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ON MULTICRITERIA PROBLEMS WITH MODIFICATION OF ATTRIBUTES*

Abstract

In this paper we propose a mathematical model for multicriteria decision problems with alternatives which may change their properties in a direct response to external actions. We assume that the change of attributes may be controlled by the decision-maker taking into account that an improvement of the criteria values bears certain cost. Thus we get a bi-level multicriteria optimisation problem: an optimal allocation of resources at the lower level, and finding the related nondominated outputs surpassing a reference point q at the higher level. A concrete problem of this type, motivated by technological, ecological and socio-economical applications, will be discussed in more detail, namely optimising the structure of a finite population X by assuring that after a fixed time T a maximal number of its elements is characterised by nondominated values of criteria. Assuming that X consists of N elements, the solution to this problem is equivalent to solving in parallel N discrete dynamic programming problems sharing the same resources.

Keywords

Multicriteria optimisation, decision theory, dynamic programming, discrete-event systems, discrete-time control systems.

INTRODUCTION

Real-life decision-making is a dynamic process, even if time is not expressed explicitly in the usually simplified mathematical problem formulation.

Applying standard static methods of multicriteria decision-making one assumes that the alternatives are characterised by fixed attributes, whereas the main difficulty consists in finding and accepting the nondominated compromise values. In dynamical multicriteria decision models based on optimal control the evolution of criteria values is described over certain interval of time, however, usually only the values of criteria evaluated *a posteriori* at the end of the control

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period are taken into account for the decision-making purposes. At last, some dynamical programming and trajectory optimisation models allow to consider the intermediate criteria values, but the control principle consists rather in switching between alternatives than in changing their properties.

Therefore there exists a need for appropriate mathematical models for the decision problems with alternatives which may change their properties as a direct response to the external actions. As an example of such problem may serve e.g. the situation where the crew is to be completed from among a set of candidates based on several criteria related to the knowledge, abilities etc., and the selection committee considers for each non-perfect candidate a possibility of investing some amount for the additional education, internal training, etc. to achieve the desired virtues after a period of time. To apply a quantitative analysis method, in the above example one has to evaluate the estimates of cost and duration of the additional training, as well as to elaborate a model of evolution of the attributes. Another type of problems, which may be treated within the framework here proposed are investment problems, where the initial decision determines the scope of future actions. An example of a problem of this kind, referring to the choice of the computer system, is shown in Sec.3.

In this paper we propose a family of such models which may be regarded as a generalisation of discrete choice methods to the situations where the time evolution plays an essential role. The changes of the values of criteria may occur spontaneously, as well as they may be systematically influenced by the decision-maker. It will also be assumed that the decision-maker's actions resulting in a desired improvement of the criteria values may bear certain cost and may not be immediate, i.e. the lapse of time necessary to realize the desired change may be considered as an auxiliary criterion. Thus we get a bi-level multicriteria optimisation problem consisting of the optimal allocation of resources at a lower level, and selecting the related Pareto optimal outputs of the original problem in a minimal time at the higher-level. An important feature of this approach is a proper description of possible transitions between the attributes of each criterion, which will be accomplished by introducing in Sec.1 so called transition patterns.

To make the presentation of the above ideas maximal comprehensible, the scope of this paper will be confined to the problems with a finite alternative set Ω, and the performance criterion $F = (F_1,...,F_N)$ admitting values from a partially ordered finite set V.

Let $V_i := \{v_{i,1}, v_{i,2}, ..., v_{i,c(i)}\}$ denote the set of values of the criterion F_i ordered from the least to the most preferred one by an order "$\prec_I$". Then V is the Cartesian product of V_i, i.e. $V = V_1 * ... * V_N$ with the coordinate-wise partial order "$\prec$". The changes of values of F for a fixed $\omega \in \Omega$ are results of the control actions $u(t) \in U$ for t from a discrete time interval $[t_0,T]$:

$$((F_1(\tau),\ldots,F_N(\tau)) \colon \Omega \to V) \to opt, \text{ for a } \tau \in [t_0, T] \\ F(t+1)(\omega) = \varphi(F(t)(\omega), u(t), t) \text{ for each } \omega \in \Omega,\ t \in [t, T-1] \tag{1}$$

The optimality principle "opt" in (1) should model adequately the decision situation concerned. As a representative example, in Sec. 2 we will investigate a reference point problem which may be formulated as:

"$(F_1(\tau),\ldots,F_N(\tau))(\omega)$ is Pareto-optimal in $F(\tau)(\Omega)$ and exceeds given $q \in V$ for the minimal $\tau \in [t_0,T]$ and at a minimal cost of control".

The triple (Ω, F, ϕ) will be called the *decision process*.

A control u(t) will be identified with a transition $(v_i \to v_j)$ on the time interval (t,t+1], which is the result of an external action undertaken by the supervisor of the decision process. Besides of controlled transitions we will distinguish the deterministic uncontrolled ones which may not be influenced, as e.g. passing to the following age classes, random transitions occurring spontaneously, and non-admissible transitions. The classes of controllable and random transitions need not be disjoint, although in this paper we will be concerned with deterministic control only. Thus, the evolution of attributes may be modelled in a manner similar to the discrete-event systems described in [8], whereby the values of F play the role of system's states.

Since the set of alternatives Ω, the set of values of criteria V, and time are all discrete, such decision processes will be called a *D-D-D-system*. The particular relevance of D-D-D-systems consists in the fact that they constitute a natural extension of the discrete choice and outranking methods. Moreover, there exist close relations to the multicriteria optimal stopping problem, and to the multicriteria problems with variable constraints described in [7; 2]. In the final section we will provide several illustrative examples and point out the further research problems related to D-D-D-systems.

After a suitable discretisation of the criteria values, the solution methods here presented may also be used for the discrete-continuous processes (D-D-C), where merely the criteria values are arbitrary real numbers. The analysis of continuous processes (D-C-C or C-C-C) can in most cases be accomplished within the framework of the multicriteria optimal control problems with the criteria included in the state-space vector.

1. EVOLUTION OF ATTRIBUTES: THE TRANSITION AND COST PATTERNS

Even under the above simplified assumptions the number N of the admissible criteria values may be very large. However, the task of supplying all necessary information concerning the transitions between the values of F might be considerably simplified if it were possible to find a convenient description of the transfer function ϕ, and to identify the non-admissible transitions before starting the numerical solution process. A further reason for introducing here the transition and cost patterns is to reduce the computational complexity of the general problem by decomposing it into several subproblems, each one of them referring to the single criterion F_i, i = 1,..N. This would be possible if the characterisation of transitions between the values of $(F_1,...F_N)$, including their admissibility, could be derived from the properties of the single criteria, $F_1(t)$ through $F_N(t)$, considered separately. Below we will show that this goal can be achieved under some additional assumptions concerning the set Ω and the criterion F.

Let us fix the moment of time $t\in [t_0,T]$ and let $V_i: = \{v_{i,1},v_{i,2},...v_{i,(c(i))}\}$denote the set of values of the criterion F_i ordered from the least to the most preferred one. If we know which transitions between the values of the criterion F for an $\omega\in\Omega$ are at all possible on the time interval (t,t+1], we could define for F_i and ω the *transition pattern* as a quadratic 0-1 matrix

$$P(F_i)(\omega) = [p^i_{jk}(\omega)]$$

with the following coefficients:

$$p^i_{jk}(\omega) = \begin{cases} 1 & \text{iff } \omega\in\Omega \text{ may change its classification in one time step from j-th} \\ & \text{to the k-th attribute of } F_i \\ 0 & \text{otherwise} \end{cases} \quad (2)$$
$$\text{for } j,k=1,..c(i),\ i = 1,..N$$

Observe that the dimension of $P(F_i)(\omega)$ equals to the number of elements of V_i , c(i), and its columns indicate the admissible transitions from an appropriate fixed starting value of F. Remark that to each transition pattern $P(F_i)$ one can associate the digraph $G(F_i)$ such that $P(F_i)$ is its structural matrix. In general, the transition patterns may vary on the interval $[t_0,T]$, being thus functions of both, ω and t.

Transitions from v to w on the time interval (s,t] may be regarded as pairs (v,w) and will be denoted by v->w. By a *superposition of the transitions* $\xi_1:= v_1\to v_2$ on the interval $(t_1,t_2]$, and $\xi_2:= v_2 \to v_3$ on the interval $(t_2, t_3]$ we will mean the transition$\xi:=v_1\to v_3$ on $(t_1,t_3]$, and denote it by $\xi = \xi_1 \circ \xi_2$.

Suppose now that $v_1 = (v_{11},v_{12})$. By the *composition of the transitions* $\xi_1 := v_{11} \to v_2$ on $(t_1,t_2]$ and $\xi_2 := v_{12} \to v_3$ on the same interval $(t_1,t_2]$, we will mean the transition $v_1 \to (v_2, v_3)$ on $(t_1, t_2]$. To denote compositions we will use the notation $\xi_1 \mathrm{c}\, \xi_2$.

Let

$$\xi_1 := (v_{11}, v_{12}) \to (v_2, v_{12}) \text{ and } \xi_2 := (v_{11}, v_{12}) \to (v_{11}, v_3)$$

Then, formally, $\xi_1 \circ \xi_2 = \xi_1 \mathrm{c}\, \xi_2$, the diversity between composition and superposition being expressed by the associated time intervals. Observe that the superpositions and compositions describe sequential and parallel processing of transitions, respectively. By definition, the compositions are always admissible in one time step, the superpositions may, but need not necessarily have this property. To assure a minimal number of non-zero coefficients in $P(F)(\omega,t)$, it is convenient to include in the transition patterns only the transitions which may not be represented as compositions of other admissible transitions.

Transitions lasting several time steps may often be represented as superposition of one-step transitions. If it is not so, they can still be considered within the same framework by introducing the intermediate or wait values of F. This question will be considered in more detail further in this section.

Now we will introduce several properties of the decision process (Ω,F,ϕ), which will be used in the further analysis of the initial decision-making problem.

Definition 1.1. We will say that the set of alternatives Ω is *homogeneous* with respect to F at the moment $t \in [t_0, T]$, iff

$$\forall 1 \le i \le N \ \forall x,y \in \Omega : P(F_i)(x) = P(F_j)(y) \tag{3}$$

If (3) is satisfied for all $t \in [t_0, T]$, we will call Ω homogeneous.

If Ω is not homogeneous but handling a separate transition pattern for all alternatives would be computationally inefficient then one may consider instead the Boolean product of transition patterns for all $x \in \Omega$. Generally, in models of real-life discrete dynamical systems the transition patterns depend on discretisation of time, which should be suitably chosen. Moreover, as we already noted, they may depend on time itself. In the sequel we will usually admit the assumption that the decision process (Ω,F,ϕ) is stationary, according to the following definition:

Definition 1.2. If for each $\omega \in \Omega$ and $1 \le i \le N$, $P(F_{\cdot i})(\omega)$ remains constant on the whole interval $[t_0, T]$ then the decision process (Ω, F, ϕ) will be called *stationary*.

Observe that the stationarity assumption is equivalent to the fact that the function φ from (1) does not depend on time t.

Another important set of properties concerns the independence of criteria $F_1, \ldots, F_N$.

Definition 1.3. The criteria $F_1, \ldots F_N$ are *evolution-independent* at $x \in \Omega$ and $t \in [t_0, T]$, by definition it means that any transition

$$F(t)(x) := (v_{1,i(1)}, \ldots v_{N,i(N)}) \to (v_{1,j(1)}, \ldots v_{N,j(N)}) = F(t+1)(x)$$

is admissible iff for each $1 \le k \le N$ the transitions $v_{k,i(k)} \to v_{k,j(k)}$ are admissible, i.e. iff $p^{k}_{i(k),j(k)}(x)(t) = 1$.

The criteria $F_1, \ldots F_N$ will be called globally evolution independent, or simply evolution-independent iff the above holds for all $x \in \Omega$ and $t \in [t_0, T]$. Roughly speaking, the criteria are evolution independent iff the admissibility of transitions between the values of F_k, $1 \le k \le N$, is not affected by the present values of all remaining criteria.
It is easy to observe that the following fact is true :

Proposition 1.1. If the decision process (Ω, F, ϕ) is homogeneous and stationary, and for an $x_0 \in \Omega$, $t \in [t_0, T]$, $F_1, \ldots F_N$ are evolution independent at (x_0, t) then $F_1, \ldots F_N$ are globally evolution independent.

In the sequel we will always assume that the criteria concerned are evolution independent.

Example 1.1. Suppose that Ω is the population of citizens of a city and one of the objectives s is the age scale with the attributes $v_1 = [0,20]$, $v_2 = (20,40]$, $v_3 = (40,60]$, and $v_4 = (60,\infty)$, denoting the age in years of a single individual. If all time discretisation steps in the process (1) are less than 20 years, which is usually the case, then the transition pattern $P(s) = [p_{jk}]$ is the matrix

$$P(s) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

No transitions are controllable, unless we dispose a relativistic vehicle to force remaining within the same age class (elements on the main diagonal would then correspond to controllable transitions).

An important feature of the transition patterns for evolution independent criteria consists in the fact that it is sufficient to determine the patterns for the single criteria only, while the transition patterns for the vector criterion may be calculated basing on the following

Proposition 1.2. Assume that the evolution independent optimisation criteria F_i and F_j are defined on a homogeneous population Ω with the transition patterns

$$P_i = [p^{(i)}_{k,l}] := P(F_i)(t) \in M_{c(i),c(i)} \text{ and } P_j = [p^{(j)}_{m,n}] := P(F_j)(t)\ M_{c(j),c(j)}$$

for certain fixed $t \in [t_0,T]$, respectively. Then the transition pattern $P_{ij} := P(F_i, F_j)(t)$ of the vector criterion $F_{ij} := (F_i, F_j)$, is the block matrix

$$P_{ij} = [p^{((i)}_{k,l}\ P_j\ ]^{\,c(i)}_{\,k,l=1} \in M_{c(i)c(j),c(i)c(j)} \tag{4}$$

(the block product of P_i and P_j), where c(i) and c(j) denote the number of admissible values of the criteria F_i and F_j , respectively. The values of F_{ij} , $v_{kl} = (v^{(i)}_k, v^{(j)}_l)$, labelling the rows and columns in P_{ij} , are ordered lexicographically with the first coordinate more relevant than the second.

Proof: Suppose first that the transitions $(v_{i,k(i)} \rightarrow v_{i,l(i)})$ and $(v_{j,k(j)} \rightarrow v_{j,l(j)})$ are both admissible, i.e. $p^{(i)}_{k(i),l(i)} = 1$ and $p^{(j)}_{k(j),l(j)} = 1$. Then from the evolution independency assumption (Def. 1.3) it follows that the transition

$$\xi := (v_{i,k(i)}, v_{j,k(j)}) \rightarrow (v_{i,l(i)}, v_{j,l(j)})$$

is admissible. According to the construction of P_{ij} (cf. (4)), the element corresponding to ξ in P_{ij} , $p^{(ij)}_{m,n}$ with m = k(i)c(j)+k(j) and n = l(i)c(j)+l(j), is the (k(j),l(j))-th coefficient of the block $p^{(i)}_{k(i),l(i)}\ P_j$, which is equal to 1 since $p^{(i)}_{k(i),l(i)} = 1$, and $p^{(i)}_{k(i),l(i)} P_j = P_j$.

If at least one from the above simple transitions is non-admissible then $p^{(i)}_{k(i),l(i)} = 0$ or $p^{(j)}_{k(j),l(j)} = 0$. In the first case the block $p^{(i)}_{k(i),l(i)} P_j$ of P_{ij} contains only zero elements, in the second, its (k(j),l(j))-th coefficient, $p^{(ij)}_{m,n}$, is equal to zero. However, from the definition of the evolution independent criteria it follows that any transition between values of (F_i, F_j) must be represented as a superposition of simple admissible transitions cannot be admissible, therefore the transition ξ corresponding to the zero coefficient $p^{(ij)}_{m,n}$ is not admissible.

Now, let us fix a coefficient $p^{(ij)}_{m,n}$ of P_{ij}. Then there exist $k1(m,n),l1(m,n)\in[1,c(i)]$ and $k2(m,n),l2(m,n)\in[1,c(j)]$ such that $p^{(ij)}_{mn}$ is the $(k2(m,n),l2(m,n))$-th coefficient of the block $p^{(i)}_{k1(m,n),l1(m,n)}P_j$ of P_{ij} , i.e. to $p^{(ij)}_{m,n}$ there can be associated the transition

$$(v_{i,k1(m,n)}, v_{j,k2(m,n)}) \to (v_{i,l1(m,n)}, v_{i,l2(m,n)})$$

and $p^{(ij)}_{m,n}$ determines its admissibility, as shown in the first part of the proof. Hence we conclude that the above characterisation of the of P_{ij} as transition pattern (2) for (F_i, F_j) is complete.

Corollary 1.1. If the transition patterns of two evolution independent criteria s and q, P(s) and P(q), have p(s) and p(q) non-zero elements, respectively, then the transition pattern for (s,q), P(s,q) contains at most p(s)p(q) non-zero elements.

Corollary 1.2. Consequently, the transition patterns for any finite number of evolution independent criteria, $F_1,\dots,F_n$, are sparse block matrices which can be constructed making recursively use of Proposition 1.2.

1.1. Assignment of controls and costs to admissible transitions

Let us fix $w\in\Omega$, $\omega\in\Omega$, and $t\in[t_0,T]$, and assume that the coefficients of the transition pattern $P(F)(\omega,t)$ are ordered lexicographically. Then to each admissible transition $v\to w$ we can associate a control $u_m(t)$ and its cost $J(v,w,\omega,t) := J(u_m,\omega,t)$, where the integer $m:= m(v,w)$ is the ordinal number of the appropriate control from the list $U:= \{u_1,\dots,u_M\}$ responsible for the change from v to w.

If $J(v,w,\omega,t)$ does not depend on the past transitions then, analogously to the transition patterns, for each $\omega\in\Omega$, $t\in[t_0,T]$, and each criterion F_i, one can define the cost pattern $J(F_i)(\omega,t)$, as a function associating to each transition $v_{i,k}\to v_{i,l}$ that is feasible between t and t+1, the cost of applying the control $u_m := u(i,k,l,\omega,t)$ that causes the change from $v_{i,k}$ to $v_{i,l}$,

$$J(F_i, v_{i,k}, v_{i,l}, \omega, t) := J_i(u_m, \omega, t)$$

Hence, the cost pattern for F_i can be represented as the $c(i)*c(i)$ real matrix defined as follows:

$$J^{i}_{kl}(\omega,t) := \begin{cases} J_i(u(i,k,l,\omega,t),\omega,t) & \text{iff } v_{i,k} \text{ is admissible} \\ \infty & \text{– otherwise} \end{cases} \tag{6}$$

Consequently, the cost pattern for $F_1, ..., F_N$, $J(F_1, ..F_N)(\omega,t)$, is the c(1)...c(N) x c(1)...c(N) real matrix storing the costs of transitions between the values of the vector criterion $F := (F_1, ...F_N)$, i.e. $J_{i(1),...,i(N),\, j(1),...,j(N)}(t, \omega)$ is the cost of changing the value $(v_{1,i(1)}, ..., v_{N,i(N)})$ of F to $(v_{1,j(1)}, ...v_{N,j(N)})$, or it is undefined iff such transition is non-admissible. Hence it follows that the structure of the cost patterns is closely related to the transitions patterns whereby only those coefficients of J which correspond to a "1" in P are finite. Thus, in a machine implementation of the above decision process, the transition patterns may serve as addresses of those elements of $J(F_1, ..F_N)$ which has to be stored in the memory. Moreover, observe that the zero coefficients of J correspond usually to non-controllable transitions.

For the evolution-independent criteria $F_1, ...F_N$ an important role is played by the following condition:

Definition 1.4. The cost function $J(F_1, ..F_N)$ satisfies the *cost-additivity* condition iff for any $\omega \in \Omega$, $t \in [t_0, T]$, the cost of any admissible transition v->w, where $v := (v_{1,i(1)}, ...v_{N,i(N)})$ and $w := (v_{1,j(1)}, ...v_{N,j(N)})$, is the sum of changing the single criteria values, i.e.:

$$J(v,w,\omega,t) = \sum_{k=1}^{N} J(v_{k,i(k)}, v_{k,j(k)}, \omega, t) \tag{7}$$

The above condition lets us consider each transition as a composition of simple transitions during the computation of the optimal improvement strategy for the values of F. This, in turn, allows to omit the operations on J(F), using instead $J(F_i)$, for I = 1,…,N.

1.2. Handling the transitions distributed over time

Let us start this subsection from the following definition :

Definition 1.5. A transition $v := (v_{1,k(1)}, ...v_{N,k(N)}) \rightarrow (v_{1,l(1)}, ...v_{N,l(N)})$, which:
(i) cannot be represented as a superposition or composition of other admissible transitions,
(ii) may be realised after θ, $\theta>1$, time units at the soonest,
(iii) while realising *v*, F does not admit any other admissible values,
will be called *irreducible*.

To consider the irreducible transitions within the uniform decision model, one can proceed as follows:

Algorithm 1.1.
Repeat for all irreducible transitions v with the realisation time $\theta:=\theta(v)$:

Case 1:
If k(j) = l(j) for jî{1,...N}\{i} then

Step 1.1. Define the wait values $v_{i,k,1}$,...$v_{i,k,}$ θ_{-1} and attach them to V_i .
Set $V_i := V_i \cup \{v_{i,k,1}, ... v_{i,k,}\ \theta_{-1}\}$.
Step 1.2. Order the wait values in any way, according to their real-life interpretation, but without affecting the existing partial order "$\prec_I$", i.e.,
if $v_{k(i)} \prec_i v_{l(i)}$, or $v_{l(i)} \prec_i v_{k(i)}$ then $v_{k(i)} \prec_i v_{i,k,1} \prec_i ... \prec_i v_{i,k,}\ \theta_{-1} \prec_i v_{l(i)}$, or
$v_{l(i)} \prec_i v_{i,k},\ \theta_{-1} \prec_i ... \prec_i v_{i,k,1} \prec_i v_{l(i)}$, respectively.
Step 1.3. Update the transition pattern $P(F_i)$, respectively.
If n is the last irreducible transition STOP else proceed with the next v.

In particular, one proceeds in this manner in a single-criteria model with the objective F_i if a transition from $v_{j,k}$ to $v_{i,l}$ is irreducible.

Case 2:
If v:= $(v_{1,(k(1)}, ..., v_{N,k(N)}))$, w:= $(v_{1,l(1)}, ..., v_{N,l(N)})$, v:=(v→w), and k(i)≠ l(i) for at least two different values of i∈{1,...N}, then

Step 2.1. Represent the transition $v := (v_{i,k} \rightarrow v_{j,l})$ in the form

$$v = v_{1i} \text{ o} \ldots \text{o } v_{i\rho} \tag{8}$$

where v_{ij} is an admissible transition or a non-reducible simple transition, taking into account as far as possible the real-life behaviour of the system during the transition.
If the representation in form (8) is impossible go to Step 2.3.
Step 2.2. Perform for each irreducible simple transition the Steps 1.1-1.3 from the Case 1.
If v is the last irreducible transition STOP else proceed with the next v.
Step 2.3. Introduce the wait values v $v_{,1}$,...v $v_{,}\theta_{-1}$, directly as elements of V, similarly as in the Steps 1.1-1.2, but do not associate them with any V_i.
If n is the last irreducible transition STOP, else proceed with the next v.

Observe that in the case dealt with in the Step 2.3, the criteria F_1 ,...F_N are not evolution-independent and the further analysis of the problem cannot be based only on the simple transition and cost patterns $P(F_i)$.

Same intermediate values may be shared by different irreducible transitions, and one can show that a minimal set of such values may be found.

2. A NEW CLASS OF MULTICRITERIA DECISION-MAKING PROBLEMS

As an example of decision models applying the above sketched family of quantitative structures, we will discuss in more detail the following basic problem:

Problem 2.1. Find the alternative $\omega\in\Omega$ and an optimal allocation of resources $u(1),...,u(\tau)$, to achieve or surpass by the value of $F(\tau)(\omega)$, in the minimal time τ and at minimal cost, one of the reference points $q\in\theta$ defined in the space of criteria values.

In a more rigorous setting, let Ω be the finite set of admissible alternatives at time $t_0 := 1$, and F_1 ,..F_N the criteria functions defined on Ω with values in the discrete sets V_i with the partial order "$\prec_I$" for I = 1,..N. Similarly as in the previous sections denote by F the vector criterion $F:= (F_1 ,..F_N)$, $F: \Omega \to V$, valued in the Cartesian product V:= V *...*V with the coordinatewise partial order "$\prec$". For a fixed $\omega\in\Omega$, the values of criteria on w may vary according to (1), i.e.

$$F(t+1)(\omega) = \phi(F(t)(\omega),u(t),t), \text{ for } t\in [t_0,T-1]$$

Our task consists in finding an $\omega\in\Omega$, a $\tau\in [t_0,T]$, and a sequence of controls $u(t_0),...,u(\tau)$, so that :

(i) $((F_1(t_0),..F_N(t_0))(\omega)$ is nondominated in V and

$$((F_1(\tau),..F_N(\tau))(\omega) \prec q \tag{9}$$

for certain reference point $q\in\theta$;

(ii) τ with the property (9) is minimal in $[t_0,T]$ for a fixed ω; such value will be denoted $\tau(\omega)$;

(iii) $\sum_{t0\le t\le\tau(\omega)} J(u(t), \omega,t)$ is minimal on the set Λ defined as follows:

$$\Lambda := \bigcup_{w\in\Omega} \{w\}*\{y\in U^S : \tau(w) \text{ and } F(\tau(w))(w) \text{ satisfy (i) and (ii)}\} \qquad (10)$$

where s:= τ(w)-t_0 , and y:= (u(t_0),...,u(τ(w)))

Observe that, according to (1), F(τ (w))(w) and τ (w) are indirect functions of y. The minimal value of (iii) on Λ will be denoted by $J_{min}(\omega)$.
(iv) $((F_1(\tau(\omega)),..F_N(\tau(\omega)))(\omega),\tau(\omega), J_{min}(\omega))$ is nondominated in the set $V\mathrm{x}\mathbb{R}^2$ with the coordinatewise partial order.

The general Problem 2.1 consists of two tasks : finding an optimal alternative w, and a sequence of controls y assuring the achievement of q at a minimal cost. Each alternative $\omega\in\Omega$ is characterised by the minimal time $\tau(\omega)$ and the minimal cost $J_{min}(\omega)$ of achieving or surpassing q. Consequently, if for each $\omega\in\Omega$ one knows these minimal parameters, then the final choice of ω is a bicriteria trade-off between the cost and time, which can be made using one of well-known interactive decision-making methods applied for bicriteria problems.
We will present a solution to the above problem for the decision processes satisfying the following assumptions:
(i) the decision process (Ω,F,ϕ) is stationary and homogeneous;
(ii) the criteria F are evolution independent;
(iii) all transitions between the values of F are deterministic;
(iv) the costs of transitions satisfy the cost-additivity condition;
(v) the reference set $Q\subset V$ can be represented in the form $Q:= \{p\in V: p\prec q\}$ for certain $q\in V$.

As the first step of the solution, below we will show how can one determine $\tau(\omega)$ and $J_{min}(\omega)$ for a fixed ω.

2.1. Solving single-object evolution problems

Let us admit all above assumptions (i)-(iv), let us fix an $\omega\in\Omega$, and let $f_0 := F(t_0)(\omega)$. Further, let us consider a directed network G=(V,E), where the nodes V can be identified with the set $V= V_1 \mathrm{x}...V_N$ of potential values of F, while the edges $e\in E\subset V^2$ are determined by the transition patterns $P_1,...,P_N$, $P_i := P(F_i)(\omega)$, for i = 1,..N, in the following way:

$$\begin{gathered} e = (f,g) \in E \Leftrightarrow f\neq g \text{ and } \exists!\, j\in\{1,..N\}\ f_i = g_i \text{ for } i\in\{1,..N\}\setminus\{j\} \text{ and} \\ P_j(f_j,g_j) = 1 \text{ or } f = g \text{ and } \forall\, i\in\{1,..N\}\ P_i(f_i,gi) = 1 \end{gathered} \qquad (11)$$

Thus, the edges of G correspond to the simple transitions between the values of F or may be loops. Additionally, the edges of G are equipped with quantitative labels describing the time θ_i and the cost of transition J_i, and qualitative labels c_i indicating whether the corresponding transition is forced or controllable. Hence, the following observation is straightforward :

Proposition 2.1. The transition between two values of criteria, f and g, is possible iff the nodes corresponding to f and g in G can be connected by a path.

As a corollary from Prop. 2.2 we get

Proposition 2.2. The graph G is the Cartesian product of graphs $G_1,...G_N$, which correspond to the single criteria $F_1,...F_N$ and their transition patterns $P_1,...P_N$, respectively. Its structural matrix is given as the block product of $P_1,...P_N$, $P_{1,...,N}$.

Hence it follows

Theorem 2.1. The solution to the Problem 2.1 for a single alternative $x\in\Omega$ can be found as a bicriteria shortest path in G between f_0 and the reference set Q:= $\{v\in V : q\{v\}$.

The solution algorithm which can be derived from the above Prop. 2.1. and 2.2 and Thm. 2.1 may be presented as follows:

Algorithm 2.1.
The input data:
The transition and cost patterns for F_1 ,...F_N , P_1 ,...P_N , and J_1 ,...J_N , respectively. The time horizon T, the starting value f_0 , the reference point $q\in V$, the reference set Q.
Step 1. Augment the transition patterns by the time-distributed transitions, applying the procedure presented in Sec. 1.3.
Step 2. Check whether the criteria are evolution independent.
If yes :
– construct the network G using the Prop. 2.2 and 2.2;
otherwise :
– set manually all edges of G.
Step 3. Check whether it exists a path joining f_0 and q, or any other $p\in Q$
(i.e. check whether Q is attainable from f_0).
If not, return to the communication shell to let the decision-maker, define new reference point or to undertake another modification of the decision-maker's preference structure.

Step 4. Determine the set D containing all bicriteria shortest paths between f_0 and all $p \in Q$, using the bicriteria shortest path algorithm. Find the set of nondominated points of D, P(D).
Step 5. Select a compromise strategy from P(D) using any bicriteria trade-off procedure.

2.2. The selection problem from among multiple evolving alternatives

In the present setting we assume that at the moment t_0 the decision-maker should choose that alternative $\omega_0 \in \Omega$ which gives the best chances to be improved till the time T so as it were not worse than q. After simulating the evolution of $F(t)(\omega)$ over time $t \in [t_0,T]$, one chooses ω_0, which will be called prospective alternative, and starts investing in its development, by undertaking the actions $u(t_0),...u(T\text{-}1)$, without taking care what happens with all remaining alternatives. This solution procedure implies the following :

Theorem 2.2. To select the prospective alternative and the best strategy in Problem 2.1 for stationary homogeneous processes with evolution independent criteria, it is necessary to solve the simultaneous bicriteria shortest path problem for the set of starting points $V_0 := \{f \in V : f = F(t_0)(\omega) \text{ for all } \omega \in \Omega\}$ and Q as the set of terminal points.

As the simultaneous shortest path algorithm one can apply a combination of the well-known Dijkstra algorithm and the bicriteria shortest path method (cf. e.g. Henig, 1985ab).

The above presented procedure will be illustrated by the following example.

Example 2.1. The choice of a computer system.
Suppose that a company is offered k different computer systems, each of them satisfies its present needs. The computer differ in price, reliability, service quality, and technical characteristics such as the processor type and its clock speed, RAM, hard disk capacity and average access time, and a possibility of attaching additional equipment and peripherals like 3D video accelerators, sensors, or control devices. All above characteristics (except reliability and, perhaps, price) may be regarded as performance criteria $F_1,...,F_N$ with discrete attribute sets. Assume that from the technical and financial purposes the buyer decides to do not buy the full system configuration at once, but prefers to extend it successively according to the future needs. Thus the choice of a system at time t_0 should

not be made taking into account just the values of $F(t_0)$, but one should apply a model of the future acquisition process within the decision process. According to the scheme presented in this paper, a system configuration would be modelled as a node in the graph G, while to an extension of the system would correspond an edge labelled by the expected price of this system extension. The final objective may consist in getting a fully configured and ready-to-use system at time T at possibly minimal price. The "full configuration" mentioned may be interpreted as a reference point in the sense of Algorithm 2.1 which allows to apply the solution methods specified in Thms. 3.1 and 2.2 and the Algorithms 1.1 and 2.1.

Finally, let us discuss the solution methods for decision processes which do not satisfy the stationarity, homogeneity, or evolution independence assumptions.

(i) For non-stationary homogeneous processes the graph G will be a function of time t. To solve the Problem 2.1 one has to apply a bicriteria shortest path algorithm for variable-structure networks [8].

(ii) If the decision process is not homogeneous, one cannot apply the simultaneous shortest path algorithm for all alternatives $\omega \in \Omega$ in G, since the structure of the graph G depends on ω. A solution to the Problem 2.1 may be found by solving the single evolution problem described in Sec. 2.1 for each graph $G(\omega)$, and aggregate the solutions as in Step 4 of Algorithm 2.1.

(iii) If the criteria are not evolution independent then the edges of the graph G may not be associated to any combination of edges in the graphs $G_1,...,G_N$. Manual editing of the transition pattern $P_{1,...N}$ is required.

(iv) Processes which are neither homogeneous nor stationary, nor the criteria are evolution independent may be analysed applying simultaneously the appropriate combination of procedures outlined in (i) and (ii).

3. OPTIMISING THE POPULATION STRUCTURE

The above presented framework may be applied to solve a variety of decision problems. Here, we will formulate the problem of optimising the structure of a finite population Ω by assuring that after a fixed time T a maximal number of elements of Ω is characterised by nondominated values of criteria.

We assume that the elements of a finite population Ω are classified according to N ordered classification criteria $F_1,...F_N$. Each element $\omega\in\Omega$ may pass to another class on the time interval [t,t+1] if according to (1) a control $u_\alpha(t)\in U(t)$ has been applied to ω_α. All transitions can be described by M:= #Ω equations of type (1) sharing the same resources:

$$\sum_{1\le\alpha\le M} b_\alpha(t)\, J(u_\alpha(t), \omega_\alpha, t) \le u_t \tag{12}$$

where $b_\alpha(t)\ge 0$ for all α. Let us note that the case $b_\alpha(t):= 1/x_{ti}$ with

$$x:= \#\{\omega\in\Omega: F(t)(w) = F(t)(\omega)\}$$

corresponds to the situation where the same control u(t) acts simultaneously on all elements of Ω characterised by the same values of criteria.

At the macroscopic level, the evolution of Ω may be described by the following discrete-time controlled dynamical system:

$$\begin{aligned} x_{t+1} &= A_t x_t + B_t w_t + \eta_t \\ z_{t+1} &= C_t x_t + D_t w_t + \varsigma_t \end{aligned} \tag{13}$$

for $t=t_0,...,T-1$, subject to the constraints

$$\sum_{t_0\le t\le T} (u_t + w_t) \le \xi(t_0,T) \tag{14}$$

where x_t, w_t, and z_t are the state, macroscopic control, and observation vectors, respectively, A_t, B_t, C_t, and D_t are real matrices, and η_t and ζ_t are random factors perturbing the growth/migration and observation processes for $t = t_0,..T$. The matrices A_t and B_t may be derived by aggregating the equations (1) for all $\omega\in\Omega$ and $t\in [t_0,T]$. The macroscopic controls w_t allow an "external" migration by attaching to (or removing from) Ω elements independently from the "internal" transitions controlled at the lower level (1), and may bear certain additional costs.

The state vectors $x_t = (x_{1t},...,x_{nt})$ contain the numbers of elements of Ω characterised by the same values of $F_1,...F_N$, for $t = t_0,...T$ (cf. Skulimowski and Schmid, 1992). Thus, there is a one-to-one correspondence I between the indices of the state variables and the elements of V, so for each $t\in [t_0,T]$ one may order the state variables $x_{1t},...,x_{nt}$ by the partial order generated from V. The values of F corresponding to the state variables are called the interpretation vector (cf. the above quoted paper of Skulimowski and Schmid) and denoted by $I(F,x_t)$. Assuming that a population Ω is characterised by certain distribution of attributes at an initial moment t_0 represented by the state vector $x^0 := x(t_0)$, the aim of control is to achieve an optimal distribution of elements of Ω at time T, using for that a minimal quantity of resources represented by u_t and w_t. Below we propose two of a variety of possible optimisation problem statements. According to an initial remark, the first one of them is related to the nondominated values of F.

Let

$$K := \{v \in V : \exists\, \omega \in \Omega \text{ such that } v = F(T)(\omega)\}$$

and let $P(K)$ be the set of nondominated elements of K. Denote by $\Pi(K)$ the set of nondominated indices of the corresponding state variables, i.e. $\Pi(K) := \Gamma^{-1}(P(K))$. By definition, the relative population structure will be optimal iff

$$\Big(\sum_{j \in \Pi(T)} x_{Tj}\Big) / \Big(\sum_{1 \le k \le T} x_{Tj}\Big) \text{ is maximal} \tag{15}$$

$$\sum_{t_{0 \le t \le T}} (q_t u_t + r_t w_t) \text{ is minimal} \tag{16}$$

and

$$m_0 \le \sum_{1 \le k \le n} x_{Tk} \le m_1 \tag{17}$$

where q_t and r_t are positive real coefficients.

The above problem formulation may have a ecological, sociological or economical motivation, namely, assuming that a population Ω remains stable if under a classification F a maximal number of its members cannot get in touch with another individuals which are better (in the partial order in V) than themselves in all relevant aspects (represented as the criteria $F_1, \ldots F_N$). In this setting, it is less important what is the shape of K and where it is situated at time T.

Introducing a loss function ψ:VxV$\rightarrow$ IR$_+$, which is right strictly order increasing, i.e.

$$v_1 \prec v_2,\ v_2 \prec v_3, \text{ and } v_1 \neq v_3 \Rightarrow \psi(v_1, v_2) \le \psi(v_1, v_3)$$

as e.g. a strictly convex distance function, we can evaluate the deviations from the ideal value $v^* := (v_{1,c(1)}, v_{2,c(2)}, \ldots v_{N,c(N)})$ at time T for each $\omega \in \Omega$. Consequently, the deviation of the whole set Ω can be characterised by the following criterion σ:

$$\sigma(u(t_0)(\Omega), \ldots u(T-1)(\Omega), w_{t0}, \ldots w_{T-1}) := \sum_{1 \le \alpha \le M} \psi(v^*, F(T)(\omega_\alpha)) \rightarrow \min \tag{18}$$

which may be more suitable for economical applications such as e.g. balancing the portfolio structure than (15)-(17). While optimising σ, we strive to approach the most preferred element of V for a possibly maximal number of elements of Ω. As the result, the set of alternatives actually characterised by nondominated values of F need not be numerous, but in average, their values are better approximating the ideal value v^* than in case of optimising the criterion (15). Let us note that always

$$\sigma = \sum_{1 \le k \le n} x_{Tk}\, \psi(v^*, I(F, x_{Ti})) \tag{19}$$

From a computational point of view a solution to the above problems consists in solving parallel N discrete optimal control problems coupled by the common resource or expense limitation (12). Thus, this problem requires non-standard solution algorithms based on dynamical programming which have been proposed in Sec. 2.2. Roughly speaking, if the decision process is homogeneous and the criteria are evolution independent, one can construct the network G presented in Secs. 2.1 and 2.2, assigning additionally the varying labels x_{ti} to the nodes $v \in V$ determined by the interpretation vector $I(F,x_t)$.

The further procedure consists in finding shortest paths (in terms of the cost function J) to the nondominated values of $F(T)(\Omega)$, calculating the values of the macroscopic criterion σ, and choosing a subset $\Omega_1 \subset \Omega$ of elements which values are to be improved.

Let us remark that the above specified class of systems requires a state-space description with a usually large number of state variables representing the quantities of elements of each class, or other characteristics as functions of time.

The above description and assumptions reflects a complicated nature of certain real-life systems, where the growth coefficients may be derived *a posteriori* from empirical experience. As examples of such systems may serve e.g. the populations of concurrent technologies or innovations, inhabitants of a town, portfolio of a company, or a wildlife reservation.

4. CONCLUDING REMARKS

The motivation for introducing the above theory originates from the real-life multicriteria decision problems, such as portfolio management, technology transfer and foresight (cf. Skulimowski, 2006) or personnel choice, where the classical decision support methods do not allow to include the time aspects into the problem analysis. The new theoretical issues should constitute a basis for a more frequent including the dynamics in the analysis of multicriteria choice problems, allowing thus a more efficient use of all preference information available which is a basis for an adequate modelling of real-life decision situations.

The approach proposed can be applied even in the simplest problems with discrete set of alternatives, and finite sets of admissible attribute values of each criterion. Methodologically, there is no conflict with other preference information since we merely extend the set of criteria values at t_0, $F(\Omega)$, to the set

$$\Psi := F(\Omega)x\{t_0\} \cup F(\Omega)(t_1)x\{t_1\}*J(U_1) \cup ... \cup F(\Omega)(T)x\{T\}*J(U_T) \tag{20}$$

where $F(\Omega)(t_i)$ is the set of reachable values of F at time t_i, and U_i is the set of all sequences of strategies $(u_{j(1)}, ... u_{j(i)})$ on $[t_0, t_1]$ applicable to one of ω. This converts the initial problem $(F:X \rightarrow V) \rightarrow \min$ into a new problem with set-valued objective $\phi : \Omega \rightarrow 2^{\Psi}$, each alternative ω being characterised by the set of values of F, cost function J, and time :

$$\phi(\omega) := \{F(\omega)(t_0)\} \cup F(\omega)(t_1)x\{t_1\}xJ(U_1(\omega)) \cup \ldots \cup F(\omega)(T)x\{T\}xJ(U_T(\omega)) \tag{21}$$

where $U_i(\omega)$ contains the sequences of controls from U applicable to ω on $[t_0, t_1]$. Observe that $\psi = U \cup \{\phi(\omega) : \omega \in \Omega\}$.

However, the choice problem, as applied to the set Ψ, remains the same as in the classical multicriteria discrete choice problem, i.e. one has to select a compromise value $\psi_c := (v_c, J_c, \tau)$ from Ψ. Once this is done, one has to find the set

$$\phi^{-1}(\psi_c) := \{\omega \in \Omega : F(\omega)(\tau) = v_c \text{ and } J(F(\omega), u_1, ... u_\tau) = J_c\} \tag{22}$$

If the set (22) contains more than one alternative, all they are equivalent with respect to the choice criteria admitted. Therefore we expect that the above presented issues might be implemented as direct extensions of well known discrete choice and outranking decision models. Moreover, in most problems, J can be aggregated with one or more of the criteria $F_1, ..., F_N$, which simplifies the formulation of the problem (21)-(22).

The solution method applied to the Problem 2.1 shows a remarkable coincidence with the approach to the multicriteria optimal control of discrete-event systems presented in [8]. On the other hand, however, the approach to simultaneously control the evolution of a population Ω outlined in Sec. 3, resulting in a discrete-time control system model (13) could be applied to control large-scale discrete-event systems which allow an appropriate decomposition of the state-space.

In the present paper we concentrated our attention on deterministic processes, although in real-life situations some of the transitions may be stochastic. The analysis of such systems which involves the optimal control of discrete Markov processes (cf. e.g. [7]) may be considered as generalisation of the methods here presented and needs further investigation.

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