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DUALITY IN FUZZY MULTIPLE OBJECTIVE LINEAR PROGRAMMING WITH POSSIBILITY AND NECESSITY RELATIONS

INTRODUCTION

The problem of duality has been investigated since the early stage of fuzzy linear programming (FLP), see [3; 9; 11; 12]. In this paper we first introduce a broad class of fuzzy multiple objective linear programming problems (FMOLP problems) and define the concepts of β-feasible and (α, β)-maximal and minimal solutions of FMOLP problems. The class of classical MOLP problems can be embedded into the class of FMOLP ones, moreover, for FMOLP problems we define the concept of duality and prove the weak and strong duality theorems – generalizations of the classical ones. The results are compared to the existing literature [6; 7; 13; 8]. To illustrate the introduced concepts and results we present and discuss a simple numerical example.

1. PRELIMINARIES

Let X be a nonempty topological space. By $\mathcal{F}(X)$ we denote the set of all fuzzy subsets A of X, where every fuzzy subset A of X is uniquely determined by the membership function $\mu_A : X \to [0,1]$, and $[0,1] \subset \mathbf{R}$ is a unit interval, $\mathbf{R}$ is the Euclidean space of real numbers. We say that the fuzzy subset A is *crisp* if μ_A is a characteristic function of A, i.e. $\mu_A : X \to \{0,1\}$. It is clear that the set of all subsets of X, $\mathcal{P}(X)$, can be isomorphically embedded into $\mathcal{F}(X)$.

Let

$$\begin{aligned} [A]_\alpha = &\quad \{x \in X | \mu_A(x) \geqslant \alpha\} \text{ for } \alpha \in (0,1], \\ [A]_0 = &\quad cl\{x \in X | \mu_A(x) > 0\} \ . \end{aligned}$$

where clB means a topological closure of B, $B \subset X$. For $\alpha \in [0,1]$, $[A]_\alpha$ are called *α-cuts.* $[A]_0$ is usually called a *support of* A. A fuzzy subset A of X is *closed, bounded, compact* or *convex*, if $[A]_\alpha$ are closed, bounded, compact or convex subsets of X for every $\alpha \in [0,1]$, respectively. By the *strict α-cut* we denote $(A)_\alpha = \{x \in X | \mu_A(x) > \alpha\}$. Moreover, A is said to be *normal* if $[A]_1$is nonempty. It is a well known fact that a fuzzy subset A of X is convex if and only if its membership function μ_A is quasiconcave on X, see e.g. [6] and also

Definition 4 below.

In the set theory, a *binary relation* P *on* X is a subset of the Cartesian product $X \times X$, that is, $P \subset X \times X$. Here, a *valued relation* P *on* X is a fuzzy subset of $X \times X$. Evidently, any binary relation P *on* X can be isomorphically embedded into the class of valued relations on X by its characteristic function (i.e. membership function) μ_P. In this sense, any binary relation is valued.

Definition 1 A fuzzy subset $\tilde{P}$ of $\mathcal{F}(X) \times \mathcal{F}(X)$ is called a fuzzy relation on X, i.e. $\tilde{P} \in \mathcal{F}(\mathcal{F}(X) \times \mathcal{F}(X))$.

Definition 2 Let P be a valued relation on X. A fuzzy relation $\tilde{Q}$ on X is called a fuzzy extension of relation P, if for each $x, y \in X$, it holds

$$\mu_{\tilde{Q}}(x, y) = \mu_P\,(x, y)\,. \tag{1}$$

A fuzzy relations on X will be denoted by the tilde, e.g. $\tilde{P}$.

From now on, throughout this paper we shall consider $X = \mathbf{R}^n$, where $\mathbf{R}^n$ is the n-dimensional Euclidean space, particularly $X = \mathbf{R}^1 = \mathbf{R}$.

In the following definition we first present possibility and necessity indices introduced originally in [2] and then define a suitable class of fuzzy numbers called here fuzzy quantities. Then, we shall derive some basic properties of this class.

Definition 3 Let A, B be fuzzy sets with the membership functions $\mu_A : \mathbf{R} \to [0,1]$, $\mu_B : \mathbf{R} \to [0,1]$, respectively. Let

$$Pos(A \preceq B) = \sup\{\min(\mu_A\,(x)\,, \mu_B\,(y))|x \leqslant y, x, y \in \mathbf{R}\}, \tag{2}$$

$$Nec(A \prec B) = \inf\{\max(1 - \mu_A\,(x)\,, 1 - \mu_B\,(y))|x > y, x, y \in \mathbf{R}\}. \tag{3}$$

Here (2) is called the *possibility index*, (3) is called the necessity index.

The possibility and necessity index has been originally introduced in [2], where also mathematical analysis and interpretation of the one has been discussed. The indices (2), (3) can be understood as special fuzzy relations on $\mathbf{R}$ introduced by Definition 1. We write alternatively

$$Pos(A \quad \preceq \quad B) = \mu_{Pos}\,(A, B) = (A \preceq^{Pos} B), \tag{4}$$

$$Nec(A \quad \prec \quad B) = \mu_{Nec}\,(A, B) = (A \prec^{Nec} B), \tag{5}$$

where $\mu_\Omega : \mathcal{F}(\mathbf{R}) \times \mathcal{F}(\mathbf{R}) \to [0,1], \Omega \in \{Pos, Nec\}$ are the membership functions of the fuzzy relations on $\mathbf{R}$. By $A \succeq^{Pos} B$ or $A \succ^{Nec} B$ we mean $B \preceq^{Pos} A$ or $B \prec^{Nec} A$, respectively.

It can be easily verified that all possibility and necessity indices are fuzzy extensions of the classical binary relation $\leqslant$ according to Definition 2.

2. FUZZY QUANTITIES

To define a suitable class of fuzzy parameters of FMOLP problems we start with definitions of particular membership functions.

Definition 4 Let A be a normal and compact fuzzy subset of $\mathbf{R}$ with the membership function μ_A. A is called the *fuzzy quantity* if there exist $a, b, c, d \in \mathbf{R}$, $-\infty < a \leqslant b \leqslant c \leqslant d < +\infty$, such that

$$\begin{array}{cc} \mu_A(t) = 0 & \text{if } t < a \text{ or } t > d, \\ \mu_A(t) & \text{is strictly increasing if } a \leqslant t \leqslant b, \\ \mu_A(t) = 1 & \text{if } b \leqslant t \leqslant c, \\ \mu_A(t) & \text{is strictly decreasing if } c \leqslant t \leqslant d. \end{array}$$

The set of all fuzzy quantities is denoted by $\mathcal{F}_0(\mathbf{R})$.

By the definition, $\mathcal{F}_0(\mathbf{R})$ contains well known classes of fuzzy numbers: crisp (real) numbers, crisp intervals, triangular fuzzy numbers, trapezoidal and bell-shaped fuzzy numbers etc. However, $\mathcal{F}_0(\mathbf{R})$ does not contain fuzzy sets with "step-like" membership functions. The proof of the following proposition is easy and can be found also in [5], or [4].

Proposition 5 Let $A, B \in \mathcal{F}(\mathbf{R})$, $\alpha \in [0, 1]$. Then
(i) $\mu_{Pos}(A, B) \geqslant \alpha$ iff $\inf[A]_\alpha \leqslant \sup[B]_\alpha$,
(ii) $\mu_{Nec}(A, B) \geqslant \alpha$ iff $\sup(A)_{1-\alpha} \leqslant \inf(B)_{1-\alpha}$.

Proposition 6 Let $A \in \mathcal{F}_0(\mathbf{R})$ be a fuzzy quantity, $\alpha \in [0, 1]$. Then

$$\inf[A]_\alpha = \inf(A)_\alpha, \sup[A]_\alpha = \sup(A)_\alpha. \tag{6}$$

Corollary 7 Let $A, B \in \mathcal{F}_0(\mathbf{R})$ be fuzzy quantities $\alpha \in (0, 1)$. Then combining Propositions 5 and 6 we obtain (i),(ii) as follows

$$\mu_{Pos}(A, B) \geqslant \alpha \text{ iff } \inf[A]_\alpha \leqslant \sup[B]_\alpha, \tag{7}$$

$$\mu_{Nec}(A, B) \geqslant \alpha \text{ iff } \sup[A]_{1-\alpha} \leqslant \inf[B]_{1-\alpha}. \tag{8}$$

Corollary 8 Let $A, B \in \mathcal{F}_0(\mathbf{R})$ be fuzzy quantities $\alpha \in (0, 1)$. Then from (7) and (8) we obtain the following formulae

$$\mu_{Pos}(B, A) < \alpha \text{ iff } \sup[A]_\alpha < \inf[B]_\alpha, \tag{9}$$

$$\mu_{Nec}(B, A) < \alpha \text{ iff } \inf[A]_{1-\alpha} < \sup[B]_{1-\alpha}. \tag{10}$$

Corollary 8 will be useful in deriving properties of α-efficient solutions of fuzzy linear problems we deal in the next section.

3. MULTIPLE OBJECTIVE LINEAR PROGRAMMING PROBLEM WITH FUZZY COEFFICIENTS

In this section we introduce a fuzzy multiple objective linear programming problem (FMOLP problem) where coefficients are fuzzy quantities.

Let $\mathcal{K} = \{1, 2, ..., k\}$,$\mathcal{M} = \{1, 2, ..., m\}$, $\mathcal{N} = \{1, 2, ..., n\}$, k, m, n be positive integers. The *multiple objective linear programming problem* (MOLP problem) is a problem

$$\begin{aligned} &\text{maximize} \quad z_q = c_{q1}x_1 + \cdots + c_{qn}x_n \ , q \in \mathcal{K}, \\ &\text{subject to} \\ &\qquad a_{i1}x_1 + \cdots + a_{in}x_n \ \leqslant \ b_i, \, i \in \mathcal{M}, \\ &\qquad\qquad x_j \geqslant 0, \, j \in \mathcal{N}. \end{aligned} \tag{11}$$

In contrast to the classical MOLP problem, here, the coefficients c_{qj}, a_{ij} and b_i will be fuzzy quantities. The fuzzy quantities will be denoted by symbols with the tilde above. Let $\mu_{\tilde{c}_{qj}} : \mathbf{R} \to [0, 1]$, $\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0, 1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0, 1]$, $q \in \mathcal{K}$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, be membership functions of the fuzzy quantities $\tilde{c}_{qj}$, $\tilde{a}_{ij}$ and $\tilde{b}_i$, respectively. Applying the Extension principle we can easily prove the following property.

Proposition 9 Let $\tilde{c}_{qj}$, $\tilde{a}_{ij} \in \mathcal{F}_0(\mathbf{R}), x_j \geqslant 0$, $q \in \mathcal{K}$, $i \in \mathcal{M}$, $j \in \mathcal{N}$. Then the fuzzy sets $\tilde{c}_{q1}x_1\tilde{+}\cdots\tilde{+}\tilde{c}_{qn}x_n$, $\tilde{a}_{i1}x_1\tilde{+}\cdots\tilde{+}\tilde{a}_{in}x_n$ defined by the Extension principle are again fuzzy quantities.

Let $\tilde{P}$ be a fuzzy relation - fuzzy extension of the usual binary relation $\leqslant$ on $\mathbf{R}$.

The *fuzzy multiple objective linear programming problem* (FMOLP problem) associated with a standard MOLP problem (11) is denoted as

$$\begin{aligned} &\text{"maximize"} \ \tilde{z}_q = \ \tilde{c}_{q1}x_1\tilde{+}\cdots\tilde{+}\tilde{c}_{qn}x_n \ , q \in \mathcal{K}, \\ &\text{"subject to"} \\ &\qquad (\tilde{a}_{i1}x_1\tilde{+}\cdots\tilde{+}\tilde{a}_{in}x_n) \ \tilde{P} \ \tilde{b}_i, \, i \in \mathcal{M}, \\ &\qquad\qquad x_j \geqslant 0, \, j \in \mathcal{N}. \end{aligned} \tag{12}$$

In (12) the value $\tilde{a}_{i1}x_1\tilde{+}\cdots\tilde{+}\tilde{a}_{in}x_n \in \mathcal{F}_0(\mathbf{R})$ is "compared" with a fuzzy quantity $\tilde{b}_i \in \mathcal{F}_0(\mathbf{R})$ by some fuzzy relation $\tilde{P}$. The "maximization" of the objective functions denoted by "maximize" $\tilde{z}_q = \tilde{c}_{q1}x_1\tilde{+}\cdots\tilde{+}\tilde{c}_{qn}x_n$ (in quotation marks) will be investigated later on. Now, we shall deal with the constraints of FMOLP problem (12), see also [5; 55; 8].

4. FEASIBLE REGION, β-FEASIBLE SOLUTION

Definition 10 Let $\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0,1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, be membership functions of fuzzy quantities $\tilde{a}_{ij}$ and $\tilde{b}_i$, respectively. Let $\tilde{P}$ be a fuzzy extension of a binary relation $\leqslant$ on $\mathbf{R}$.
A fuzzy set $\tilde{X}$, whose membership function $\mu_{\tilde{X}}$ is defined for all $x \in \mathbf{R}^n$ by

$$\mu_{\tilde{X}}(x) = \begin{cases} \min\{\mu_{\tilde{P}}(\tilde{a}_{11}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{1n}x_n, \tilde{b}_1), \cdots, \mu_{\tilde{P}}(\tilde{a}_{m1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{mn}x_n, \tilde{b}_m)\} \\ \qquad \text{if } x_j \geqslant 0 \text{ for all } j \in \mathcal{N}, \\ 0 \qquad \text{otherwise}, \end{cases} \tag{13}$$

is called the *fuzzy set of feasible region* or shortly *feasible region* of the FMOLP problem (12).
For $\beta \in (0,1]$, a vector $x \in [\tilde{X}]_\beta$ is called the *β-feasible solution* of the FMOLP problem (12).

Notice that the feasible region $\tilde{X}$ of a FMOLP problem is a fuzzy set. On the other hand, β-feasible solution is a vector belonging to the β-cut of the feasible region $\tilde{X}$. It is not difficult to show, that if all coefficients $\tilde{a}_{ij}$ and $\tilde{b}_i$ are crisp fuzzy quantities, i.e. they are isomorfic to the corresponding real numbers, then the fuzzy feasible region is isomorfic to the set of all feasible solutions of the corresponding classical LP problem, see [5], or [6].

Let $\tilde{d}$ be a fuzzy quantity, i.e. $\tilde{d} \in \mathcal{F}_0(\mathbf{R})$, $\beta \in [0,1]$. We shall use the following notation:

$$\begin{aligned} \tilde{d}^{\mathrm{L}}(\beta) &= \inf\left\{t | t \in [\tilde{d}]_\beta\right\} = \inf[\tilde{d}]_\beta, \\ \tilde{d}^{\mathrm{R}}(\beta) &= \sup\left\{t | t \in [\tilde{d}]_\beta\right\} = \sup[\tilde{d}]_\beta. \end{aligned} \tag{14}$$

Proposition 11 Let $\tilde{a}_{ij}$ and $\tilde{b}_i$ be fuzzy quantities and let $x_j \geqslant 0$ for all $i \in \mathcal{M}$, $j \in \mathcal{N}$, let $\beta \in (0,1)$. Moreover, let $\preceq^{Pos}$, $\prec^{Nec}$ be fuzzy extensions of the binary relation $\leqslant$ defined by (4) and (5). Then for $i \in \mathcal{M}$ it holds
(i) $\mu_{\tilde{\preceq}_{Pos}}(\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n, \tilde{b}_i) \geqslant \beta$ if and only if

$$\sum_{j \in \mathcal{N}} \tilde{a}_{ij}^{\mathrm{L}}(\beta) x_j \leqslant \tilde{b}_i^{\mathrm{R}}(\beta), \tag{15}$$

(ii) $\mu^{\prec}_{Nec}(\tilde{a}_{i1}x_1\tilde{+}\cdots\tilde{+}\tilde{a}_{in}x_n,\tilde{b}_i) \geqslant \beta$ if and only if

$$\sum_{j\in\mathcal{N}} \tilde{a}^{\mathrm{R}}_{ij}(1-\beta)x_j \leqslant \tilde{b}^{\mathrm{L}}_i(1-\beta). \tag{16}$$

Proof The proof follows directly from notation 14, Corollary 7, (7), and (8).

Corollary 12 (i) Let $\tilde{P} = \preceq^{Pos}$. A vector $x = (x_1, \cdots, x_n)$ is a β-feasible solution of the FMOLP problem (12) if and only if it is a nonnegative solution of the system of inequalities

$$\sum_{j\in\mathcal{N}} \tilde{a}^{\mathrm{L}}_{ij}(\beta)x_j \leqslant \tilde{b}^{\mathrm{R}}_i(\beta), i \in \mathcal{M}.$$

(ii) Let $\tilde{P} = \prec^{Nec}$. A vector $x = (x_1, \cdots, x_n)$ is a β-feasible solution of the FMOLP problem (12) if and only if it is a nonnegative solution of the system of inequalities

$$\sum_{j\in\mathcal{N}} \tilde{a}^{\mathrm{R}}_{ij}(1-\beta)x_j \leqslant \tilde{b}^{\mathrm{L}}_i(1-\beta), i \in \mathcal{M}.$$

5. MAXIMIZING OBJECTIVE FUNCTIONS

Now, we return to the problem of "maximization" of objective functions $\tilde{z}_q = \tilde{c}_{q1}x_1\tilde{+}\cdots\tilde{+}\tilde{c}_{qn}x_n$ in (12). We look for the "best" fuzzy quantities $\tilde{z}_q$ with respect to the given fuzzy constraints, or, in other words, with respect to the fuzzy set of feasible region of (12).

Definition 13 Let $\tilde{P}$ be a fuzzy relation on $\mathbf{R}$, let $\alpha \in (0,1]$.
Let $\tilde{a}, \tilde{b}$ be fuzzy quantities, we write

$$\tilde{a}\ \tilde{P}_\alpha\ \tilde{b}, \text{ if } \mu_{\tilde{P}}(\tilde{a},\tilde{b}) \geqslant \alpha. \tag{17}$$

and call $\tilde{P}_\alpha$ the *α-relation on $\mathbf{R}$ associated to $\tilde{P}$*. We also write

$$\tilde{a}\ \tilde{P}^*_\alpha\ \tilde{b}, \text{ if } \tilde{a}\tilde{P}_\alpha\tilde{b} \text{ and } \mu_{\tilde{P}}(\tilde{b},\tilde{a}) < \alpha, \tag{18}$$

and call $\tilde{P}^*_\alpha$ the *strict α-relation on $\mathbf{R}$ associated to $\tilde{P}$.*

Notice that $\tilde{P}_\alpha$ and $\tilde{P}^*_\alpha$ are binary relations on the set of fuzzy quantities $\mathcal{F}_0(\mathbf{R})$ being constructed from a fuzzy relation $\tilde{P}$ on the level $\alpha \in (0,1]$. $\tilde{P}^*_\alpha$ is a strict relation to the relation $\tilde{P}_\alpha$.

If $\tilde{a}$ and $\tilde{b}$ are crisp fuzzy numbers corresponding to real numbers a and b, respectively, and $\tilde{P}$ is a fuzzy extension of relation $\leqslant$, then $a\tilde{P}_\alpha b$ if and only if $a \leqslant b$. Then for $\alpha \in (0,1)$, $a\tilde{P}^*_\alpha\, b$ if and only if $a < b$.

The following proposition is a simple consequence of the above results applied to particular fuzzy relations $\tilde{P} = \preceq^{Pos}$ and $\tilde{P} = \prec^{Nec}$, see [5]

Proposition 14 Let $\tilde{a}$ and $\tilde{b}$ be fuzzy quantities, $\alpha \in (0,1]$.
(i) Let $\tilde{P} = \preceq^{Pos}$ be a fuzzy relation on $\mathbf{R}$ defined by (4). Then

$$\begin{array}{lll} \tilde{a}\ \tilde{P}_\alpha\ \tilde{b} & \text{iff} & \tilde{a}^{\mathrm{L}}(\alpha) \leqslant \tilde{b}^{\mathrm{R}}(\alpha), \\ \tilde{a}\ \tilde{P}^*_\alpha\ \tilde{b} & \text{iff} & \tilde{a}^{\mathrm{R}}(\alpha) < \tilde{b}^{\mathrm{L}}(\alpha). \end{array} \tag{19}$$

(ii) Let $\tilde{P} = \prec^{Nec}$ be a fuzzy relation on $\mathbf{R}$ defined by (5). Then

$$\begin{array}{lll} \tilde{a}\ \tilde{P}_\alpha\ \tilde{b} & \text{iff} & \tilde{a}^{\mathrm{R}}(1-\alpha) \leqslant \tilde{b}^{\mathrm{L}}(1-\alpha), \\ \tilde{a}\ \tilde{P}^*_\alpha\ \tilde{b} & \text{iff} & \tilde{a}^{\mathrm{R}}(1-\alpha) \leqslant \tilde{b}^{\mathrm{L}}(1-\alpha) \text{ and} \\ & & \tilde{a}^{\mathrm{L}}(1-\alpha) < \tilde{b}^{\mathrm{R}}(1-\alpha). \end{array} \tag{20}$$

Proof The proof follows from Definition 13, (14), Corollary 7 and Corollary 8 applied to fuzzy relations $\tilde{P} = \preceq^{Pos}$ and $\tilde{P} = \prec^{Nec}$.

An interpretation of the α-relation and strict α-relation on R associated to $\tilde{P}$ when comparing fuzzy quantities $\tilde{a}$ and $\tilde{b}$ is as follows. For a given level of satisfaction $\alpha \in (0,1]$, a fuzzy quantity $\tilde{a}$ "is not better than" fuzzy quantity $\tilde{b}$ with respect to fuzzy relation $\preceq^{Pos}$ if the smallest value of $[\tilde{a}]_\alpha$ is less or equal to the largest value of $[\tilde{b}]_\alpha$. In a sense, it is the *optimistic approach* to the comparison of fuzzy quantities $\tilde{a}$ and $\tilde{b}$ which means that among values of $[\tilde{a}]_\alpha$ and $[\tilde{b}]_\alpha$ there exists a value a of $[\tilde{a}]_\alpha$ and value b of $[\tilde{b}]_\alpha$ such that $a \leqslant b$. Moreover, fuzzy quantity $\tilde{a}$ "is worse than" fuzzy quantity $\tilde{b}$ with respect to fuzzy relation $\preceq^{Pos}$ if the largest value of $[\tilde{a}]_\alpha$ is strictly less than the smallest value of $[\tilde{b}]_\alpha$.

On the other hand, a fuzzy quantity $\tilde{a}$ "is not better than" fuzzy quantity $\tilde{b}$ with respect to fuzzy relation $\prec^{Nec}$ if the largest value of $[\tilde{a}]_{1-\alpha}$ is less or equal to the smallest value of $[\tilde{b}]_{1-\alpha}$. This could be called the *pessimistic approach* to the comparison of fuzzy quantities. The meaning of that is as follows: among all values a of $[\tilde{a}]_{1-\alpha}$ and b of $[\tilde{b}]_{1-\alpha}$ it holds $a \leqslant b$. Moreover, fuzzy quantity $\tilde{a}$ "is worse than" fuzzy quantity $\tilde{b}$ with respect to fuzzy relation $\prec^{Nec}$ if $\tilde{a}$ "is not better than" $\tilde{b}$ and the smallest value of $[\tilde{a}]_{1-\alpha}$ is strictly less than the largest value of $[\tilde{b}]_{1-\alpha}$.

Now, modifying the well known concept of efficient solution in multicriteria optimization we define "maximization" (or "minimization") of the objective functions of FMOLP problem (12). We shall consider a fuzzy relation $\tilde{P}$ on

$\mathbf{R}$ being a fuzzy extension of the usual binary relation $\leqslant$ on $\mathbf{R}$. Here, $\tilde{P}$ is used both for the objective function, and for the constraints. However, we allow for independent, i.e. different satisfaction levels: $\alpha \neq \beta$,where α is considered for the objective functions and β for the constraints, $\alpha, \beta \in (0,1]$. For our convenience we denote the value of the objective functions of (12) alternatively as follows $\tilde{z}_q = \tilde{c}_q^T x = \tilde{c}_{q1}x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_{qn}x_n = \sum_{j \in \mathcal{N}} \tilde{c}_{qj}x_j$.

Definition 15 Let $\tilde{c}_{qj}$, $\tilde{a}_{ij}$ and $\tilde{b}_i$, $q \in \mathcal{K}$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, be fuzzy quantities on $\mathbf{R}$. Let $\tilde{P}$ be a fuzzy relation on $\mathbf{R}$, being also a fuzzy extension of the usual binary relation $\leqslant$ on $\mathbf{R}$, let $\alpha, \beta \in (0,1]$. A β-feasible solution of (12) $x \in [\tilde{X}]_\beta$ is called the (α, β)-*maximal solution of* (12) if there is no $x' \in [\tilde{X}]_\beta$, $x \neq x'$, such that $\tilde{c}_q^T x \tilde{P}_\alpha \tilde{c}_q^T x'$ for all $q \in \mathcal{K}$ and $\tilde{c}_q^T x \tilde{P}_\alpha^* \tilde{c}_q^T x'$ for at least one $q \in \mathcal{K}$. Here, $\tilde{P}_\alpha^*$ is the strict α-relation on $\mathbf{R}$ associated to $\tilde{P}$.

Notice that any (α, β)-maximal solution of the FLP problem is a β-feasible solution of the FMOLP problem with some additional property concerning the values of the objective functions. Clearly, if all coefficients of FMOLP problem (12) are crisp fuzzy quantities, then (α, β)-maximal solution of the FLP problem is isomorphic to the classical Pareto-optimal solution of the corresponding LP problem (11). Comparing to the approach of satisficing solution, see [5; 6], we do not need any exogenously given additional fuzzy goal in order to optimize the objective functions.

In the following lemmas and corollary we show some important properties of (α, β)-maximal solutions of (12) in case of special fuzzy extensions of the binary relation $\leqslant$, particularly $\preceq^{Pos}$, and $\prec^{Nec}$. The corresponding proofs are straightforward.

Lemma 16 Let $\tilde{c}_{qj}$, $q \in \mathcal{K}$, $j \in \mathcal{N}$, be fuzzy quantities on $\mathbf{R}$ and let $\alpha \in (0,1)$. Let $\tilde{P} = \preceq^{Pos}$ be a fuzzy relation on $\mathbf{R}$ defined by (4) and let $\tilde{P}_\alpha^*$ be the strict α-relation on $\mathbf{R}$ associated to $\tilde{P}$. Nonnegative vectors $x = (x_1, \cdots, x_n), x' = (x_1', \cdots, x_n')$ satisfy

$$\tilde{c}_q^T x \tilde{P}_\alpha^* \tilde{c}_q^T x'$$

if and only if

$$\sum_{j \in \mathcal{N}} \tilde{c}_{qj}^{\mathrm{R}}(\alpha) x_j < \sum_{j \in \mathcal{N}} \tilde{c}_{qj}^{\mathrm{L}}(\alpha) x_j'. \tag{21}$$

Corollary 17 If (21) is satisfied then

$$\sum_{j \in \mathcal{N}} \tilde{c}_{qj}^{\mathrm{L}}(\alpha) x_j < \sum_{j \in \mathcal{N}} \tilde{c}_{qj}^{\mathrm{L}}(\alpha) x_j', \tag{22}$$

$$\sum_{j\in\mathcal{N}} \tilde{c}_{qj}^{\mathrm{R}}(\alpha)x_j < \sum_{j\in\mathcal{N}} \tilde{c}_{qj}^{\mathrm{R}}(\alpha)x'_j. \tag{23}$$

Lemma 18 Let $\tilde{c}_{qj}$, $q\in\mathcal{K}$, $j\in\mathcal{N}$, be fuzzy quantities on $\mathbf{R}$ and let $\alpha\in(0,1)$. Let $\tilde{P}=\prec^{Nec}$ be a fuzzy relation on $\mathbf{R}$ defined by (5) and let $\tilde{P}^*_\alpha$ be the strict α-relation on $\mathbf{R}$ associated to $\tilde{P}$. Nonnegative vectors $x=(x_1,\cdots,x_n), x'=(x'_1,\cdots,x'_n)$ satisfy

$$\tilde{c}_q^T x \tilde{P}^*_\alpha \tilde{c}_q^T x'$$

if and only if

$$\sum_{j\in\mathcal{N}} \tilde{c}_{qj}^{\mathrm{R}}(1-\alpha)x_j \leqslant \sum_{j\in\mathcal{N}} \tilde{c}_{qj}^{\mathrm{L}}(1-\alpha)x'_j, \tag{24}$$

$$\sum_{j\in\mathcal{N}} \tilde{c}_{qj}^{\mathrm{L}}(1-\alpha)x_j < \sum_{j\in\mathcal{N}} \tilde{c}_{qj}^{\mathrm{R}}(1-\alpha)x'_j. \tag{25}$$

The following two propositions give some sufficient conditions for $x^*$to be an (α,β)-maximal solution of FMOLP problem (12).

Proposition 19 Let $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be fuzzy quantities for all $q\in\mathcal{K}$, $i\in\mathcal{M}$ and $j\in\mathcal{N}$, let $\alpha,\beta\in(0,1)$.
Let $\tilde{X}$ be a feasible region of FMOLP problem (12) with $\tilde{P}=\preceq^{Pos}$. Let c_{qj} be such that $\tilde{c}_{qj}^{\mathrm{L}}(\alpha)\leqslant c_{qj}\leqslant \tilde{c}_{qj}^{\mathrm{R}}(\alpha)$ for all $q\in\mathcal{K}$, $j\in\mathcal{N}$. If $x^*=(x_1^*,\cdots,x_n^*)$ is a Pareto-optimal solution of the MOLP problem

$$\begin{array}{ll} \text{maximize} & z_q=\sum_{j\in\mathcal{N}} c_{qj}x_j, q\in\mathcal{K}, \\ \text{subject to} & \sum_{j\in\mathcal{N}} \tilde{a}_{ij}^{\mathrm{L}}(\beta)x_j \leqslant \tilde{b}_i^{\mathrm{R}}(\beta), i\in\mathcal{M}, \\ & x_j\geqslant 0, \quad j\in\mathcal{N}, \end{array} \tag{26}$$

then x^* is an (α,β)-maximal solution of FMOLP problem (12).

The next proposition is a modification of Proposition 19 for $\tilde{P}=\prec^{Nec}$.

Proposition 20 Let $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be fuzzy quantities for all $q\in\mathcal{K}$, $i\in\mathcal{M}$ and $j\in\mathcal{N}$, $\alpha,\beta\in(0,1)$. Let $\tilde{X}$ be a feasible region of FMOLP problem (12) with $\tilde{P}=\prec^{Nec}$. Let c_{qj} be such that $\tilde{c}_{qj}^{\mathrm{L}}(\alpha)\leqslant c_{qj}\leqslant \tilde{c}_{qj}^{\mathrm{R}}(\alpha)$ for all $q\in\mathcal{K}$, $j\in\mathcal{N}$. If $x^*=(x_1^*,\cdots,x_n^*)$ is a Pareto-optimal solution of the MOLP problem

$$\begin{array}{ll} \text{maximize} & z=\sum_{j\in\mathcal{N}} c_{qj}x_j, q\in\mathcal{K}, \\ \text{subject to} & \sum_{j\in\mathcal{N}} \tilde{a}_{ij}^{\mathrm{R}}(\beta)x_j \leqslant \tilde{b}_i^{\mathrm{L}}(\beta), i\in\mathcal{M}, \\ & x_j\geqslant 0, \quad j\in\mathcal{N}, \end{array} \tag{27}$$

then x^* is an $(1-\alpha,1-\beta)$-maximal solution of FMOLP problem (12).

6. DUAL PROBLEM

In this section we shall investigate the well known concept of duality in LP for FMOLP problems based on possibility and necessity fuzzy relations $\preceq^{Pos}$, and $\prec^{Nec}$. Similar approach for single objective FLP problems can be found in [8], some results for different concept of "optimal" solution of FLP based on satisficing solutions, can be found in [5] and [4]. Here, we derive some innovation of weak and strong duality theorems which extend the known results for LP problems.

Consider the following FMOLP problem

$$
\begin{array}{lll}
 & \text{"maximize"} & \tilde{z}_q = \tilde{c}_{q1}x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_{qn}x_n, q \in \mathcal{K}, \\
\text{(P)} & \text{"subject to"} & (\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n)\ \tilde{P}\ \tilde{b}_i, i \in \mathcal{M}, \\
 & & x_j \geqslant 0, \quad j \in \mathcal{N},
\end{array} \tag{28}
$$

where $\tilde{c}_{qj}$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ are fuzzy quantities with membership functions $\mu_{\tilde{c}_{jq}} : \mathbf{R} \to [0,1]$, $\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0,1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$, $q \in \mathcal{K}, i \in \mathcal{M}$, $j \in \mathcal{N}$.

FMOLP problem (28) will be called the *primal FMOLP problem (P)*. The feasible region of (P) is introduced by Definition 10 and (α, β)-maximal solution is defined by Definition 15.

The *dual FMOLP problem (D)* can be formulated as follows

$$
\begin{array}{lll}
 & \text{"minimize"} & \tilde{w} = \tilde{b}_1 y_1 \tilde{+} \cdots \tilde{+} \tilde{b}_m y_m \\
\text{(D)} & \text{"subject to"} & \tilde{c}_{qj}\ \tilde{Q}(\tilde{a}_{1j}y_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{mj}y_m), q \in \mathcal{K}, j \in \mathcal{N}, \\
 & & y_i \geqslant 0, \quad i \in \mathcal{M}.
\end{array} \tag{29}
$$

Here, either $\tilde{P} = \preceq^{Pos}$, $\tilde{Q} = \prec^{Nec}$, or $\tilde{P} = \prec^{Nec}$, $\tilde{Q} = \preceq^{Pos}$. In problem (P), "maximization" is considered with respect to fuzzy relation $\tilde{P}$, in problem (D), "minimization" is considered with respect to fuzzy relation $\tilde{Q}$. Notice that the dual problem (D) is a single criterion FLP problem. The pair of FMOLP problems (P) and (D), i.e. (28) and (29), is called the *primal - dual pair of FMOLP problems*. Now, we define a concept of feasible region of (D), that is, a modification of Definition 10.

Definition 21 Let $\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0,1]$ and $\mu_{\tilde{c}_{qj}} : \mathbf{R} \to [0,1]$, $q \in \mathcal{K}, i \in \mathcal{M}$, $j \in \mathcal{N}$, be membership functions of fuzzy quantities $\tilde{a}_{ij}$ and $\tilde{c}_{qj}$, respectively. Let $\tilde{P}$ be a fuzzy extension of a binary relation P on $\mathbf{R}$.

A fuzzy set $\tilde{Y}$, whose membership function $\mu_{\tilde{Y}}$ is defined for all $y \in \mathbf{R}^m$ by

$$\mu_{\tilde{Y}}(y) = \begin{cases} \min\{\mu_{\tilde{P}}(\tilde{c}_{11}, \tilde{a}_{11}y_1\tilde{+}\cdots\tilde{+}\tilde{a}_{m1}y_m), \cdots, \mu_{\tilde{P}}(\tilde{c}_{kn}, \tilde{a}_{1n}y_1\tilde{+}\cdots\tilde{+}\tilde{a}_{mn}y_m)\} \\ \qquad \text{if } y_i \geqslant 0 \text{ for all } i \in \mathcal{M}, \\ 0 \qquad \text{otherwise,} \end{cases} \tag{30}$$

is called a *fuzzy set of feasible region* or shortly *feasible region* of dual FMOLP problem (29).
For $\beta \in (0, 1]$, a vector $y \in [\tilde{Y}]_\beta$ is called the *β-feasible solution* of dual FMOLP problem (29).

Now, we define an "optimal solution" of the dual FMOLP problem (D).

Definition 22 Let $\tilde{c}_{qj}$, $\tilde{a}_{ij}$ and $\tilde{b}_i$, $q \in \mathcal{K}, i \in \mathcal{M}$, $j \in \mathcal{N}$, be fuzzy quantities on $\mathbf{R}$. Let $\tilde{Q}$ be a fuzzy relation on $\mathbf{R}$ - fuzzy extension of the usual binary relation $\leqslant$ on $\mathbf{R}$, and let $\alpha, \beta \in (0, 1]$. A β-feasible solution of (29) $y \in [\tilde{Y}]_\beta$ is called the *(α, β)-minimal solution of* (29) if there is no $y' \in [\tilde{Y}]_\beta$, $y' \neq y$, such that $\tilde{b}^T y' \ \tilde{Q}^*_\alpha \ \tilde{b}^T y$. Here, $\tilde{Q}^*_\alpha$ is the strict α-relation on $\mathbf{R}$ associated to $\tilde{Q}$.

Let P be the usual binary relation $\leqslant$ on $\mathbf{R}$. Now, we shall investigate FMOLP problems (28) and (29) with pairs of dual fuzzy relations in the constraints, particularly, either $\tilde{P} = \preceq^{Pos}$, $\tilde{Q} = \prec^{Nec}$ or $\tilde{P} = \prec^{Nec}$, $\tilde{Q} = \preceq^{Pos}$, see (4), (5). The values of objective functions $\tilde{z}_q$ and $\tilde{w}$ are "maximized " and "minimized", with respect to fuzzy relation $\tilde{P}$ and $\tilde{Q}$, respectively.

The feasible region of the primal FMOLP problem (P) is denoted by $\tilde{X}$, the feasible region of the dual FMOLP problem (D) by $\tilde{Y}$. Clearly, $\tilde{X}$ is a fuzzy subset of $\mathbf{R}^n$, $\tilde{Y}$ is a fuzzy subset of $\mathbf{R}^m$.

The following proposition is a useful modification of Proposition 11.

Proposition 23 Let $\tilde{c}_{qj}$ and $\tilde{a}_{ij}$ be fuzzy quantities and let $y_i \geqslant 0$ for all $q \in \mathcal{K}, i \in \mathcal{M}$, $j \in \mathcal{N}$, $\alpha \in (0, 1)$. Let $\preceq^{Pos}$ and $\prec^{Nec}$ be fuzzy extensions of the binary relation $\leqslant$ defined by (4), (5). Then for $q \in \mathcal{K}, j \in \mathcal{N}$, it holds

(i) $\mu_{\preceq^{Pos}}(\tilde{c}_{qj}, \tilde{a}_{1j}y_1\tilde{+}\cdots\tilde{+}\tilde{a}_{mj}y_m) \geqslant \beta$ iff

$$\sum_{i \in \mathcal{M}} \tilde{a}^{\mathrm{R}}_{ij}(\beta) y_i \geqslant \tilde{c}^{\mathrm{L}}_{qj}(\beta), \tag{31}$$

(ii) $\mu_{\prec^{Nec}}(\tilde{c}_{qj}, \tilde{a}_{1j}y_1\tilde{+}\cdots\tilde{+}\tilde{a}_{mj}y_m) \geqslant \beta$ iff

$$\sum_{i \in \mathcal{M}} \tilde{a}^{\mathrm{L}}_{ij}(1-\beta) y_i \geqslant \tilde{c}^{\mathrm{R}}_{qj}(1-\beta). \tag{32}$$

Corollary 24 (i) Let $\tilde{P} = \preceq^{Pos}$. A vector $y = (y_1, \cdots, y_m)$ is an β-feasible solution of the FLP problem (29) if and only if it is a nonnegative solution of the system of inequalities

$$\sum_{i \in \mathcal{M}} \tilde{a}_{ij}^{\mathrm{R}}(\beta) y_i \geqslant \tilde{c}_{qj}^{\mathrm{L}}(\beta), q \in \mathcal{K}, j \in \mathcal{N}.$$

(ii) Let $\tilde{P} = \prec^{Nec}$. A vector $y = (y_1, \cdots, y_m)$ is an α-feasible solution of the FLP problem (29) if and only if it is a nonnegative solution of the system of inequalities

$$\sum_{i \in \mathcal{M}} \tilde{a}_{ij}^{\mathrm{L}}(1-\beta) y_i \geqslant \tilde{c}_{qj}^{\mathrm{R}}(1-\beta), j \in \mathcal{N}.$$

Lemma 25 Let $\tilde{b}_i$, $i \in \mathcal{M}$, be fuzzy quantities on $\mathbf{R}$. Let $\tilde{P} = \preceq^{Pos}$ be fuzzy relations on $\mathbf{R}$ defined by (4) and $\alpha \in (0,1)$. Then $y = (y_1, \cdots, y_m), y' = (y'_1, \cdots, y'_m)$ are nonnegative vectors such that $\tilde{b}^T y' \; \tilde{P}^*_\alpha \; \tilde{b}^T y$, where $\tilde{P}^*_\alpha$ is the strict α-relation on $\mathbf{R}$ associated to $\tilde{P}$, if and only if

$$\sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{R}}(\alpha) y'_i < \sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{L}}(\alpha) y_i. \tag{33}$$

Corollary 26 If (**??**) is satisfied then

$$\sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{L}}(\alpha) y'_i < \sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{L}}(\alpha) y_i, \tag{34}$$

$$\sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{R}}(\alpha) y'_i < \sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{R}}(\alpha) y_i. \tag{35}$$

Lemma 27 Let $\tilde{b}_i$, $i \in \mathcal{M}$, be fuzzy quantities on $\mathbf{R}$. Let $\tilde{P} = \prec^{Nec}$ be a fuzzy relation on $\mathbf{R}$ defined by (4) and $\alpha \in (0,1)$. The vectors $y = (y_1, \cdots, y_m), y' = (y'_1, \cdots, y'_m)$ are nonnegative with $\tilde{b}^T y' \; \tilde{P}^*_\alpha \; \tilde{b}^T y$, where $\tilde{P}^*_\alpha$ is the strict α-relation on $\mathbf{R}$ associated to $\tilde{P}$, if and only if

$$\sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{R}}(1-\alpha) y'_i \leqslant \sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{L}}(1-\alpha) y_i, \tag{36}$$

$$\sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{L}}(1-\alpha) y'_i < \sum_{i \in \mathcal{M}} \tilde{b}_i^{\mathrm{R}}(1-\alpha) y_i. \tag{37}$$

The following propositions give sufficient conditions for y^* to be an (α, β)-minimal solution of FMOLP problem (D).

Proposition 28 Let $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be fuzzy quantities for all $q \in \mathcal{K}, i \in \mathcal{M}$ and $j \in \mathcal{N}$, $\alpha, \beta \in (0,1)$. Let $\tilde{Y}$ be a feasible region of FMOLP problem (29) with $\tilde{P} = \preceq^{Pos}$. Let b_i be such that $\tilde{b}_i^{\mathrm{L}}(\alpha) \leqslant b_i \leqslant \tilde{b}_i^{\mathrm{R}}(\alpha)$ for all $i \in \mathcal{M}$. If $y^* = (y_1^*, \cdots, y_m^*)$ is an optimal solution of the LP problem

$$\begin{array}{ll} \text{minimize} & w = \sum_{i\in\mathcal{M}} b_i y_i \\ \text{subject to} & \sum_{i\in\mathcal{M}} \tilde{a}_{ij}^{\mathrm{R}}(\beta) y_i \geqslant \tilde{c}_{qj}^{\mathrm{L}}(\beta), q \in \mathcal{K}, j \in \mathcal{N}, \\ & y_i \geqslant 0, \quad i \in \mathcal{M}, \end{array} \tag{38}$$

then y^* is a (α, β)-minimal solution of FMOLP problem (D).

The following proposition is a simple and useful modification of Proposition 28.

Proposition 29 Let $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be fuzzy quantities for all $q \in \mathcal{K}, i \in \mathcal{M}$ and $j \in \mathcal{N}$, $\alpha, \beta \in (0,1)$. Let $\tilde{X}$ be a feasible region of FMOLP problem (P) with $\tilde{P} = \prec^{Nec}$. Let b_i be such that $\tilde{b}_i^{\mathrm{L}}(\alpha) \leqslant b_i \leqslant \tilde{b}_i^{\mathrm{R}}(\alpha)$ for all $i \in \mathcal{M}$. If $y^* = (y_1^*, \cdots, y_m^*)$ is an optimal solution of the LP problem

$$\begin{array}{ll} \text{minimize} & w = \sum_{i\in\mathcal{M}} b_i y_i \\ \text{subject to} & \sum_{i\in\mathcal{M}} \tilde{a}_{ij}^{\mathrm{L}}(\beta) y_i \geqslant \tilde{c}_{qj}^{\mathrm{R}}(\beta), q \in \mathcal{K}, j \in \mathcal{N}, \\ & y_i \geqslant 0, \quad i \in \mathcal{M}, \end{array} \tag{39}$$

then y^* is an $(1-\alpha, 1-\beta)$-minimal solution of FMOLP problem (D).

7. WEAK AND STRONG DUALITY THEOREMS

Now, we focus our attention to duality theory for FMOLP problems (see also [7; 12; 8]). In the following duality theorems we present always two versions: (i) for fuzzy relation $\preceq^{Pos}$ in problem (P) and (ii) for fuzzy relation $\prec^{Nec}$in problem (P). In order to prove duality results we assume that the level of satisfaction α of the objective function is equal to the level of satisfaction β of the constraints. Otherwise, the duality theorems in our formulation do not hold.

Moreover, we assume that each objective function is associated with a weight $w_q > 0, q \in \mathcal{K}$, such that $\sum_{q\in\mathcal{K}} w_q = 1$, where w_q may be interpreted as a relative importance of the q-th objective function.

Theorem 30 *First Weak Duality Theorem.* Let $\tilde{c}_{qj}$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be fuzzy quantities, $q \in \mathcal{K}, i \in \mathcal{M}$ and $j \in \mathcal{N}$, $\alpha \in (0, 1)$.
(i) Let $\tilde{X}$ be a feasible region of FMOLP problem (28) with $\tilde{P} = \preceq^{Pos}$, $\tilde{Y}$ be a feasible region of FMOLP problem (29) with $\tilde{Q} = \prec^{Nec}$.
If a vector $x = (x_1, \ldots, x_n) \geqslant 0$ belongs to $[\tilde{X}]_\alpha$ and $y = (y_1, \ldots, y_m) \geqslant 0$ belongs to $[\tilde{Y}]_{1-\alpha}$, then

$$\sum_{j \in \mathcal{N}} \tilde{c}^{\mathrm{R}}_{qj}(\alpha) x_j \leqslant \sum_{i \in \mathcal{M}} \tilde{b}^{\mathrm{R}}_i(\alpha) y_i. \tag{40}$$

(ii) Let $\tilde{X}$ be a feasible region of FMOLP problem (28) with $\tilde{P} = \prec^{Nec}$, $\tilde{Y}$ be a feasible region of FMOLP problem (29) with $\tilde{Q} = \preceq^{Pos}$.
If a vector $x = (x_1, \ldots, x_n) \geqslant 0$ belongs to $[\tilde{X}]_{1-\alpha}$ and $y = (y_1, \ldots, y_m) \geqslant 0$ belongs to $[\tilde{Y}]_\alpha$, then

$$\sum_{j \in \mathcal{N}} \tilde{c}^{\mathrm{L}}_{qj}(\alpha) x_j \leqslant \sum_{i \in \mathcal{M}} \tilde{b}^{\mathrm{L}}_i(\alpha) y_i. \tag{41}$$

Proof (i) Let $q \in \mathcal{K}, x \in [\tilde{X}]_\alpha$ and $y \in [\tilde{Y}]_{1-\alpha}$, $x_j \geqslant 0$, $y_i \geqslant 0$ for all $i \in \mathcal{M}$, $j \in \mathcal{N}$. Then by Proposition 23 (ii), multiplying both sides by nonnegative x_j and summing up for $j \in \mathcal{N}$ we obtain

$$\sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{M}} \tilde{a}^{\mathrm{L}}_{ij}(\alpha) y_i x_j \geqslant \sum_{j \in \mathcal{N}} \tilde{c}^{\mathrm{R}}_{qj}(\alpha) x_j. \tag{42}$$

In a similar way, by Proposition 11 (i) we obtain

$$\sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{M}} \tilde{a}^{\mathrm{L}}_{ij}(\alpha) x_j y_i \leqslant \sum_{i \in \mathcal{M}} \tilde{b}^{\mathrm{R}}_i(\alpha) y_i. \tag{43}$$

Combining inequalities (42) and (43), we obtain

$$\sum_{j \in \mathcal{N}} \tilde{c}^{\mathrm{R}}_{qj}(\alpha) x_j \leqslant \sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{M}} \tilde{a}^{\mathrm{L}}_{ij}(\alpha) x_j y_i \leqslant \sum_{i \in \mathcal{M}} \tilde{b}^{\mathrm{R}}_i(\alpha) y_i,$$

which is the desired result.

(ii) Let $q \in \mathcal{K}, x \in [\tilde{X}]_{1-\alpha}$ and $y \in [\tilde{Y}]_\alpha$, $x_j \geqslant 0, y_i \geqslant 0$ for all $i \in \mathcal{M}$, $j \in \mathcal{N}$. Then by Proposition 23 (i), multiplying both sides by x_j and summing up we obtain

$$\sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{M}} \tilde{a}^{\mathrm{R}}_{ij}(\alpha) y_i x_j \geqslant \sum_{j \in \mathcal{N}} \tilde{c}^{\mathrm{L}}_{qj}(\alpha) x_j.$$

In a similar way, by Proposition 11 (ii) with α instead of $1-\alpha$ we obtain

$$\sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{M}}\tilde{a}^{\mathrm{R}}_{ij}(\alpha)x_jy_i \leqslant \sum_{i\in\mathcal{M}}\tilde{b}^{\mathrm{L}}_i(\alpha)y_i.$$

Combining the last two inequalities, we obtain

$$\sum_{j\in\mathcal{N}}\tilde{c}^{\mathrm{L}}_{qj}(\alpha)x_j \leqslant \sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{M}}\tilde{a}^{\mathrm{R}}_{ij}(\alpha)x_jy_i \leqslant \sum_{i\in\mathcal{M}}\tilde{b}^{\mathrm{L}}_i(\alpha)y_i.$$

Corollary 31 Let $w_q>0$, for all $q\in\mathcal{K}$, such that $\sum_{q\in\mathcal{K}}w_q=1$.

(i) If a vector $x=(x_1,\ldots,x_n)\geqslant 0$ belongs to $[\tilde{X}]_\alpha$ and $y=(y_1,\ldots,y_m)\geqslant 0$ belongs to $[\tilde{Y}]_{1-\alpha}$, then

$$\sum_{q\in\mathcal{K}}\sum_{j\in\mathcal{N}}w_q\tilde{c}^{\mathrm{R}}_{qj}(\alpha)x_j \leqslant \sum_{i\in\mathcal{M}}\tilde{b}^{\mathrm{R}}_i(\alpha)y_i. \tag{44}$$

(ii) If a vector $x=(x_1,\ldots,x_n)\geqslant 0$ belongs to $[\tilde{X}]_{!-\alpha}$ and $y=(y_1,\ldots,y_m)\geqslant 0$ belongs to $[\tilde{Y}]_\alpha$, then

$$\sum_{q\in\mathcal{K}}\sum_{j\in\mathcal{N}}w_q\tilde{c}^{\mathrm{L}}_{qj}(\alpha)x_j \leqslant \sum_{i\in\mathcal{M}}\tilde{b}^{\mathrm{L}}_i(\alpha)y_i. \tag{45}$$

Theorem 32 *Second Weak Duality Theorem.* Let $\tilde{c}_{qj}$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be fuzzy quantities for all $q\in\mathcal{K}, i\in\mathcal{M}$ and $j\in\mathcal{N}$, $\alpha\in(0,1)$, moreover, $w_q>0, q\in\mathcal{K}$, such that $\sum_{q\in\mathcal{K}}w_q=1$.

(i) Let $\tilde{X}$ be a feasible region of FMOLP problem (28) with $\tilde{P}=\preceq^{Pos}$, $\tilde{Y}$ be a feasible region of FMOLP problem (29) with $\tilde{Q}=\prec^{Nec}$.
If for some $x=(x_1,...,x_n)\geqslant 0$ belonging to $[\tilde{X}]_\alpha$ and $y=(y_1,...,y_m)\geqslant 0$ belonging to $[\tilde{Y}]_{1-\alpha}$ it holds

$$\sum_{q\in\mathcal{K}}\sum_{j\in\mathcal{N}}w_q\tilde{c}^{\mathrm{R}}_{qj}(\alpha)x_j = \sum_{i\in\mathcal{M}}\tilde{b}^{\mathrm{R}}_i(\alpha)y_i, \tag{46}$$

for some $q\in\mathcal{K}$, then x is an (α,α)-maximal solution of FMOLP problem (P), (28) and y is an $(1-\alpha,1-\alpha)$-minimal solution of FMOLP problem (D), (29).

(ii) Let $\tilde{X}$ be a feasible region of FMOLP problem (28) with $\tilde{P}=\prec^{Nec}$, $\tilde{Y}$ be a feasible region of FMOLP problem (29) with $\tilde{Q}=\preceq^{Pos}$.
If for some $x=(x_1,...,x_n)\geqslant 0$ belonging to $[\tilde{X}]_{1-\alpha}$ and $y=(y_1,...,y_m)\geqslant 0$ belonging to $[\tilde{Y}]_\alpha$ it holds

$$\sum_{q\in\mathcal{K}}\sum_{j\in\mathcal{N}}w_q\tilde{c}^{\mathrm{L}}_{qj}(\alpha)x_j = \sum_{i\in\mathcal{M}}\tilde{b}^{\mathrm{L}}_i(\alpha)y_i, \tag{47}$$

for some $q \in \mathcal{K}$, then x is an $(1-\alpha, 1-\alpha)$-maximal solution of FMOLP problem (P), (28) and y is an (α, α)-minimal solution of FMOLP problem (D), (29).

Proof (i) Let $x \in [\tilde{X}]_\alpha$ and $y \in [\tilde{Y}]_{1-\alpha}$, $x_j \geqslant 0$, $y_i \geqslant 0$ for all $i \in \mathcal{M}$, $j \in \mathcal{N}$. Then by Theorem 30 (i), inequality (40) is satisfied.

By equality (46), x is a Pareto-optimal solution of LP problem (26) with $\beta = \alpha$, $z_q = \sum\limits_{j\in\mathcal{N}} c_{qj}x_j = \sum\limits_{j\in\mathcal{N}} \tilde{c}^{\mathrm{R}}_{qj}(\alpha)x_j$ and y is an optimal solution of MOLP problem (39) with $\beta = \alpha$, $w = \sum\limits_{i\in\mathcal{M}} b_iy_i = \sum\limits_{i\in\mathcal{M}} \tilde{b}^{\mathrm{R}}_i(\alpha)y_i$. By Proposition 19, x is an (α, α)-maximal solution of FMOLP problem (P) and by Proposition 29, y is an $(1-\alpha, 1-\alpha)$-minimal solution of FMOLP problem (D).

(ii) Let $x \in [\tilde{X}]_{1-\alpha}$ and $y \in [\tilde{Y}]_\alpha$, $x_j \geqslant 0, y_i \geqslant 0$ for all $i \in \mathcal{M}$, $j \in \mathcal{N}$. Then by Theorem 30 (ii), inequality (41) is satisfied.

By equality (47), x is an optimal solution of MOLP problem (27) with $\beta = \alpha$, $z_q = \sum\limits_{j\in\mathcal{N}} c_{qj}x_j = \sum\limits_{j\in\mathcal{N}} \tilde{c}^{\mathrm{L}}_{qj}(\alpha)x_j$ and y is an optimal solution of MOLP problem (39) with $\beta = \alpha$, $w = \sum\limits_{i\in\mathcal{M}} b_iy_i = \sum\limits_{i\in\mathcal{M}} \tilde{b}^{\mathrm{L}}_i(\alpha)y_i$. By Proposition 20, x is an $(1-\alpha, 1-\alpha)$-maximal solution of FLP problem (P) and by Proposition 28, y is an (α, α)-minimal solution of FLP problem (D).

Remarks.

1. In the crisp and single-objective case, Theorems 30 and 32 are the standard LP Weak Duality Theorems.

2. Let $\alpha \geqslant 0,5$. Then $[\tilde{X}]_\alpha \subset [\tilde{X}]_{1-\alpha}$, $[\tilde{Y}]_\alpha \subset [\tilde{Y}]_{1-\alpha}$, hence in the First Weak Duality Theorem we can change the assumptions as follows: $x \in [\tilde{X}]_\alpha$ and $y \in [\tilde{Y}]_\alpha$. However, the statements of the theorem remain unchanged. The same holds for the Second Weak Duality Theorem.

Finally, let us turn to the *strong duality*. Motivated by the pairs of Propositions 19, 29 and Propositions 20, 28 in Theorem 32, we consider a pair of dual LP problems corresponding to FLP problems (28) and (29) with fuzzy relations $\tilde{P} = \preceq^{Pos}$, $\tilde{Q} = \prec^{Nec}$, assuming $\alpha = \beta$, particularly

$$\text{(P1)} \quad \begin{array}{ll} \text{maximize} & z_q = \sum_{j\in\mathcal{N}} \tilde{c}^{\mathrm{R}}_{qj}(\alpha)x_j, q \in \mathcal{K}, \\ \text{subject to} & \sum_{j\in\mathcal{N}} \tilde{a}^{\mathrm{L}}_{ij}(\alpha)x_j \leqslant \tilde{b}^{\mathrm{R}}_i(\alpha), i \in \mathcal{M}, \\ & x_j \geqslant 0, \quad j \in \mathcal{N}, \end{array} \tag{48}$$

$$\text{(D1)} \quad \begin{array}{ll} \text{minimize} & w = \sum_{i\in\mathcal{M}} \tilde{b}^{\mathrm{R}}_i(\alpha)y_i \\ \text{subject to} & \sum_{i\in\mathcal{M}} \tilde{a}^{\mathrm{L}}_{ij}(\alpha)y_i \geqslant \tilde{c}^{\mathrm{R}}_{qj}(\alpha), q \in \mathcal{K}, j \in \mathcal{N}, \\ & y_i \geqslant 0, \quad i \in \mathcal{M}. \end{array} \tag{49}$$

Moreover, we consider a pair of dual LP problems with fuzzy relations $\tilde{P} =\prec^{Nec}$, $\tilde{P}^D =\preceq^{Pos}$:

$$\text{(P2)}\quad \begin{array}{ll} \text{maximize} & z_q = \sum_{j\in\mathcal{N}} \tilde{c}^{\mathrm{L}}_{qj}(\alpha)x_j, q\in\mathcal{K}, \\ \text{subject to} & \sum_{j\in\mathcal{N}} \tilde{a}^{\mathrm{R}}_{ij}(\alpha)x_j \leqslant \tilde{b}^{\mathrm{L}}_i(\alpha), i\in\mathcal{M}, \\ & x_j \geqslant 0, \quad j\in\mathcal{N}, \end{array} \tag{50}$$

$$\text{(D2)}\quad \begin{array}{ll} \text{minimize} & w = \sum_{i\in\mathcal{M}} \tilde{b}^{\mathrm{L}}_i(\alpha)y_i \\ \text{subject to} & \sum_{i\in\mathcal{M}} \tilde{a}^{\mathrm{R}}_{ij}(\alpha)y_i \geqslant \tilde{c}^{\mathrm{L}}_{qj}(\alpha), q\in\mathcal{K}, j\in\mathcal{N}, \\ & y_i \geqslant 0, \quad i\in\mathcal{M}. \end{array} \tag{51}$$

Notice that in case of single objective problem, (P1) and (D1) are classical dual LP problems and the same holds for (P2) and (D2).

Theorem 33 *Strong Duality Theorem.* Let $\tilde{c}_{qj}$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be fuzzy quantities for all $q\in\mathcal{K}, i\in\mathcal{M}$ and $j\in\mathcal{N}$, let $w_q > 0$, for all $q\in\mathcal{K}$, such that $\sum_{q\in\mathcal{K}} w_q = 1$.
(i) Let $\tilde{X}$ be a feasible region of FMOLP problem (28) with $\tilde{P} =\preceq^{Pos}$, $\tilde{Y}$ be a feasible region of FMOLP problem (29) with $\tilde{Q} =\prec^{Nec}$. If for some $\alpha\in(0,1)$, $[\tilde{X}]_\alpha$ and $[\tilde{Y}]_{1-\alpha}$ are nonempty, then there exists x^*- an (α,α) -maximal solution of FMOLP problem (P), and there exists y^* - an $(1-\alpha,1-\alpha)$-minimal solution of FMOLP problem (D) such that

$$\sum_{q\in\mathcal{K}}\sum_{j\in\mathcal{N}} w_q\tilde{c}^{\mathrm{R}}_{qj}(\alpha)x^*_j = \sum_{i\in\mathcal{M}} \tilde{b}^{\mathrm{R}}_i(\alpha)y^*_i. \tag{52}$$

(ii) Let $\tilde{X}$ be a feasible region of FMOLP problem (28) with $\tilde{P} =\prec^{Nec}$, $\tilde{Y}$ be a feasible region of FMOLP problem (29) with $\tilde{Q} =\preceq^{Pos}$. If for some $\alpha\in(0,1)$, $[\tilde{X}]_{1-\alpha}$ and $[\tilde{Y}]_\alpha$ are nonempty, then there exists x^*- an $(1-\alpha,1-\alpha)$-maximal solution of FLP problem (P), and y^* - an $(\alpha\,,\alpha)$-minimal solution of FLP problem (D) such that

$$\sum_{q\in\mathcal{K}}\sum_{j\in\mathcal{N}} w_q\tilde{c}^{\mathrm{L}}_{qj}(\alpha)x^*_j = \sum_{i\in\mathcal{M}} \tilde{b}^{\mathrm{L}}_i(\alpha)y^*_i. \tag{53}$$

Proof (i) Clearly, $[\tilde{X}]_\alpha$ is the set of all $\alpha-$feasible solutions of MOLP problem (P1) and $[\tilde{Y}]_{1-\alpha}$ is the set of all $(1-\alpha)-$feasible solutions of MOLP problem (D1), we assume that they are both nonempty. As (P1) and (D1) are dual MOLP problems in the usual sense, there exists $x^*\in[\tilde{X}]_\alpha$- a Pareto-optimal solution of (P1), and $y^*\in[\tilde{Y}]_{1-\alpha}$- an optimal solution of (D1), such that (52) holds.

It remains to prove that x^* is an (α,α)-maximal solution of FMOLP problem (28), and $y^*$is an $(1-\alpha,1-\alpha)$-minimal solution of FMOLP problem (29).

By Proposition 19, x^* is an (α, α)-maximal solution of FLP problem (P1) and by Proposition 29, y^* is an $(1-\alpha, 1-\alpha)$-minimal solution of FLP problem (D1).

Part (ii) can be proven analogically using Propositions 20 and 28.

Remarks.

1. In the crisp and single-objective case, Theorem 33 is a standard LP (Strong) Duality Theorem.

2. Let $\alpha \geqslant 0,5$. Then $[\tilde{X}]_\alpha \subset [\tilde{X}]_{1-\alpha}$, $[\tilde{Y}]_\alpha \subset [\tilde{Y}]_{1-\alpha}$, hence in the Strong Duality Theorem we can assume $x \in [\tilde{X}]_\alpha$ and $y \in [\tilde{Y}]_\alpha$. Evidently, the statement of the theorem remains unchanged.

3. Theorem 33 provides only the existence of the (α, α)-maximal solution (or $(1-\alpha, 1-\alpha)$-maximal solution) of FMOLP problem (P), and $(1-\alpha, 1-\alpha)$-minimal solution ((α, α)-minimal solution) of FMOLP problem (D) such that (52) or (53) holds. However, the proof of the theorem gives also the method for finding the solutions by solving (MO)LP problems (P1) and (D1).

4. The following questions remain open and can be investigated in the future:

(1) How the theorems could be modified for more general fuzzy extensions of $\leqslant$.

(2) Duality theorems allowing for different satisfaction levels α and β would be interesting.

8. ILLUSTRATIVE EXAMPLE

In this section we discuss a simple illustrative example to clarify the introduced concepts and results, to provide some interpretation and features of possible applications. Last but not least, to solve the multi-objective FLP problem (P) by the single-objective FLP problem (D).

Let two new products A and B be manufactured. The manufacturing process is composed of two sub-processes, Processes 1 and 2. The estimated processing resources (e.g. processing time, materials) for manufacturing a batch of Product A for each process are the following: $\tilde{a}_{11}$ units for Process 1 and $\tilde{a}_{21}$ units for Process 2. On the other hand, the processing resources for manufacturing a batch of Product B for each process are as follows: $\tilde{a}_{12}$ units for Process 1, $\tilde{a}_{22}$ units at Process 2. The working resource for Process 1 is restricted by $\tilde{b}_1$ units, for Process 2 by $\tilde{b}_2$ units. The "profit" rates (1000 CZK/batch) of Products A and B are estimated as $\tilde{c}_{11}$ and $\tilde{c}_{12}$, respectively. The "utility" rates (1000 CZK/batch) of Products A and B are estimated as $\tilde{c}_{21}$ and $\tilde{c}_{22}$, respectively. The weights of the criteria are $w_1 = 0,6$ and $w_2 = 0,4$. All mentioned parameters $\tilde{a}_{ij}$, $\tilde{b}_i$ and

$\tilde{c}_{qj}$ are subjected to uncertainty and they are expressed by fuzzy quantities. We shall investigate what quantity of Products A and B should be manufactured in order to "maximize" the total "profit" and total "utility". For this purpose we formulate the following FMOLP problem (primal problem)

$$
\text{(PE)} \quad
\begin{array}{ll}
\text{"maximize"} & \tilde{z}_1 = \tilde{c}_{11}x_1 \tilde{+} \tilde{c}_{12}x_2, \\
& \tilde{z}_2 = \tilde{c}_{21}x_1 \tilde{+} \tilde{c}_{22}x_2 \\
\text{"subject to"} & (\tilde{a}_{11}x_1 \tilde{+} \tilde{a}_{12}x_2)\ \tilde{P}\ \tilde{b}_1, \\
& (\tilde{a}_{21}x_1 \tilde{+} \tilde{a}_{22}x_2)\ \tilde{P}\ \tilde{b}_2, \\
& x_1, x_2 \geqslant 0,
\end{array}
\tag{54}
$$

where $\tilde{c}_{qj} = (c_{qj}^L, c_{qj}, c_{qj}^R)$, $\tilde{a}_{ij} = (a_{ij}^L, a_{ij}, a_{ij}^R)$ and $\tilde{b}_i = (b_i^L, b_i, b_i^R)$ are triangular fuzzy quantities (with triangular piecewise linear membership functions) given by the triples, as usual. Here, we shall consider the following triangular fuzzy quantities

$$
\begin{array}{ll}
\tilde{c}_{11} = (3,4,5), & \tilde{c}_{12} = (2,4,6), \\
\tilde{c}_{21} = (2,3,4), & \tilde{c}_{21} = (3,4,5), \\
\tilde{a}_{11} = (1,3,5), & \tilde{a}_{12} = (1,1,1), \\
\tilde{a}_{21} = (1,3,5), & \tilde{a}_{22} = (3,3,3), \\
\tilde{b}_1 = (8,11,14), & \tilde{b}_2 = (11,12,15).
\end{array}
\tag{55}
$$

Notice that $\tilde{a}_{12}$ and $\tilde{a}_{22}$ are crisp fuzzy numbers.

The dual FLP problem to (PE) is formulated as follows

$$
\text{(DE)} \quad
\begin{array}{ll}
\text{"minimize"} & \tilde{w} = \tilde{b}_1 y_1 \tilde{+} \tilde{b}_2 y_2 \\
\text{"subject to"} & \tilde{c}_{11} \tilde{Q} (\tilde{a}_{11}y_1 \tilde{+} \tilde{a}_{21}y_2), \\
& \tilde{c}_{12}\ \tilde{Q} (\tilde{a}_{12}y_1 \tilde{+} \tilde{a}_{22}y_2), \\
& \tilde{c}_{21} \tilde{Q} (\tilde{a}_{11}y_1 \tilde{+} \tilde{a}_{21}y_2), \\
& \tilde{c}_{22} \tilde{Q} (\tilde{a}_{12}y_1 \tilde{+} \tilde{a}_{22}y_2), \\
& y_1, y_2 \geqslant 0,
\end{array}
\tag{56}
$$

Here, $\tilde{P}$ and $\tilde{Q}$ is a pair of dual fuzzy relations, particularly $\tilde{P} = \preceq^{Pos}$ and $\tilde{Q} = \prec^{Nec}$, see (4), (5).

Given $\alpha, \beta \in (0,1)$, $\alpha = \beta$, by (48) and (49) we obtain the following couple of dual problems

$$
\begin{array}{ll}
\text{maximize} & z_1 = \tilde{c}_{11}^{\mathrm{R}}(\alpha)x_1 \tilde{+} \tilde{c}_{12}^{\mathrm{R}}(\alpha)x_2, \\
& z_2 = \tilde{c}_{21}^{\mathrm{R}}(\alpha)x_1 \tilde{+} \tilde{c}_{22}^{\mathrm{R}}(\alpha)x_2, \\
\text{subject to} & \tilde{a}_{11}^{\mathrm{L}}(\alpha)x_1 \tilde{+} \tilde{a}_{12}^{\mathrm{L}}(\alpha)x_2 \leqslant \tilde{b}_1^{\mathrm{R}}(\alpha), \\
& \tilde{a}_{21}^{\mathrm{L}}(\alpha)x_1 \tilde{+} \tilde{a}_{22}^{\mathrm{L}}(\alpha)x_2 \leqslant \tilde{b}_2^{\mathrm{R}}(\alpha), \\
& x_1, x_2 \geqslant 0,
\end{array}
\tag{57}
$$

$$\begin{array}{ll}\text{minimize} & w = \tilde{b}_1^{\mathrm{R}}(\alpha)y_1 \tilde{+} \tilde{b}_2^{\mathrm{R}}(\alpha)y_2,\\ \text{subject to} & \tilde{a}_{11}^{\mathrm{L}}(\alpha)y_1 \tilde{+} \tilde{a}_{21}^{\mathrm{L}}(\alpha)y_2 \geqslant \tilde{c}_{11}^{\mathrm{R}}(\alpha),\\ & \tilde{a}_{12}^{\mathrm{L}}(\alpha)y_1 \tilde{+} \tilde{a}_{22}^{\mathrm{L}}(\alpha)y_2 \geqslant \tilde{c}_{12}^{\mathrm{R}}(\alpha),\\ & \tilde{a}_{11}^{\mathrm{L}}(\alpha)y_1 \tilde{+} \tilde{a}_{21}^{\mathrm{L}}(\alpha)y_2 \geqslant \tilde{c}_{21}^{\mathrm{R}}(\alpha),\\ & \tilde{a}_{12}^{\mathrm{L}}(\alpha)y_1 \tilde{+} \tilde{a}_{22}^{\mathrm{L}}(\alpha)y_2 \geqslant \tilde{c}_{22}^{\mathrm{R}}(\alpha),\\ & y_1, y_2 \geqslant 0,\end{array} \tag{58}$$

As $\preceq^{Pos}$ is an "optimistic" fuzzy relation and $\prec^{Nec}$ is a "pessimistic" one, this couple can be called "optimistic-pessimistic" dual couple. Particularly, substituting (55) into (57), (58), we obtain

$$\begin{array}{ll}\text{maximize} & z_1 = (5-\alpha)x_1 + (6-2\alpha)x_2,\\ & z_2 = (4-\alpha)x_1 + (5-\alpha)x_2,\\ \text{subject to} & (1+2\alpha)x_1 + x_2 \leqslant 14-3\alpha,\\ & (1+\alpha)x_1 + 3x_2 \leqslant 15-3\alpha,\\ & x_1, x_2 \geqslant 0,\end{array} \tag{59}$$

$$\begin{array}{ll}\text{minimize} & w = (14-3\alpha)y_1 + (15-3\alpha)y_2,\\ \text{subject to} & (1+2\alpha)y_1 + (1+\alpha)y_2 \geqslant 5-\alpha,\\ & y_1 + 3y_2 \geqslant 6-2\alpha,\\ & (1+2\alpha)y_1 + (1+\alpha)y_2 \geqslant 4-\alpha,\\ & y_1 + 3y_2 \geqslant 5-\alpha,\\ & y_1, y_2 \geqslant 0.\end{array} \tag{60}$$

On the other hand, let $\tilde{P} = \prec^{Nec}$ and $\tilde{Q} = \preceq^{Pos}$. Then by (50) and (51), with $\alpha = \beta$ we obtain the following couple of dual problems

$$\begin{array}{ll}\text{maximize} & z_1 = \tilde{c}_{11}^{\mathrm{L}}(\alpha)x_1 \tilde{+} \tilde{c}_{12}^{\mathrm{L}}(\alpha)x_2,\\ & z_2 = \tilde{c}_{21}^{\mathrm{L}}(\alpha)x_1 \tilde{+} \tilde{c}_{22}^{\mathrm{L}}(\alpha)x_2,\\ \text{subject to} & \tilde{a}_{11}^{\mathrm{R}}(\alpha)x_1 \tilde{+} \tilde{a}_{12}^{\mathrm{R}}(\alpha)x_2 \leqslant \tilde{b}_1^{\mathrm{L}}(\alpha),\\ & \tilde{a}_{21}^{\mathrm{RL}}(\alpha)x_1 \tilde{+} \tilde{a}_{22}^{\mathrm{R}}(\alpha)x_2 \leqslant \tilde{b}_2^{\mathrm{L}}(\alpha),\\ & x_1, x_2 \geqslant 0,\end{array} \tag{61}$$

$$\begin{array}{ll}\text{minimize} & w = \tilde{b}_1^{\mathrm{L}}(\alpha)y_1 \tilde{+} \tilde{b}_2^{\mathrm{L}}(\alpha)y_2,\\ \text{subject to} & \tilde{a}_{11}^{\mathrm{R}}(\alpha)y_1 \tilde{+} \tilde{a}_{21}^{\mathrm{R}}(\alpha)y_2 \geqslant \tilde{c}_{11}^{\mathrm{L}}(\alpha),\\ & \tilde{a}_{12}^{\mathrm{R}}(\alpha)y_1 \tilde{+} \tilde{a}_{22}^{\mathrm{R}}(\alpha)y_2 \geqslant \tilde{c}_{12}^{\mathrm{L}}(\alpha),\\ & \tilde{a}_{11}^{\mathrm{R}}(\alpha)y_1 \tilde{+} \tilde{a}_{21}^{\mathrm{R}}(\alpha)y_2 \geqslant \tilde{c}_{21}^{\mathrm{L}}(\alpha),\\ & \tilde{a}_{12}^{\mathrm{R}}(\alpha)y_1 \tilde{+} \tilde{a}_{22}^{\mathrm{R}}(\alpha)y_2 \geqslant \tilde{c}_{22}^{\mathrm{L}}(\alpha),\\ & y_1, y_2 \geqslant 0,\end{array} \tag{62}$$

This couple can be called ”pessimistic-optimistic” dual couple. Again, substituting (55) into (62), we obtain

$$\begin{array}{ll} \text{maximize} & z_1 = (3+\alpha)x_1 + (2+2\alpha)x_2, \\ & z_2 = (2+\alpha)x_1 + (3+\alpha)x_2, \\ \text{subject to} & (5-2\alpha)x_1 + x_2 \leqslant 8+3\alpha, \\ & (3-\alpha)x_1 + 3x_2 \leqslant 11+\alpha, \\ & x_1, x_2 \geqslant 0, \end{array} \tag{63}$$

$$\begin{array}{ll} \text{minimize} & w = (8+3\alpha)y_1 + (11+\alpha)y_2, \\ \text{subject to} & (5-2\alpha)y_1 + (3-\alpha)y_2 \geqslant 3+\alpha, \\ & y_1 + \quad 3y_2 \geqslant 2+2\alpha, \\ & (5-2\alpha)y_1 + (3-\alpha)y_2 \geqslant 2+\alpha, \\ & y_1 + \quad 3y_2 \geqslant 3+\alpha, \\ & y_1, y_2 \geqslant 0. \end{array} \tag{64}$$

Let $\alpha = \beta = 0,7$ be an appropriate level of satisfaction (degree of satisfaction or, necessity degree) for the objective function and for the constraints. By Simplex method we obtain the following numerical results. The optimal solutions of problems (59), (60), i.e. ”optimistic-pessimistic” dual couple are displayed in Table 1.

Table 1

$\tilde{P}$ =	$\preceq^{Pos}$		$\tilde{Q} =$	$\prec^{Nec}$
$x_1^* =$	4,15		$y_1^* =$	0,74
$x_2^* =$	1,95		$y_2^* =$	1,25
$z^* =$	24,91		$w^* =$	24,91

The optimal solutions of problems (63), (64), i.e. the ”pessimistic - optimistic” dual couple are displayed in Table 2.

Table 2

$\tilde{P}$ =	$\prec^{Nec}$		$\tilde{Q} =$	$\preceq^{Pos}$
$x_1^{**} =$	2,19		$y_1^{**} =$	0,39
$x_2^{**} =$	2,22		$y_2^{**} =$	1,00
$z^{**} =$	15,65		$w^{**} =$	15,65

As is evident from Table 1, the value $z^* =$ 24,91 of the optimal solution of the ”optimistic” primal problem is greater than the value $z^{**} =$ 15,65 of the optimal solution of the ”pessimistic” primal one. This result is in a correspondence with our expectation. By Strong Duality Theorem $x^* = (4,15; 1,95)$ is a

(0,7; 0,7)-maximal solution of FLP problem (PE), and y^* =(0,74; 1,25) is a (0,3; 0,3)-minimal solution of FLP problem (DE) such that (52) holds, i.e. $z^* = w^*$. Moreover, y^* =(0,74; 1,25) is a vector of dual (shadow) prices of the resources $\tilde{b}_i$ at disposition. The vector y^* is a $(1-\alpha, 1-\alpha)$-minimal solution of the "pessimistic" dual problem with the meaning that the smallest value of $\tilde{a}_{1j}y_1\tilde{+}\tilde{a}_{2j}y_2$ with the degree of satisfaction at least $1-\alpha$, is less or equal to the largest value of $\tilde{c}_j$ with the degree of satisfaction at least $1-\alpha =$ 0,3.

Analogical explanation could be formulated for the other dual couple (PE) and (DE) with $\tilde{P} = \prec^{Nec}$ and $\tilde{Q} = \preceq^{Pos}$, i.e. for "pessimistic - optimistic" dual couple. Again by Strong Duality Theorem $x^{**} =$ (2,19; 2,22) is a (0,3; 0,3)-maximal solution of FLP problem (PE), and $y^{**} =$ (0,39; 1,00) is a (0,7; 0,7)-minimal solution of FLP problem (DE) such that (52) holds, i.e. $z^{**} = w^{**} =$ 15,65. Here, y^{**} =(0,39; 1,00) is a vector of dual (shadow) prices of the resources $\tilde{b}_i$ at disposition. The vector y^* is a (0,7; 0,7)-minimal solution of the "optimistic" dual problem with the meaning that the largest value of $\tilde{a}_{1j}y_1\tilde{+}\tilde{a}_{2j}y_2$ with the degree of satisfaction at least 0,7 is at most equal to the smallest value of $\tilde{c}_j$ with the degree of satisfaction at least 0,7.

CONCLUSION

In this paper we introduced a class of FMOLP problems and defined the concepts of β-feasible and (α, β)-maximal and minimal solutions. Our approach here is different to the approaches used in [5] and [6]. Particularly, in [5] and [6] we investigated different concept of "optimal" solution of FLP problem, namely, the concept of satisficing solution, for the comparison of these approaches (see [8]).

In [8], we used a similar concept of α-efficient solutions, however, it was applied in a different way to the single objective FLP problem. Here, we present a more detailed analysis of the MFLP problems focused on duality theory, moreover, an illustrative example is discussed.

In [11] a problem of LP with coefficients belonging to given usual sets have been investigated and duality results have been derived (see also [1]).

Recently, in [13], duality in FLP is investigated for a special relation used for comparing fuzzy numbers, based on other two possibility and necessity indices, namely (2) and (3). A fuzzy relation investigated in [13] is a fuzzy extension of the usual binary relation $\leqslant$, in the sense of Definition 2, however, it is different to fuzzy relations $\preceq^{Pos}$or $\prec^{Nec}$ investigated here.

It is possible to investigate duality in FLP problems even in more general settings. There exist several ways of generalization. For instance, it is possible to extend the duality results to some other classes of fuzzy relations, or, to find some necessary conditions that fuzzy relations for comparing fuzzy numbers should satisfy in order to provide a duality result, or, eventually a duality gap. Moreover, in [6], the concept of dual couples of t-norms and t-conorms has been formulated and dual fuzzy relations have been defined. The role of dual relations in the couple of dual FLP problems should be also clarified and a more general duality theory could be derived. The other way of generalization is based on introducing interactive fuzzy coefficients, or oblique fuzzy vectors (see e.g. [6]).

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