paper we will adopt the similar idea of providing robust conclusion by applying ROR to the $TOPSIS$ method. $TOPSIS$ ranks alternatives according to their closeness to two reference points: ideal and anti-ideal solutions in the normalized and weighted criteria space. The best alternative should have simultaneously the shortest distance to the ideal solution and the farthest distance to the anti-ideal solution (see Hwang and Yoon, 1981; Chen and Hwang, 1992).

To eliminate the impact of different physical scales on the final recommendation, a method like $TOPSIS$, and other methods based on the distance measure, need to normalize the multi-criteria evaluations before introducing the distance measure. This normalization, that transforms original evaluations into a common scale, is perceived as a weakness of such methods, because it is responsible for their non-meaningfulness (see Martel and Roy, 2006).

In the proposed extension of the $TOPSIS$ method we will consider the preference model in the form of a set of additive value functions compatible with the preference information given by the DM. We will propose a new way of calculating relative closeness score in the value space that takes into account all compatible value functions and provides a robust conclusion.

The paper is organized in the following way. Section 2 recalls some concepts of robust ordinal regression, as well as some elements of the GRIP method which is presently the most general of all UTA-like methods. Section 3 recalls basic concepts of the $TOPSIS$ method. Section 4 presents a new method that combines the robust ordinal regression and the ranking score based on the aggregate distance measure coming from the $TOPSIS$ method. Section 5 presents a didactic example. Last section contains conclusions.

2 Robust Ordinal Regression

Let us consider a multiple criteria decision problem where a finite set of m alternatives $A = \{a_1, \ldots, a_m\}$, is evaluated on a finite family $F = \{g_1, \ldots, g_n\}$ of n criteria. Let $I = \{1, \ldots, n\}$ denote the set of criteria indices, and assume, without loss of generality, that the greater $g_i(a)$, the better alternative a on criterion g_i, for all $i \in I$, $a \in A$, i.e. g_i are all gain-type criteria $(i = 1, \ldots, n)$.

A DM is willing to rank the alternatives of A from the best to the worst, according to his/her preferences. The ranking can be complete or partial, depending on the preference information provided by the DM and on the way of exploiting this information. The family of criteria F is supposed to satisfy some consistency conditions, i.e. completeness (all relevant criteria are considered), monotonicity (the better the evaluation of an alternative on considered criteria, the more it is preferable to another), and non-redundancy (no superfluous criteria are considered) (see Roy and Bouyssou, 1993). Such a decision-making problem statement is called a *multiple criteria ranking problem*.

It is well known that the only objective information coming out from the above problem statement is a *dominance relation* in the set A. In fact, alternative a_k dominates alternative a_l if a_k is at least as good as a_l on all criteria from F, and there is at least one criterion from F such that a_k is better than a_l. Moreover, there is no doubt that then a_k should be comprehensively considered at least as good as a_l, independently of the specific preferences of the DM. Instead, when a_k is not dominating a_l, the statement that a_k is at least as good as a_l depends on the preferences of the DM. According to the dominance relation, alternative $a_k \in A$ is preferred to alternative $a_l \in A$ (denoted as $a_k \succ a_l$) if and only if $g_i(a_k) \geq g_i(a_l)$ for all $i \in I$, with at least one strict inequality; a_k is indifferent to a_l (denoted as $a_k \sim a_l$) if and only if $g_i(a_k) = g_i(a_l)$ for all $i \in I$; finally, a_k is incomparable with a_l (denoted as $a_k \; ? \; a_l$) otherwise, i.e. if $g_i(a_k) > g_i(a_l)$ for at least one criterion $i \in I$ and $g_j(a_k) < g_j(a_l)$ for at least one other criterion $j \in I$. Since incomparability is the most frequent situation, the dominance relation is usually very poor.

To enrich this relation, the DM has to provide preference information which is used to construct a preference model (also called an aggregation model) making the alternatives more comparable. This preference model induces a preference structure on the set A, whose proper exploitation permits to work out a ranking proposed to the DM.

The robust ordinal regression approach (ROR) extends the simple ordinal regression by taking into account not a single instance of the preference model compatible with the DM's preference information, but the whole set of compatible instances of the preference model. As a result of considering the whole set of compatible instances of the preference model, one gets two kinds of results with respect to each pair of alternatives $a_k, a_l \in A$:

- *necessary preference relation* $a_k \succsim^N a_l$, if and only if a_k is at least as good as a_l according to all instances of the preference model compatible with the preference information,
- *possible preference relation* $a_k \succsim^P a_l$, if and only if a_k is at least as good as a_l according to at least one instance of the preference model compatible with the preference information.

The necessary preference relation can be considered as robust with respect to the preference information. The robustness of the necessary preference relation refers to the fact that a given pair of alternatives compares in the same way whatever the instance of the preference model compatible with the preference information. Indeed, when no preference information is given, the necessary preference relation boils down to the dominance relation, and the possible preference relation is a complete relation. Every new item of the preference information, e.g., a pairwise comparison of some reference alternatives for which the dominance relation does not hold, is enriching the necessary preference relation and it is impoverishing the possible preference relation, so that they converge with the growth of the preference information. Such an approach gives also space for interactivity with the DM.

In what follows, the evaluation of each alternative $a \in A$ on each criterion $g_i \in F$ will be denoted by $g_i(a)$. Let G_i denote the value set (scale) of the criterion $g_i, i \in I$. Consequently, $G = \prod_{i \in I} G_i$ represents the evaluation space, and $a \in G$ denotes a profile of an alternative in such a space. We consider a *weak preference relation* $\succsim$ on A which means, for each pair of vectors, $a_k, a_l \in G,\ a_k \succsim a_l \Leftrightarrow$ "$a_k$is at least as good as a_l". This weak preference relation can be decomposed into an asymmetric and a symmetric part, as follows:

1) $a_k \succ a_l \equiv [a_k \succsim a_l$ and not $a_l \succsim a_k] \Leftrightarrow$ "a_k is preferred to a_l",

2) $a_k \sim a_l \equiv [a_k \succsim a_l$ and $a_l \succsim a_k] \Leftrightarrow$ "a_k is indifferent to a_l".

From a pragmatic point of view, it is reasonable to assume that $G_i = [\alpha_i, \beta_i]$, i.e. the evaluation scale on each criterion g_i is bounded, such that $\alpha_i < \beta_i$ are the worst and the best (finite) evaluations, respectively. Thus, $g_i : A \to G_i, i \in I$, therefore, each alternative $a \in A$ is associated with an evaluation vector $[g_1(a), \ldots, g_n(a)] \in G$.

The idea of considering the whole set of value functions compatible with the preference information provided by the DM was originally introduced in UTA^{GMS} (see Greco et al. (2008)). In this method, the preference information is given in the form of a partial preorder $\succsim$ on a subset of reference alternatives $A^R \subseteq A$ (i.e., a set of pairwise comparisons of reference alternatives), called a *reference preorder*. The reference alternatives are usually those contained in the set A for which the DM is able to express holistic preferences.

The method $GRIP$ (Generalized Regression with Intensities of Preference) proposed by Figueira et al. (2009) can be seen as an extension of UTA^{GMS} permitting to take into account additional preference information in form of comparisons of intensities of preference between some pairs of reference alternatives. These comparisons are expressed in two possible ways (not exclusive):

- comprehensively, on all criteria, that defines a partial preorder $\succsim^*$ on $A^R \times A^R$, such that, given $a_k, a_l, a_p, a_q \in A^R$, $(a_k, a_l) \succsim^* (a_p, a_q)$ means "a_k is preferred to a_l at least as much as a_p is preferred to a_q",

- partially, on any criterion, that defines a partial preorder $\succsim_i^*$ on $A^R \times A^R$, such that, given $a_k, a_l, a_p, a_q \in A^R$, $(a_k, a_l) \succsim_i^* (a_p, a_q)$ means "a_k is preferred to a_l at least as much as a_p is preferred to a_q, on criterion $g_i,\ i \in I$".

In what follows, after Figueira et al. (2009), we also consider the weak preference relation $\succsim_i$ on A being a complete preorder whose meaning is: for all $a_k, a_l \in A$, $a_k \succsim_i a_l \Leftrightarrow$ "a_k is at least as good as a_l on criterion g_i, $i \in I$." This relation is not provided by the DM, but it is obtained directly from the evaluation of alternatives a_k and a_l on criterion g_i, i.e., $a_k \succsim_i a_l \Leftrightarrow g_i(a_k) \geq g_i(a_l)$.

In ROR involving additive value functions as a preference model, it has the form $U(a) = \sum_{i \in I} u_i(a)$, where u_i are marginal value functions which are either:

(a) *piecewise-linear*,

(b) *general monotone, non-decreasing.*

In case (a), each range $[\alpha_i, \beta_i]$ is divided into $k_i \geq 1$ equal sub-intervals $[x_i^0, x_i^1]$, $[x_i^1, x_i^2]$, $\ldots$, $[x_i^{k_i-1}, x_i^{k_i}]$, where $x_i^j = \alpha_i + \frac{j}{k_i}(\beta_i - \alpha_i)$, $j = 0, \ldots, k_i$, and $i \in I$. The marginal value of an alternative $a \in A$ is obtained by linear interpolation:

$$u_i(a) = u_i(x_i^j) + \frac{g_i(a) - x_i^j}{x_i^{j+1} - x_i^j}(u_i(x_i^{j+1}) - u_i(x_i^j)), \quad g_i(a) \in [x_i^j, x_i^{j+1}]$$

The piecewise-linear additive model is completely defined by the marginal values at the breakpoints, i.e., $u_i(x_i^0) = u_i(\alpha_i)$,$u_i(x_i^1)$,$u_i(x_i^2)$,$\ldots$, $u_i(x_i^{k_i}) = u_i(\beta_i)$, $i \in I$. The number of linear pieces k_i is fixed a priori for each marginal value function u_i, $i \in I$.

In case (b), the characteristic points of the marginal value functions u_i, $i \in I$ are fixed in evaluation points of considered alternatives. Let l_i be the permutation on the set of indices of alternatives from A^R that reorders them according to the increasing evaluation on criterion i, i.e.:

$$\alpha_i \leq x_{l_i(1)} \leq x_{l_i(2)} \leq \ldots \leq x_{l_i(m-1)} \leq x_{l_i(m)} \leq \beta_i, \ i \in I$$

The general, non-decreasing additive model is completely defined by the marginal values at the characteristic points, i.e., $u_i(\alpha_i)$,$u_i(x_{k_i(1)})$, $u_i(x_{k_i(2)})$, $\ldots$, $u_i(x_{k_i(m)}), u_i(\beta_i)$. Note that in this case, no linear interpolation is required to express the marginal value of any reference alternative.

A value function is called *compatible* if it is capable of restoring the partial preorder $\succsim$ on A^R, as well as the given relation of intensity of preference among ordered pairs of reference alternatives. Moreover, each compatible value function induces a complete preorder (ranking) on the whole set A.

In particular, for any two alternatives $a_k, a_l \in A$, a compatible value function U ranks a_k and a_l in one of the following ways:

- a_k is preferred to a_l because $U(a_k) > U(a_l)$,

- a_l is preferred to a_k because $U(a_k) < U(a_l)$,
- a_k is indifferent to a_l because $U(a_k) = U(a_l)$.

With respect to $a_k, a_l \in A$, it is thus reasonable to ask the following two questions:

- are a_k and a_l ranked in the same way by *all* compatible value functions?
- is there *at least one* compatible value function ranking a_k at least as good as a_l (or a_l at least as good as a_k)?

Having answers to these questions for all pairs of alternatives $(a_k, a_l) \in A \times A$, one gets a *necessary weak preference relation* $\succsim^N$, if $U(a_k) \geq U(a_l)$ for all compatible value functions, and a *possible weak preference relation* $\succsim^P$ in A, if $U(a_k) \geq U(a_l)$ for at least one compatible value function.

Let us remark that preference relations $\succsim^N$ and $\succsim^P$ are meaningful only if there exists at least one compatible value function. Therefore, whenever the contrary is not explicitly stated, we suppose that there exists at least one compatible value function. Observe also that in this case, for any $a_k, a_l \in A^R$:

$$a_k \succsim a_l \Rightarrow a_k \succsim^N a_l, \text{ and } a_k \succ a_l \Rightarrow \text{ not } (a_l \succsim^P a_k).$$

In fact, if $a_k \succsim a_l$, then for any compatible value function, $U(a_k) \geq U(a_l)$ and, therefore, $a_k \succsim^N a_l$. Moreover, if $a_k \succ a_l$, then for any compatible value function, $U(a_k) > U(a_l)$ and, consequently, there is no compatible value function such that $U(a_l) \geq U(a_k)$, which means that not $(a_l \succsim^P a_k)$.

Formally, an additive compatible value function is an additive value function $U(a) = \sum_{i \in I} u_i(a)$ satisfying the following set of constraints corresponding to the DM's preference information:

a) $U(a_k) \geq U(a_l) + \varepsilon$ if $a_k \succ a_l$,

b) $U(a_k) = U(a_l)$ if $a_k \sim a_l$,

c) $U(a_k) - U(a_l) \geq U(a_p) - U(a_q) + \varepsilon$ if $(a_k, a_l) \succ^* (a_p, a_q)$,

d) $U(a_k) - U(a_l) = U(a_p) - U(a_q)$ if $(a_k, a_l) \sim^* (a_p, a_q)$,

e) $u_i(a_k) - u_i(a_l) \geq u_i(a_p) - u_i(a_q) + \varepsilon$ if $(a_k, a_l) \succ_i^* (a_p, a_q),\ i \in I$,

f) $u_i(a_k) - u_i(a_l) = u_i(a_p) - u_i(a_q)$ if $(a_k, a_l) \sim_i^* (a_p, a_q),\ i \in I$,

where $a_k, a_l, a_p, a_q \in A^R$, and $\varepsilon > 0$. Moreover, the following monotonicity and normalization constraints are also taken into account:

g) $u_i(a_k) \geq u_i(a_l)$ if $a_k \succsim_i a_l,\ \forall a_k, a_l \in A,\ i \in I$,

h) $u_i(\alpha_i) = 0$, where $\alpha_i = \min\{g_i(a) \colon a \in A\}$,

i) $\sum_{i\in I} u_i(\beta_i) = 1$, where $\beta_i = \max\{g_i(a)\colon a \in A\}$.

If the constraints from a) to i) are satisfied, then there exists at least one compatible value function, and the partial preorders $\succsim$ and $\succsim^*$ on A^R and $A^R \times A^R$ can be extended on A and $A \times A$, respectively.

To avoid the use of an arbitrary value of ε, we consider it as an auxiliary variable, and we test the feasibility of constraints a), c) and e). In this way, we take into account all possible value functions, even those for which the threshold ε is very small. In fact, the value of ε is not meaningful by itself and it is useful only because it permits to discriminate preference from indifference.

Therefore, to conclude about the truth or falsity of binary relations $\succsim^N$ and $\succsim^P$, for all $a_k, a_l \in A$ and $i \in I$, one has to solve a series of linear programming problems with ε as the objective function to be maximized, as explained below (Greco et al., 2008):

1) $a_k \succsim^P a_l \Leftrightarrow \varepsilon^* > 0$,
where $\varepsilon^* = \max\, \varepsilon$, subject to the constraints a), b), c), d), e), f), g), h), i), plus the constraint: $U(a_k) - U(a_l) \geq 0$;

2) $a_k \succsim^N a_l \Leftrightarrow \varepsilon^* \leq 0$,
where $\varepsilon^* = \max\, \varepsilon$, subject to the constraints a), b), c), d), e), f), g), h), i), plus the constraint: $U(a_l) - U(a_k) \geq \varepsilon$;

Analogously, one can test relations $\succsim^{*^N}$, $\succsim^{*^P}$, $\succsim_i^{*^N}$ and $\succsim_i^{*^P}$, for all $a, b, c, d \in A$ and $i \in I$ (see Figueira et al., 2009).

3 *TOPSIS* method

TOPSIS (Technique for Order Preference by Similarity to an Ideal Solution) is a multiple criteria decision method to rank alternatives or to select the best alternative from a finite set of alternatives. It was initially proposed by Hwang and Yoon (1981) for solving a multiple attribute decision making problem with no articulation of preference information. The original *TOPSIS* concept and its various extensions have been widely used in the literature (Lai et al., 1994; Chen, 2000; Deng et al., 2000; Chu, 2002; Braglia et al., 2003; Liao, 2003; Chen and Tzeng, 2004; Olson, 2004; Opricovic and Tzeng, 2004; Abo-Sinna and Amer, 2005).

The basic principle of *TOPSIS* is that the best alternative should have simultaneously the shortest distance from the ideal alternative and the farthest distance from the anti-ideal alternative (see Chen and Hwang, 1992). Assuming, without loss of generality, that all criteria are of gain-type, the ideal alternative (called a positive ideal solution) is an alternative that maximizes each individual criterion, whereas the anti-ideal alternative (called a negative ideal solution) minimizes each individual criterion.

Here, we are adopting the same notation as in Section 2, with the exception that alternatives are numbered and the identifying index is j, i.e. $g_i(a_j) = g_{ji}$, $i = 1, \dots, n$; $j = 1, \dots, m$. All evaluations of the alternatives on particular criteria are presented in a decision matrix denoted by $\left[g_{ji}\right]_{m \times n}$. It is also assumed that the DM has determined the relative weights of criteria (denoted as w_i, for $i = 1, \dots, n$), satisfying $\sum_{i=1}^{n} w_i = 1$.

Then, the steps of the *TOPSIS* method can be expressed in the following way:

1. Construct the normalized decision matrix $\left[r_{ji}\right]_{m \times n}$. The normalized value r_{ji} is calculated as:

$$r_{ji} = \frac{g_{ji}}{\sqrt{\sum_{j=1}^{m} g_{ji}^2}}, \quad i = 1, \dots, n;\ j = 1, \dots, m$$

   In this step the various criteria dimensions are transformed into non-dimensional attributes, which allows comparison across the criteria. It should be noted that different kinds of normalization procedure usually produce different rankings of alternatives. More information about the normalization can be found in (Opricovic and Tzeng, 2004) and in (Martel and Roy, 2006).

2. Construct the weighted normalized decision matrix $\left[v_{ji}\right]_{m \times n}$. The weighted normalized value v_{ji} is calculated as:

$$\left[v_{ji}\right] = \left[w_i\right]\left[r_{ji}\right], \quad i = 1, \dots, n;\ j = 1, \dots, m$$

3. Determine the ideal and anti-ideal solutions (alternatives):

$$a^+ = \left\{v_1^+, \dots, v_n^+\right\} = \left\{\max_j v_{ji},\ j = 1, \dots, m\right\},$$

$$a^- = \left\{v_1^-, \dots, v_n^-\right\} = \left\{\min_j v_{ji},\ j = 1, \dots, m\right\}.$$

4. Calculate the distance measures of each alternative, using the n-dimensional Euclidean distance. The distance of the alternative $a_j \in A$ to the ideal solution (denoted as $d^+(a_j)$) and the anti-ideal solution (denoted as $d^-(a_j)$) is given as:

$$d^+(a_j) = \sqrt{\sum_{i=1}^{n} (v_{ji} - v_i^+)^2}, \quad j = 1, \dots, m,$$

$$d^-(a_j) = \sqrt{\sum_{i=1}^{n}(v_{ji} - v_i^-)^2}, \quad j = 1, \ldots, m.$$

5. Calculate the relative closeness of each alternative to the ideal solution. The relative closeness of the alternative $a_j \in A$ (denoted as $c^*(a_j)$) with respect to a^+ is defined as:

   $$c^*(a_j) = \frac{d^-(a_j)}{d^+(a_j) + d^-(a_j)}, \quad j = 1, \ldots, m$$

   Since $d^+(a_j) \geqslant 0$ and $d^-(a_j) \geqslant 0$, then $c^*(a_j) \in [0, 1]$ for each $a_j \in A$. Moreover, $c^*(a_j) = 1$ if $a_j = a^+$ and $c^*(a_j) = 0$ if $a_j = a^-$.

6. Rank the alternatives in the descending order of relative closeness c^*. The best alternative is the one with the greatest relative closeness to the ideal solution.

Note that the *TOPSIS* method introduces two reference points, but the way of calculating relative closeness does not take into account the relative importance of the distances from these points.

4 ROR applied to *TOPSIS*

As announced in the Introduction, in this section we present a new method for ranking a finite set of alternatives that combines the robust ordinal regression approach and the relative closeness ranking score based on the aggregate distance measure function coming from the *TOPSIS* method.

In the proposed approach, the preference model is composed of a set of additive value functions compatible with the preference information given by the DM. We consider marginal value functions having one of two forms: *piecewise-linear*, or *general monotone, non-decreasing*. The preference information is composed of a non-complete set of pairwise comparisons of reference alternatives, and of comparisons of intensities of preference between some pairs of reference alternatives. These comparisons are expressed in two possible, not exclusive ways: comprehensively, i.e. with respect to all criteria, and partly, i.e. with respect to particular criteria. Based on this set of compatible value functions, we solve a series of non-linear programming problems to introduce an aggregate function representing closeness in the *value space* of each alternative to the two reference points: the ideal, and the anti-ideal solutions. The ranking score determined by this distance measure function is then used to rank all alternatives.

Below we adopt notation introduced in previous sections. The detailed procedure of the proposed approach can be described in the following steps:

1. Determine the feasibility of the preference information provided by the DM, by solving the linear programming problem $\varepsilon^{*}= \max \varepsilon$, subject to the constraints from a) to i) (see Section 2), and testing the positive value of ε^{*}. Moreover, we want to use an additive value function compatible with preference information given by the DM that uses marginal value functions having as simple form as possible. To achieve this, we test the representability of the preference information in the simplest, linear form of the additive value function. If this test fails, we change the preference model to more complex, i.e. increase the number of linear pieces in piecewise-linear marginal value functions, or use the general monotone, non-decreasing marginal value functions.

2. Calculate the distance measures of each alternative, using the n-dimensional Euclidean distance in the value space with respect to the ideal and anti-ideal points. From among all compatible value functions we choose one that minimizes the Euclidean distance from the alternative considered to the ideal point a^{+} in the value space. We assume that the ideal point in the value space has all co-ordinates equal to 1, i.e., $a^{+}=\left[1,\ldots,1\right]$. Such a point is certainly non-attainable for any preference information. It is a stable reference point for all alternatives throughout the entire procedure. Analogously, from among all compatible value functions we choose one that maximizes the Euclidean distance from the alternative considered to the anti-ideal point a^{-} in the value space. The anti-ideal point is defined as $a^{-}=\left[0,\ldots,0\right]$ in the value space. It is also constant for the whole procedure. Therefore, the distance of the alternative $a_j \in A$ to the ideal point (denoted as $d^{+}(a_j)$) in the value space can be obtained as the result of the following non-linear programming problem:

$$\text{Minimize: } d^{+}(a_j)=\sqrt{\sum_{i=1}^{n}\left(u_i(a_j)-1\right)^{2}}$$

subject to the following set of constraints corresponding to the DM's preference information:

a') $c^{*}(a_k) \geq c^{*}(a_l)+\varepsilon$ if $a_k \succ a_l$,

b') $c^{*}(a_k)=c^{*}(a_l)$ if $a_k \sim a_l$,

c') $c^{*}(a_k)-c^{*}(a_l) \geq c^{*}(a_p)-c^{*}(a_q)+\varepsilon$ if $(a_k,a_l) \succ^{*} (a_p,a_q)$, d') $c^*(a_k) - c^*(a_l) = c^*(a_p) - c^*(a_q)$ if $(a_k, a_l) \sim^* (a_p, a_q)$,

e') $u_i(a_k) - u_i(a_l) \geq u_i(a_p) - u_i(a_q) + \varepsilon$ if $(a_k, a_l) \succ_i^* (a_p, a_q), i \in I$,

f') $u_i(a_k) - u_i(a_l) = u_i(a_p) - u_i(a_q)$ if $(a_k, a_l) \sim_i^* (a_p, a_q), i \in I$,

where $a_k, a_l, a_p, a_q \in A^R$, ε is a small positive value, and $c^*(a_h)$ is the relative closeness of the alternative $a_h \in A^R$ to the ideal solution in the value space, given as:

$$c^*(a_h) = \frac{\sqrt{\sum_{i=1}^{n}\big(u_i(a_h)\big)^2}}{\sqrt{\sum_{i=1}^{n}\big(u_i(a_h)\big)^2} + \sqrt{\sum_{i=1}^{n}\big(u_i(a_h) - 1\big)^2}}$$

Moreover, the monotonicity and normalization constraints g), h) and i) (see Section 2) are also taken into account. Analogously, the distance of the alternative $a_j \in A$ to the anti-ideal point (denoted as $d^-(a_j)$) in the value space can be obtained as the result of the following non-linear programming problem:

$$\text{Maximize: } d^-(a_j) = \sqrt{\sum_{i=1}^{n}\big(u_i(a_j)\big)^2}$$

subject to constraints a'), b'), c'), d'), e'), f'), g), h) and i).

3. Calculate the ranking score of each alternative which is the relative closeness to the ideal solution in the value space. The relative closeness of the alternative $a_j \in A$ (denoted by $c^*(a_j)$) with respect to the ideal solution a^+ is defined as:

$$c^*(a_j) = \frac{d_*^-(a_j)}{d_*^+(a_j) + d_*^-(a_j)}, \quad j = 1, \ldots, m,$$

where $d_*^+(a_j)$ and $d_*^-(a_j)$ are optimal distances resulting from step 2.

4. Rank the alternatives in the descending order of ranking score c^*. The best alternative is the one with the greatest relative closeness to the ideal solution in the value space.

It should be noted that the proposed way of calculating distance in the value space takes into account all value functions compatible with the given preference information. Moreover, it permits to avoid the normalization procedure that is used in *TOPSIS* to transform original evaluations on different scales into a common scale.

5 A didactic example

To illustrate the method proposed in the previous Section, we present the following problem inspired from practice.

Suppose that a DM in the public transport company has to rank order 12 buses taking into account the following criteria considered during the periodical technical inspection of the buses:

- MaxSpeed [gain] – maximum speed [km/h],
- Torque [gain] – torque [Nm],
- FuelCons [cost] – fuel consumption [l/100km],
- OilCons [cost] – oil consumption [l/100km],
- HorsePow [gain] – maximum horsepower of the engine [KM].

Criteria "MaxSpeed", "FuelCons" evaluate the overall performance of the bus, while others concentrate on the characteristics of the engine. Evaluations of all buses considered on the above criteria are given in Table 1.

BusId	MaxSpeed	Torque	FuelCons	OilCons	HorsePow
b01	90	426	27	2	112
b02	90	425	27	1	112
b03	87	400	23	4	96
b04	86	448	26	1	120
b05	83	402	26	2	128
b06	82	428	33	2	121
b07	80	445	26	1	122
b08	71	480	23	0	119
b09	75	449	26	1	120
b10	74	430	25	2	115
b11	72	479	35	1	145
b12	68	440	26	2	126

Table 1

Evaluations of 12 buses on 5 criteria

Suppose the DM provided the following preference information:

1. Pairwise comparisons of some buses:
   - bus b06 is preferred to bus b03: b06 $\succ$ b03,
   - bus b01 is indifferent to bus b02: b01 $\sim$ b02,

2. Overall intensity of preference:
   - bus b04 is preferred to bus b08 stronger than bus b07 is preferred to bus b06: (b04, b08) $\succ$ (b07, b06),
3. Intensity of preference on criterion "Torque":
   - with respect to "Torque", bus b02 is preferred to bus b03 stronger than bus b04 is preferred to bus b06:
     (b02, b03) $\succ_{Torque}$ (b04, b06).

Applying the proposed method to the above preference information, and using the linear model of preferences, we got distances to the ideal solution (d^+) and to the anti-ideal solution (d^-) in the value space, presented in Table 2.

Table 2
Distances of 12 buses to the ideal and the anti-ideal solutions in the value space

BusId	d^+	d^-
b01	2.0968	0.3116
b02	2.0968	0.3116
b03	2.2227	0.0302
b04	2.0518	0.4654
b05	2.0239	0.6197
b06	2.0519	0.4843
b07	2.0469	0.5035
b08	2.0652	0.4454
b09	2.0601	0.4646
b10	2.0900	0.3678
b11	1.9893	0.9483
b12	2.0394	0.5806

Table 3
Final ranking of 12 buses according to relative closeness scores in the value space

Rank	c^*	BusId
1	0.3296	b11
2	0.2406	b05
3	0.2223	b12
4	0.2027	b01
4	0.2027	b02
5	0.2025	b07
6	0.1945	b06
7	0.1912	b04
8	0.1876	b09
9	0.1830	b08
10	0.1669	b03
11	0.1527	b10

Based on the above distance measures we calculate the relative closeness score (c^*) of each alternative to the ideal solution in the value space. Then, we rank all alternatives according to the descending order of this relative closeness. The final ranking including relative closeness scores is presented in Table 3.

6 Conclusion

We presented a new MCDA ranking method that combines the advantages of the robust ordinal regression approach and the ranking score based on the aggregate distance measure function coming from the $TOPSIS$ method. In this method, the preference model is composed of a set of additive value functions compatible with the preference information given by the DM. Based on this set of compatible value functions, we introduce an aggregate function representing relative closeness to the reference point (ideal solution) in the value space. The ranking score determined by this distance measure function is then used to rank all alternatives.

The main advantages of the proposed method are:

1. It models the user's preferences in terms of additive value functions, that can be composed of linear, piecewise-linear or very general monotonic marginal value functions.

2. It takes into account the preference information expressed by the user in a very simple and intuitive way i.e. in the form of comparisons of some reference alternatives and/or in the form of intensities of preference between some pairs of reference alternatives. Moreover, intensities of preference can be specified comprehensively, on all criteria, or partly, on specific criteria.

3. It permits to detect inconsistent preference information with respect to an assumed form of the preference model. When the ordinal regression fails to find any compatible value function, the DM can remove this impossibility in one of two ways: the preference model can be changed to a more complex one (i.e. from piecewise-linear to general additive), or the inconsistent subset of pairwise comparisons can be detected and removed.

4. Calculating distance in the value space permits to avoid the normalization procedure that is used in $TOPSIS$ to transform original evaluations on different scales into a common unit. This normalization makes $TOPSIS$, and other methods based on the distance measure function, non-meaningful.

5. It can deal with ordinal scales of criteria while in the original $TOPSIS$ method the calculation of distance in the original ordinal criteria space does not make sense. Here, the ordinal criteria scales are translated

into marginal value scales which are interval scales. These scales give a correct interpretation to the distance in the value space.

6. It provides robust conclusions. The proposed relative closeness score used to rank the alternatives is based on all compatible value functions, rather than on only one or a few among the many possible functions, as it is usual in MCDA.

In brief, the robust ordinal regression applied to $TOPSIS$ not only facilitates the preference elicitation but also solves the problem of non-meaningfulness of $TOPSIS$.

Acknowledgements

The author wish to acknowledge financial support from the Polish National Science Centre, grant No. N N519 441939.

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