Sebastian Sitarz

SENSITIVITY ANALYSIS IN LINEAR VECTOR OPTIMIZATION

Abstract

In this paper the element-wise analysis approach to sensitivity analysis in linear vector optimization is presented. Two cases are considered: sensitivity analysis of efficient solutions and sensitivity analysis of dominating solutions. The results obtained allow to create methods based on the analysis of a simplex tableau. The presented approach allowed to obtain the intervals of the parameter for which a given solution is efficient or dominating.

Keywords

Multicriteria linear programming, sensitivity analysis, postoptimal analysis.

INTRODUCTION

Sensitivity analysis is an important tool in decision-making theory. Many research papers have been published in this field in recent years. Kuk et al. [4] consider three types of perturbation maps: perturbation maps, proper perturbation maps, and weak perturbation maps, corresponding to three kinds of solution concepts: minimality, proper minimality, and weak minimality with respect to a fixed ordering cone for a vector optimization problem. Sensitivity analysis for multiobjective linear programming problems based on scalarization was presented by Vetschera [7], although the volume-based sensitivity analysis was used. In the paper of Thuan and Luc [6] it is proved that if the data of a linear multiobjective programming problem are smooth functions of a parameter, then in the parameter space there is an open dense subset where the efficient solution set of the problem can be locally represented as a union of some faces whose vertices and directions are smooth functions of the parameter. Yildirim [8] presents a unifying geometric framework to extend the optimal partition approach to sensitivity analysis in convex conic optimization. Gunawan and Azarm [3] present a method to measure the multiobjective robustness of a design alternative using the sensitivity region concept and an approach using that measure to obtain robust Pareto solutions of multicriteria programming problems.

The paper presents sensitivity analysis of a chosen efficient (or dominating) solution in vector linear optimization. The case of parameterizing the objective function coefficient is considered. The methods presented allow to use analysis of simplex tableau. Here, the author examines the sensitivity of a single efficient (or dominating) solution. The postoptimization problem is presented in the following ways:

1. If and when a given efficient solution remains an efficient solution after a certain change of the objective function.

2. If and when a given dominating solution remains a dominating solution after a certain change of objective function.

The paper is structured as follows. Section 2 presents the basic theory and notation of linear vector optimization. Section 3 describes the formulation of the considered problems: sensitivity analysis of efficiency and sensitivity analysis of domination. For illustration, a numerical example is presented in Section 4. The last section consists of concluding remarks and further research.

1. LINEAR VECTOR OPTIMIZATION

We will consider the following vector linear optimization problem:

$$\text{VMax } \{\mathbf{Cx}: \mathbf{x}\in \mathrm{X}\} \tag{1}$$

where

$\mathrm{X}=\{\mathbf{x}\in \mathrm{R}^n: \mathbf{Ax}\le\mathbf{b}, \mathbf{x}\ge 0\}$ or $\mathrm{X}=\{\mathbf{x}\in \mathrm{R}^n: \mathbf{Ax}=\mathbf{b}, \mathbf{x}\ge 0\}$ – feasible region in decision space.

$\mathbf{x}\in \mathrm{R}^n$ – vector of decision variables

$\mathbf{C}\in \mathrm{R}^{n,k}$ – matrix of objective function coefficients

$\mathbf{A}\in \mathrm{R}^{n,m}$ – full row rank matrix of constraint coefficients

$\mathbf{b}\in \mathrm{R}^m$ – right hand side vector

We call $\hat{\mathbf{x}}\in \mathrm{X}$ the dominating solution of (1) if

$$\underset{\mathbf{x}'\in \mathrm{X}}{\forall}\ \mathbf{C}\hat{\mathbf{x}} \ge \mathbf{Cx}'$$

We will denote the set of all dominating solutions of the problem (i) by $\mathrm{X}_{\mathrm{D}(i)}$.

We call $\mathbf{x}^*\in \mathrm{X}$ the efficient solution of (1) if

$$\sim \underset{\mathbf{x}'\in \mathrm{X}}{\exists}\ \mathbf{Cx}^* \le \mathbf{Cx}' \wedge\ \mathbf{Cx}^* \ne \mathbf{Cx}'$$

We will denote the set of all efficient solutions of the problem (i) by $\mathrm{X}_{\mathrm{S}(i)}$.

1.1. Simplex tableau

We will use the following notation for the problem (1):

$\mathbf{A}_B = [\mathbf{a}^{j_1}, \mathbf{a}^{j_2}, \ldots, \mathbf{a}^{j_m}]$ – basic columns of **A**

$B = \{j_1, \ldots, j_m\}$ – index set of base

$\mathbf{A}_N$ – nonbasic columns of **A**

$\mathbf{x} = [\mathbf{x}_B, \mathbf{x}_N]$ – basic solution associated with B, ($\mathbf{x}_B = \mathbf{A}_B^{-1}\mathbf{b} \geq \mathbf{0}$, $\mathbf{x}_N = \mathbf{0}$)

$\mathbf{C}_B$ – basic columns of **C**

$\mathbf{C}_N$ – nonbasic columns of **C**

$\overline{\mathbf{C}} = \mathbf{C} - \mathbf{C_B A_B}^{-1}\mathbf{A}$ – reduced cost matrix

$\overline{\mathbf{C}}_N = \mathbf{C}_N - \mathbf{C_B A_B}^{-1}\mathbf{A}$ – reduced cost matrix associated with nonbasic variables.

Using these symbols, we will denote a simplex tableau as presented in Table 1.

Table 1

Organization of a simplex tableau of the problem (1)

	$\mathbf{x}$	
$\mathbf{x_B}$	$\mathbf{A_B^{-1}A}$	$\mathbf{A_B^{-1}b}$
	$\overline{\mathbf{C}}$	

1.2. Testing efficiency

Consider the following single objective linear programming problem associated with the basic solution x*:

$$\begin{aligned} & \text{Max } \mathbf{1}^T\mathbf{v} \\ & -\overline{\mathbf{C}}_\mathbf{N}\,\mathbf{y} + \mathbf{I}\mathbf{v} = \mathbf{0} \\ & \left[\mathbf{A}_B^{-1}\mathbf{A}_N\right]_D \mathbf{y} + \mathbf{I}\mathbf{s} = \mathbf{0} \\ & \mathbf{0} \leq \mathbf{y},\ \mathbf{0} \leq \mathbf{v},\ \mathbf{0} \leq \mathbf{s} \end{aligned} \tag{2}$$

where

$\mathbf{1} = [1,1,\ldots,1]^T$ vector of ones,

I – identity matrix.

$\left[\mathbf{A}_B^{-1}\mathbf{A}_N\right]_D$ – the rows of $\mathbf{A}_B^{-1}\mathbf{A}_N$ associated with degenerated basic variables.

Theorem 1 [5]

The solution **x*** is efficient if and only if the problem (2) has a bounded objective function value of zero.

2. SENSITIVITY ANALYSIS

We want to determine a region of the parameter t such that the feasible solution x* is an efficient (dominating) solution of the following problem:

$$\text{VMax } \{ \mathbf{D}_t^{ij}\, \mathbf{x} \colon \mathbf{x} \in X \} \tag{3}$$

where $\mathbf{D}_t^{ij}$ is this matrix obtained from matrix **C** by changing element c_{ij} into parameter t. To make the notation clear we will omit the indexes "ij" and "t", ie.: $\mathbf{D} := \mathbf{D}_t^{ij}$. The elements d_{kl} of matrix **D** are described as follows:

$$d_{kl} = \begin{cases} c_{kl}, & \text{if } (k,l) \neq (i,j) \\ t, & \text{if } (k,l) = (i,j) \end{cases}$$

Moreover, we will denote the reduced cost matrix of the problem (3) by $\overline{\mathbf{D}}$.

2.1. Testing efficiency

The problem test (2) constructed for the problem (3) has the following form:

$$\begin{aligned} &\text{Max } \mathbf{1}^{\mathrm{T}}\mathbf{v} \\ &-\overline{\mathbf{D}}_{\mathrm{N}}\, \mathbf{y} + \mathbf{I}\mathbf{v} = \mathbf{0} \\ &\left[\mathbf{A}_{\mathrm{B}}^{-1}\mathbf{A}_{\mathrm{N}}\right]_{\mathrm{D}} \mathbf{y} + \mathbf{I}\mathbf{s} = \mathbf{0} \\ &\mathbf{0} \leq \mathbf{y},\ \mathbf{0} \leq \mathbf{v},\ \mathbf{0} \leq \mathbf{s} \end{aligned} \tag{4}$$

Let us discuss the effect of parameterizing the coefficient c_{ij} in the problem (1) on the problem (4). We will analyze the problem in two cases: when $j \notin B$ and $j \in B$.

Case: $j \notin B$.

In this case the reduced cost matrix has the form:

$$\overline{\mathbf{D}}_{\mathrm{N}} = \mathbf{C}_{\mathrm{N}}(t) - \mathbf{C}_{\mathrm{B}}\mathbf{A}_{\mathrm{B}}^{-1}\mathbf{A}$$

In the above equation the parameter t appears only in one element c_{ij}. Hence

$$\left(\overline{d}_N\right)_{kl} = \begin{cases} \left(\overline{c}_N\right)_{kl} & \text{if } (k,\ l) \neq (i,\ j) \\ t - \mathbf{C}_B^i \left[\mathbf{A}_B^{-1}\mathbf{A}\right]^j & \text{if } (k,\ l) = (i,\ j) \end{cases}$$

where

$\mathbf{C}_\mathrm{B}^i$ – i-th row of matrix $\mathbf{C}_\mathrm{B}$

$\left[\mathbf{A}_\mathrm{B}^{-1}\mathbf{A}\right]^j$ j-th column of matrix $\left[\mathbf{A}_\mathrm{B}^{-1}\mathbf{A}\right]$

Therefore in a case where $j \notin \mathrm{B}$ the only one element of the constraint matrix [$\overline{\mathbf{D}}_\mathrm{N}$, I] depends on t.

Case: $j \in \mathrm{B}$

In this case the reduced cost matrix has the form:

$$\overline{\mathbf{D}}_\mathrm{N} = \mathbf{C}_\mathrm{N} - \mathbf{C}_\mathrm{B}(t)\,\mathbf{A}_\mathrm{B}^{-1}\,\mathbf{A}$$

In the above equation the parameter t appears only in i-th row of $\mathbf{C}_\mathrm{B}(t)$. Hence

$$\left(\overline{d}_\mathrm{N}\right)_{kl} = \begin{cases} \left(\overline{c}_\mathrm{N}\right)_{kl} & \text{if } k \neq i \\ \left(c_\mathrm{N}\right)_{il} - \mathbf{C}_\mathrm{B}^i(t)\left[\mathbf{A}_\mathrm{B}^{-1}\mathbf{A}\right]^l & \text{if } k = i \end{cases}$$

where

$\mathbf{C}_\mathrm{B}^i(t) = [c_{i1}, \ldots, c_{i,j-1},\ t,\ c_{i,j+1}, \ldots, c_{im}]$ – i-th row of matrix $\mathbf{C}_\mathrm{B}$.

$\left[\mathbf{A}_\mathrm{B}^{-1}\mathbf{A}\right]^l$ – l-th column of matrix $\left[\mathbf{A}_\mathrm{B}^{-1}\mathbf{A}\right]$

Therefore in a case where $j \in \mathrm{B}$ the only one row of the constraint matrix [$\overline{\mathbf{D}}_\mathrm{N}$, I] depends on t. This row is associated with objective function $\mathbf{c}_i^\mathrm{T}\mathbf{x}$

We have shown that parameterizing the coefficient c_{ij} in the problem (1) causes the paramaterizing constraint matrix in the problem (4). The methods of analyzing such problems were widely discussed by many authors. Below we present some of them in view of the results obtained earlier.

Consider a single objective parametric linear programming:

$$\begin{aligned} &\text{Max } \mathbf{c}^\mathrm{T}\mathbf{x} \\ &\mathbf{x} \in \mathrm{X} = \{\ (\mathbf{A} + \mathbf{A}^* t)\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\ \} \end{aligned} \qquad (5)$$

where $\mathbf{A}^*$ denotes the matrix consisting of coefficients of parameter t.

The simplest case, where only one element of the constraint matrix changes (A^* has all zero elements except one), is examined among others in Dinkelbach's book [1] which contains methods of sensitivity analysis for this case. Let us notice that such parameterization presents sensitivity analysis of efficient solution due to c_{ij} when $j \notin B$ (see chapter 3.1).

Gal's book [2] gives algorithms for finding problem solutions (5) in two cases. One case deals with parameterization of one constraint matrix row (A^* has all zero rows except one). The other case deals with parameterization of one column of constraint matrix (A^* has all zero columns except one).

2.2. Testing domination

We want to verify that the dominating solution $\hat{\mathbf{x}}$ of (1) remains the efficient solution of (3).

Point $\hat{\mathbf{x}}$ is dominating a solution of (3) if and only if it is an optimal solution for all objective functions. Thus, it is an optimal solution if the optimality condition (by means of simplex tableau) is satisfied. This condition is satisfied if $\overline{\mathbf{D}} \le 0$ (all elements of matrix $\overline{\mathbf{D}}$ should be nonpositive).

Below, we present an example of sensitivity analysis in the presented model.

3. EXAMPLE

Consider the problem:

$$\text{VMax } [4x_1 - x_2,\ \ x_1 + 3x_2]$$
$$x_1 + x_2 \le 6,$$
$$x_1 + 2x_2 \le 10,$$
$$0 \le x_1,\ 0 \le x_2$$

The set of feasible solutions X is a polyhedron with the extreme points: $\mathbf{x}^1 = [0, 0]$, $\mathbf{x}^2 = [0, 5]$, $\mathbf{x}^3 = [2, 4]$, $\mathbf{x}^4 = [6, 0]$. The set of all efficient solutions contains two edges: $\overline{\mathbf{x}^1\mathbf{x}^2}$ and $\overline{\mathbf{x}^2\mathbf{x}^3}$. Figure 1 presents the graphical illustration of this problem in the decision space.

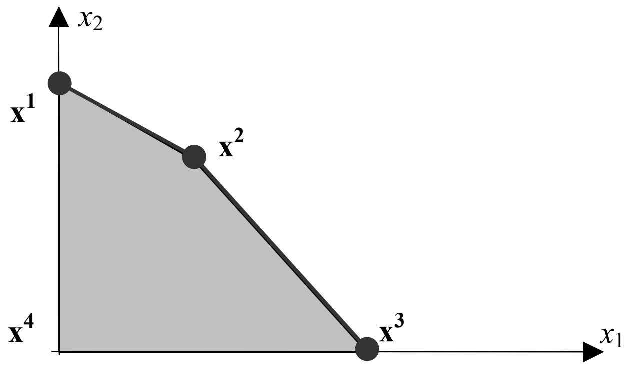

Fig. 1. Graphical illustration of the example in the decision space.

Let us analyze the sensitivity of efficiency for the extreme point $\mathbf{x}^2$=[2,4] considering the coefficient $c_{22}=t$. The simplex tableau for the problem (1) associated with extreme point $\mathbf{x}^3$ is shown in Table 2.

Table 2

Simplex tableau related to x^2 (Example 1)

	x_1	x_2	x_3	x_4	
x_1	1	0	2	-1	2
x_2	0	1	−1	1	4
	0	0	−7	3	
	0	0	−2+t	1−t	

Using the reduced cost matrix presented in table 2:

$$\overline{\mathbf{D}}_{\mathrm{N}} = \begin{bmatrix} -7 & 3 \\ -2+t & 1-t \end{bmatrix}$$

we obtain the problem test (4) for $\mathbf{x}^3$:

$$\text{Max } v_1 + v_2$$
$$7y_1 - 3y_2 + v_1 = 0$$
$$(2-\mathrm{t})y_1 + (-t+1)y_2 + v_2 = 0$$
$$y_1, y_2, v_1, v_2 \geq 0$$

The initial simplex tableau for this problem is presented in Table 3.

Table 3

Initial tableau for problem test considering point x[3] (Example 1)

	y_1	y_2	v_1	v_2	
v_1	7	−3	1	0	0
v_2	2−t	−1+t	0	1	0
	−9+t	4−t	0	0	

It is easy to determine for which t the solution presented in table 3 remains optimal:

$$(-9+t \le 0) \wedge (4-t \le 0) \quad \Leftrightarrow t \in [4, 9]$$

Let $t \le 4$. Consider two cases: $t \le 1$ and $t \ge 1$

- if $t \le 1$ we have a nonpositive column accompanied by a positive reduced cost of y_2:

$$\begin{bmatrix} -3 \\ -1+t \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for} \quad t \le 1$$

This means that our linear program has an unbounded optimal value. Thus for $t \in (\ \infty, 1]$ the extreme point $\mathbf{x}^2$ is not an efficient point.

- if $t > 1$ (and $t \le 4$). Pivoting y_2 into the basis, we obtain Table 4.

Table 4

Second simplex tableau for the problem test considering point x[2] (Example 1)

	y_1	y_2	v_1	v_2	
v_1	$\frac{-1+4t}{-1+t}$	0	1	$\frac{3}{-1+t}$	0
y_2	$\frac{2-t}{-1+t}$	1	0	$\frac{1}{-1+t}$	0
	$\frac{1-4t}{-1+t}$	0	0	$\frac{-4+t}{-1+t}$	

To determine for which t the solution presented in Table 4 remains optimal we solve the system of inequalities:

$$\frac{1-4t}{-1+t} \le 0 \text{ and } \frac{-4+t}{-1+t} \le 0.$$

Since $1<t\le 4$ this system is satisfied for each $t\in(1, 4>$.

Let $t\ge 9$. In this case, pivoting y_1 into the basis (table 3), we obtain Table 5.

Table 5

Third simplex tableau for the problem test considering point x^2 (Example 1)

	y_1	y_2	v_1	v_2	
y_1	1	$-\frac{3}{7}$	$\frac{1}{7}$	0	0
v_2	0	$-\frac{1}{7}+\frac{4}{7}t$	$-\frac{2}{7}+\frac{1}{7}t$	1	0
	0	$\frac{1}{7}-\frac{4}{7}t$	$\frac{9}{7}-\frac{1}{7}t$	0	

To determine for which t the solution presented in Table 4 remains optimal we solve the system of inequalities:

$$\frac{1}{7}-\frac{4}{7}t \le 0 \text{ and } \frac{9}{7}-\frac{1}{7}t \le 0$$

This system is satisfied for each $t\in[9, \infty)$

To summarize: The extreme point $\mathbf{x}^2=[2, 4]$ is efficient for $c_{22}=t\in(1, +\infty)$.

Investigating other extreme points, we obtain the intervals presented in Table 6.

Table 6

Intervals of the parameter t, for which the extreme points are efficient

	c_{11}	c_{12}	c_{21}	c_{22}
$\mathbf{x}^1=[0, 5]^T$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, 1,5)$	$(2, +\infty)$
$\mathbf{x}^2=[2, 4]^T$	$(0,5, +\infty)$	$(-\infty, 8)$	$(-\infty, 3)$	$(1, +\infty)$
$\mathbf{x}^3=[6, 0]^T$	$(1, +\infty)$	$(-\infty, 4)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$
$\mathbf{x}^4=[0, 0]^T$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

Now let us analyze the sensitivity of domination for the extreme point $\mathbf{x}^2$=[2,4] considering the coefficient $c_{22}=t$. First, we look for such values of the parameter t for which the point x^2 is optimal for both criteria. Using the reduced cost matrix (all elements of matrix $\overline{\mathbf{D}}$ should be nonpositive) presented in Table 2, we obtain the following conditions:

$$-7 \le 0$$

$$3 \le 0$$

and

$$-2+t \le 0$$

$$1-t \le 0$$

This system is inconsistent. Thus, there is no such t that x^2 is a dominating point.

Investigating other extreme points, we obtain the intervals presented in Table 7.

Table 7

Intervals of the parameter t, for which the extreme points are dominating

	c_{11}	c_{12}	c_{21}	c_{22}
$\mathbf{x}^1=[0, 5]^T$	$(-\infty, 0{,}5]$	$[8, +\infty)$	∅	∅
$\mathbf{x}^2=[2, 4]^T$	∅	∅	∅	∅
$\mathbf{x}^3=[6, 0]^T$	∅	∅	$[3, +\infty)$	$(-\infty, 1]$
$\mathbf{x}^4=[0, 0]^T$	∅	∅	∅	∅

CONCLUSIONS AND FURTHER RESEARCH

The element-wise analysis approach to sensitivity analysis in linear vector optimization was presented. Two cases were considered: sensitivity analysis of efficient solutions and sensitivity analysis of dominating solutions. The results obtained allow to create methods based on analysis of a simplex tableau. The approach presented here allowed to obtain the intervals of parameter for which a given solution is efficient or dominating. We presented examples which let us analyse the described methods.

It is worth considering a case of vector perturbation (instead only of one element). Parameterizing the vector of the coefficients in cost matrix causes parameterizing of one column of constraint matrix in one-criterion linear program, which may be the subject of further research.

REFERENCES

1. Dinkelbach W.: Sensitivitätsanalysen und parametrische Programmierung. Springer, Berlin-New York 1969.
2. Gal T.: Postoptimal analyses, parametric programming, and related topics. Walter de Gruyter, Berlin 1995.
3. Gunawan S., Azarm S.: Multi-objective robust optimization using a sensitivity region concept. „Structural and Multidisciplinary Optimization" 2005, 29, pp. 50-60.
4. Kuk H., Tanino T., Tanaka: Sensitivity analysis in parametrized convex vector optimization. „Journal of Mathematical Analysis and Applications" 1996, 202, 3, pp. 511-522.
5. Steuer R.: Multiple Criteria Optimization Theory: Computation and Application. John Willey, New York 1986.
6. Thuan L.V. , Luc D.T.: On Sensitivity in Linear Multiobjective Programming. "Journal of Optimization Theory and Applications" 2000, 107, 3, pp. 615-626.
7. Vetschera R.: Volume-Based Sensitivity Analysis for Multi-Criteria Decision Models. In: Methods of Multicriteria Decision Theory. Eds. Göpfert A., Seeländer J., Tammer Chr. Hänsel-Hohenhausen, Egelsbach 1997.
8. Yildirim E.A.: Unifying Optimal Partition Approach to Sensitivity Analysis in Conic Optimization. "Journal of Optimization Theory and Applications" 2004, 122, 2, pp. 405-423.