Jaroslav Ramík

MULTI-CRITERIA APPROACHES TO DIAGNOSTICS OF ENTERPRISES USING ANALYTICAL NETWORK PROCESS*

Abstract

In this paper we investigate an economic diagnostic system in the situation of lack of data. We propose a diagnostic model working both with statistical and expert data. In case there exists statistical data, the diagnostic model should supply the results based on the well known Bayes' approaches, otherwise, the model should combine statistical and expert data by a generalized approach. Hence, this model is a generalization both the classical statistical approach and also expert one, which is allowed by Analytic Network Process.

Keywords

Multi-criteria decision making, analytic hierarchy process (AHP), analytic network process (ANP), pair-wise comparisons, subjective probability, Bayes' theory, diagnostics of enterprises.

INTRODUCTION

Solving problems of diagnostics of economical systems, particularly enterprises, we meet usually difficulties with interdependences among individual symptoms, i.e. the symptoms of economical systems and also causes of these symptoms. By the diagnostics we understand here a test (or system of tests) for predicting a state of the system in the future.

Recently, many diagnostic approaches are based on artificial intelligence, e.g. neural networks, see [3], the classical statistical approach based on Bayes theory is, however, still attractive, see e.g. [1], [2], [7]. This approach is focused on the assumption that the decision under uncertainty should utilize information about the decision environment, i.e. information about the history of the solution of the problem, expert knowledge etc.

Subjective probabilities in Bayes' theory allow for revision of the original prior information acquired from a large sample of population by means of the results of experiments, i.e. by so called posterior probabilities, see [7].

* This research has been supported by GACR Project No. 402/06/0431.

In economical diagnostics we often meet a situation with the lack of statistical data, the data are either out of reach or the sample is too small due to significant changes during the time and the dynamics of the process. That is why diagnostic systems utilizing both statistical and at the same time expert data are needed. In case there exists statistical data, the diagnostic model should supply the results based on the well known Bayes' approaches, otherwise, the model should combine statistical and expert data by a generalized approach, see [6]. In this paper we shall deal with the case of expert data i.e. a situation where no statistical data exists. This case is based on multi-criteria approach, particularly analytic hierarchy process (AHP), see [5] – [7].

1. BAYES' THEORY

Consider two-stage decision system: On the first stage we consider n disjoint events – *states of the system*: $S_1, S_2,...,S_n$, such that $S_i \cap S_j = \varnothing$ for $i \neq j$ and $\sum_{i=1}^{n} P(S_i) = 1$, $P(S_i) > 0$, $i=1,2,...,n$, is a probability of state S_i , see Fig.1. On the second stage consider m outcomes of the experiment $E_1, E_2,...,E_m$ such that $E_r \cap E_s = \varnothing$ for $r \neq s$ and $\sum_{r=1}^{m} P(E_r | S_i) = 1$, where $P(E_r | S_i)$, $i=1,2,...,n$, is a subjective probability of E_r on condition the existence of state S_i.

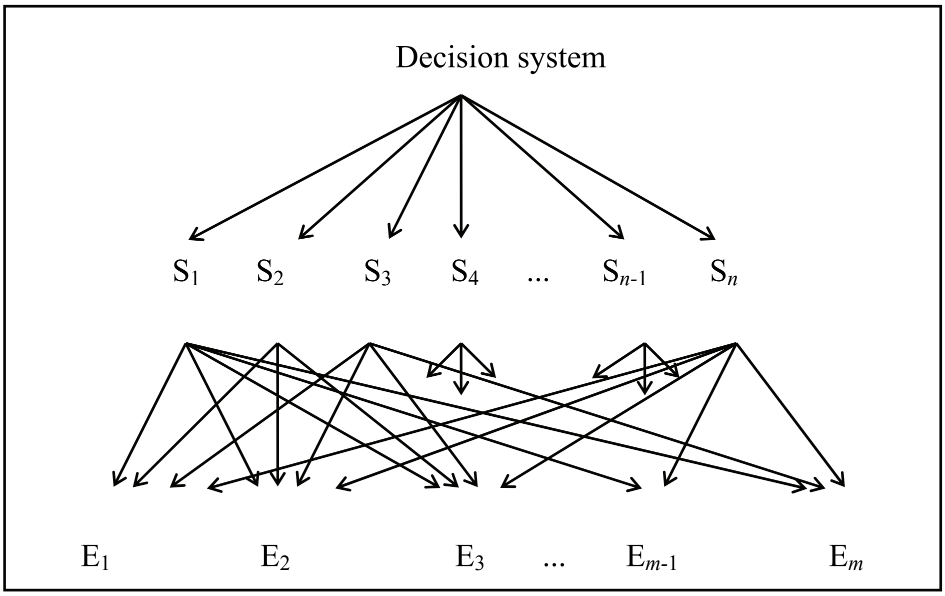

Fig. 1. Two-stage experiment – Diagnostic system

Subjective probability can be calculated by the well known *Bayes' formula*:

$$P(E_r \mid S_i) = \frac{P(E_r \cap S_i)}{P(S_i)} \tag{1}$$

By disjointness of states S_i and properties of probabilities we get:

$$P(E_r) = \sum_{i=1}^{n} P(E_r \cap S_i) \tag{2}$$

Substituting from (1) to (2) we obtain for $r = 1,2,...,m$:

$$P(E_r) = \sum_{i=1}^{n} P(E_r|S_i)P(S_i) \tag{3}$$

Further, we denote:

$$P(S) = \begin{bmatrix} P(S_1) \\ P(S_2) \\ \vdots \\ P(S_n) \end{bmatrix}, \; P(E) = \begin{bmatrix} P(E_1) \\ P(E_2) \\ \vdots \\ P(E_m) \end{bmatrix} \tag{4}$$

$$P(E \mid S) = \begin{bmatrix} P(E_1|S_1) & P(E_2|S_1) & \cdots & P(E_n|S_1) \\ P(E_1|S_2) & P(E_2|S_2) & \cdots & P(E_n|S_2) \\ \vdots & \vdots & \vdots & \vdots \\ P(E_1|S_m) & P(E_2|S_m) & \cdots & P(E_n|S_m) \end{bmatrix} \tag{5}$$

$$P(S \mid E) = \begin{bmatrix} P(S_1|E_1) & P(S_1|E_2) & \cdots & P(S_1|E_m) \\ P(S_2|E_1) & P(S_2|E_2) & \cdots & P(S_2|E_m) \\ \vdots & \vdots & \vdots & \vdots \\ P(S_n|E_1) & P(S_n|E_2) & \cdots & P(S_n|E_m) \end{bmatrix} \tag{6}$$

Then (3) can be expressed as follows:

$$P(E) = P(E \mid S)\, P(S) \tag{7}$$

$P(S_i)$ are called *prior probabilities*, they are known in advance - "a priori", usually as relative frequencies of populations. Also $P(E_r \mid S_i)$ are usually known in advance as statistical characteristics of the experiment. *Bayes' theorem*, the essence of the theory with the same name answers the question what is the probability of state S_i assuming that the outcome of the experiment is E_r. We look for *posterior probability* $P(S_i \mid E_r)$. Using the above defined notation the posterior probabilities are given by the following formula (called Bayes' theorem):

$$P(\mathrm{S}_i \mid \mathrm{E}_r) = \frac{P(\mathrm{E}_r \mid \mathrm{S}_i)\, P(\mathrm{S}_i)}{\sum_{k=1}^{n} P(\mathrm{E}_r \mid \mathrm{S}_k)\, P(\mathrm{S}_k)} \tag{8}$$

Let $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$ be a k-dimensional vector, then $diag\,(\mathbf{c}) = \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & c_k \end{bmatrix}$

is called the *diagonal matrix to vector* **c**. Then Bayes' formula (8) can be also expressed in the following matrix form:

$$P(\mathrm{S} \mid \mathrm{E}) = diag(P(\mathrm{S}))\cdot P(\mathrm{E} \mid \mathrm{S})^{\mathrm{T}}\cdot[diag(P(\mathrm{E} \mid \mathrm{S})\cdot P(\mathrm{S}))]^{-1} \tag{9}$$

2. MULTI-CRITERIA DECISIONS AND AHP/ANP

Consider a decision system with three hierarchical levels:

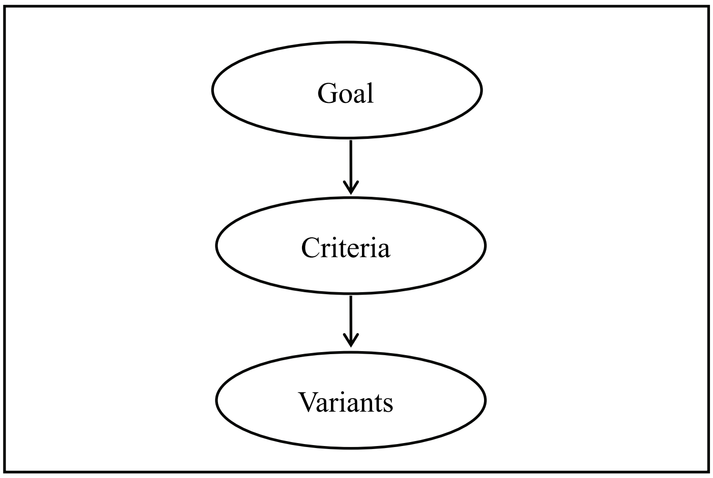

Fig. 2. Hierarchical system with 3 levels

This system is characterized by the *supermatrix* (see [7]):

$$\mathbf{W} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{W_{21}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{W_{32}} & \mathbf{I} \end{bmatrix} \tag{10}$$

here $\mathbf{W}_{21}$ is the $n\times1$ matrix (weighting vector of the criteria), $\mathbf{W}_{32}$ is the $m\times n$ matrix (the columns of this matrix are evaluations of variants by the criteria), I is the unit $m\times m$ matrix. The limit matrix $\mathbf{W}^{\infty} = \lim_{k\to+\infty} \mathbf{W}^{k}$ (see [6]) is given as follows:

$$\mathbf{W}^{\infty} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{W}_{32}\mathbf{W}_{21} & \mathbf{W}_{32} & \mathbf{I} \end{bmatrix} \tag{11}$$

Here $\mathbf{Z} = \mathbf{W}_{32}\mathbf{W}_{21}$ is the $m\times1$ matrix, i.e. the resulting priority vector of the variants. The variants can be ordered according to these priorities.

In real decision systems with 3 levels there exist typical interdependences among individual elements, e.g. criteria. Consider now the dependences among the criteria, see Fig. 3. Such a system can be solved by the method named Analytical Network Process (ANP), an extension of AHP, see [6].

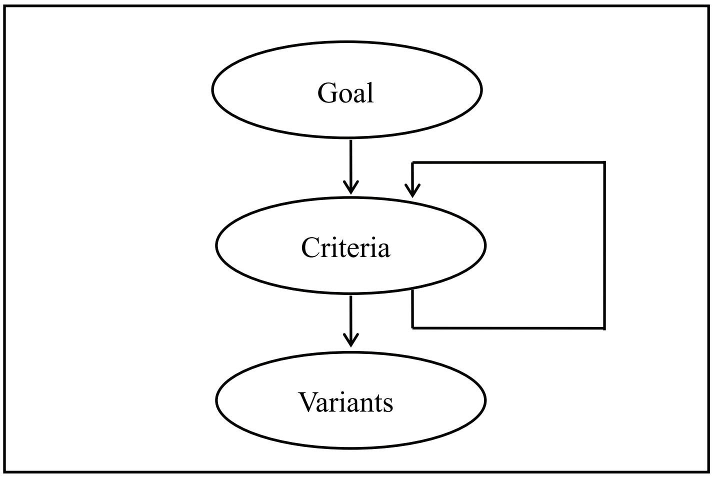

Fig. 3. Dependencies amongst criteria

This system is given by the supermatrix:

$$\mathbf{W} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{W}_{21} & \mathbf{W}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{32} & \mathbf{I} \end{bmatrix} \tag{12}$$

where the interdependences are characterized by $n\times n$ matrix $\mathbf{W}_{22}$. It is clear that matrix (12) need not be column-stochastic, i.e. sum of the elements in each column is equal to one, hence in general the limiting matrix does not exist. Stochasticity of this matrix can be saved by additional normalization of the columns of the submatrix $\begin{bmatrix} W_{22} \\ W_{32} \end{bmatrix}$, by applying e.g. the Saaty's pairwise comparison method. Then there exists a limiting matrix $\mathbf{W}^{\infty}$ such that

$$\mathbf{W}^{\infty} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{W}_{32}(\mathbf{I} - \mathbf{W}_{22})^{-1}\mathbf{W}_{21} & \mathbf{W}_{32}(\mathbf{I} - \mathbf{W}_{22})^{-1} & \mathbf{I} \end{bmatrix} \quad (13)$$

Hence the vector

$$\mathbf{Z} = \mathbf{W}_{32}(\mathbf{I} - \mathbf{W}_{22})^{-1}\mathbf{W}_{21}$$

is used for ordering the variants i.e. for the decision making process.

In the systems with 3 levels there are usually interdependences among criteria and variants, see Fig. 4.

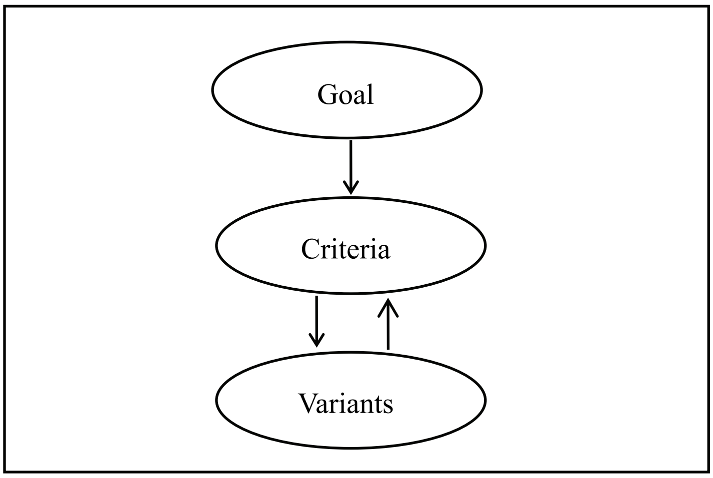

Fig. 4. Dependences amongst criteria and variants

This system is characterized by the supermatrix:

$$\mathbf{W} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{W}_{21} & \mathbf{0} & \mathbf{W}_{23} \\ \mathbf{0} & \mathbf{W}_{32} & \mathbf{0} \end{bmatrix} \quad (14)$$

where the dependences are given by the $m\times n$ matrix $\mathbf{W}_{32}$, resp. by $n\times m$ matrix $\mathbf{W}_{23}$. Evidently, matrix (14) is stochastic, however, it is neither primitive nor irreducible, hence for the limiting matrix we apply Perron-Frobenius theorem, see [4].

Let $\mathbf{W_{21}}$, $\mathbf{W_{32}}$, $\mathbf{W_{23}}$ be column stochastic matrices with positive elements. Then for the limiting matrix $\mathbf{W}^{\infty}$ of the supermatrix $\mathbf{W}$ it holds:

$$\mathbf{W}^{\infty} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ \mathbf{W}_{23}\mathbf{B}\mathbf{W}_{32}\mathbf{W}_{21} & \mathbf{A} & \mathbf{W}_{23}\mathbf{B} \\ \mathbf{B}\mathbf{W}_{32}\mathbf{W}_{21} & \mathbf{W}_{32}\mathbf{A} & \mathbf{B} \end{bmatrix} \quad (15)$$

where

$$\mathbf{A} = \lim_{k\to+\infty}\left[\mathbf{W}_{23}\mathbf{W}_{32}\right]^{k}, \ \mathbf{B} = \lim_{k\to+\infty}\left[\mathbf{W}_{32}\mathbf{W}_{23}\right]^{k} \quad (16)$$

Remark

Matrices $\mathbf{W_{32}}$ a $\mathbf{W_{23}}$ are supposed to be stochastic with positive elements, consequently they are primitive. The same holds for $\mathbf{W_{32}W_{23}}$ and $\mathbf{W_{23}W_{32}}$. Then there exist limit matrices (16), uniquely defined by positive vectors $\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$, such that $\mathbf{A} = \mathbf{a}\ \mathbf{e}_n^{\mathrm{T}}$, $\mathbf{B} = \mathbf{b}\ \mathbf{e}_m^{\mathrm{T}}$ and $\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j = 1$, where $\mathbf{e}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, resp. $\mathbf{e}_m = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, is the n-dimensional, resp. m- dimensional vector. The matrix $\mathbf{A}$ is then stochastic $n\times n$ matrix, where all its columns are identically equal to vector $\mathbf{a}$, similarly, $\mathbf{B}$ is a stochastic square $m\times m$ matrix where all its columns are identically equal to vector $\mathbf{b}$. The priority vector $\mathbf{Z}$ is located in the third row of the limit matrix $\mathbf{W}^{\infty}$, i.e.

$$\mathbf{Z} = \mathbf{B}\mathbf{W}_{32}\mathbf{W}_{21} \quad (17)$$

Consider the following matrices of prior probabilities:

$$W_{21} = \begin{bmatrix} P(\mathrm{S}_1) \\ P(\mathrm{S}_2) \\ \vdots \\ P(\mathrm{S}_n) \end{bmatrix} \quad (18)$$

$$W_{32}=\begin{bmatrix} P(E_1|S_1) & P(E_2|S_1) & \cdots & P(E_n|S_1) \\ P(E_1|S_2) & P(E_2|S_2) & \cdots & P(E_n|S_2) \\ \vdots & \vdots & \vdots & \vdots \\ P(E_1|S_m) & P(E_2|S_m) & \cdots & P(E_n|S_m) \end{bmatrix} \qquad (19)$$

And the matrix of posterior probabilities:

$$W_{23}=\begin{bmatrix} P(S_1|E_1) & P(S_1|E_2) & \cdots & P(S_1|E_m) \\ P(S_2|E_1) & P(S_2|E_2) & \cdots & P(S_2|E_m) \\ \vdots & \vdots & \vdots & \vdots \\ P(S_n|E_1) & P(S_n|E_2) & \cdots & P(S_n|E_m) \end{bmatrix} \qquad (20)$$

Bayes' theorem (9) gives the relationship among prior and posterior probabilities as follows:

$$W_{23}= \mathit{diag}(W_{21})\cdot W_{32}^{T}\cdot[\mathit{diag}(W_{32}W_{21})]^{-1} \qquad (21)$$

If in the supermatrix **W** the block W_{23} is defined by (13), then it holds:

$$W_{23}W_{32}W_{21} = W_{21} \qquad (22)$$

On the other hand, if in the supermatrix **W** the block W_{23} is defined by (16), then for the limiting matrix $\mathbf{W}^{\infty}$ we get:

$$\mathbf{W}^{\infty}=\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ \mathbf{W}_{21} & \mathbf{W}_{21}e_n^{T} & \mathbf{W}_{21}e_m^{T} \\ \mathbf{W}_{32}\mathbf{W}_{21} & \mathbf{W}_{32}\mathbf{W}_{21}e_n^{T} & \mathbf{W}_{32}\mathbf{W}_{21}e_m^{T} \end{bmatrix} \qquad (23)$$

In the limiting matrix (23) the first column is important as in the second row we have the vector of prior probabilities $P(S)= W_{21}$ and in the third row we get the vector of posterior probabilities $P(E) = W_{32}W_{21}$. The form (21) of marix W_{23}, i.e. Bayes' theorem, is a sufficient condition for $\mathbf{W}^{\infty}$ of the feedback system given by **W** in (14) can be written as (23), however, this condition is not sufficient. A natural question arises whether in **W** there exists a block W_{23} different to (21), with the same limiting matrix $\mathbf{W}^{\infty}$. The question is what is a sufficient condition for W_{23}, such that limit matrix $\mathbf{W}^{\infty}$ to matrix **W** from (21) has the form (23). The following theorem gives the answer to this question, see [6].

Theorem 1

Let W_{21}, W_{32}, W_{23} be column stochastic with positive elements in blocks of **W** defined by (14). Then $\mathbf{W}^{\infty}$ is in the form (23) if and only if the following equation holds:

$$W_{23}W_{32}W_{21} = W_{21} \qquad (24)$$

In a particular system (e.g. diagnostic system) the matrices W_{21} and W_{32} are given beforhand. In case statistical data are at disposition they are prior probabilities P(S) and P(E|S), otherwise, in case of expert data the matrices of priorities might be collected by Saaty's method of pairwise comparisons. We have to find matrix W_{23} of posterior probabilities P(S|E) (case of statistical data), or, the feedback matrix of priotrities (case of expert data).

System (24) is a reasonable model for finding matrix W_{23}, which is, however, not uniquely solvable. In the stochastic case of matrix (21) classical Bayes' approach is a suitable method for finding solution of (24). However, in case of expert data this approach need not be the unique possible solution, there exist also some other solutions, different to Bayes' one, that might also be sufficient or even more advantageous. Here ANP is a new method generalizing the classical Bayes' approach allowing for a mix of statistical and expert data. We have the following theorem.

Theorem 2

Let Q be the $(m \times n)$ column stochastic matrix with positive elements such that:

$$QW_{21} = W_{32}W_{21} \qquad (25)$$

Then matrix:

$$W_{23}^{*} = diag(W_{21}) \cdot Q^{T} \cdot [diag(W_{32}W_{21})]^{-1} \qquad (26)$$

is a column stochastic solution of the system:

$$W_{23}^{*}W_{32}W_{21} = W_{21} \qquad (27)$$

Let W_{23}^{*}, W_{32} , W_{21} satisfy (25) – (27). Then the limiting matrix $\mathbf{W}^{\infty}$ to supermatrix

$$\mathbf{W}^{*} = \begin{bmatrix} 0 & 0 & 0 \\ W_{21} & 0 & W_{23}^{*} \\ 0 & W_{32} & 0 \end{bmatrix}$$

is written in the form (23). If $Q \neq W_{32}$, then the solution of (27) is different to W_{23} in (21). This property is illustrated in the next section.

3. APPLICATION – A DIAGNOSTIC SYSTEM

In this part we apply the model described in the previous section to particular feedback system (14) of small and medium enterprises (SMEs). In the diagnostic system we consider three states: S_1 – the enterprise will bankrupt, S_2 – the enterprise will survive, S_3 – the enterprise will succeed. The prior probabilities – relative frequencies from statistical data of about 200 SMEs in Ostrava – Karviná region – are listed in the following table:

States	$P(S)$
S_1	0.20
S_2	0.70
S_3	0.10

To find out the economic state of the enterprise we applied a special test (experiment) with 4 outcomes (results):

E_1 – very bad result, E_2 – bad result, E_3 – good result and E_4 – excellent result.

In the next table the prior subjective probabilities are listed. They are based again on the above mentioned statistical data.

$P(E \mid S)$	SS_1	S_2	S_3
E_1	0.70	0.30	0.15
E_2	0.20	0.40	0.20
E_3	0.07	0.20	0.25
E_4	0.03	0.10	0.40

Probabilities of the symptoms are calculated as: $P(E) = P(E \mid S) \cdot P(S)$, the results is in the next table:

Symptoms	$P(E)$
E_1	0.365
E_2	0.340
E_3	0.179
E_4	0.116

The posterior probabilities are calculated from (9) as:

$$P(S \mid E) = diag(P(S)) \cdot P(E \mid S)^T \cdot [diag(P(E \mid S) \cdot P(S))]^{-1}.$$

The results are summarized in the following table:

$P(S \mid E)$	E_1	E_2	E_3	E_4
S_1	0.38	0.12	0.08	0.05
S_2	0.58	0.82	0.78	0.60
S_3	0.04	0.06	0.14	0.35

The values e.g. from the first column of the previous table can be interpreted as follows: If the outcome of the test of an enterprise is very bad (symptom E_1), then the probability that the enterprise would bankrupt (state S_1) is equal to 0.38, the probability that this enterprise would survive (state S_2) is 0.58 and probability the same enterprise would be successful (state S_3) is only 0.04. Analogically we could interprete the other three columns of the table, i.e. the other outcomes of the test. Now, let $W_{21} = P(E)$, $W_{32} = P(E \mid S)$, $W_{23} = P(S \mid E)$.

As an example consider the matrix Q defined below which satisfies (25) and (27), hence:

$$W_{23}^{*} \cdot W_{32} \cdot W_{21} = W_{21},$$

$$Q \cdot W_{21} = W_{32} \cdot W_{21},$$

with the following matrices:

$$W_{23}^{*} = diag(W_{21}) \cdot Q^{T} \cdot [diag(W_{32} W_{21})]^{-1}.$$

Q	S_1	S_2	S_3
E_1	0.52	0.33	0.33
E_2	0.38	0.37	0.02
E_3	0.03	0.20	0.33
E_4	0.07	0.10	0.32

W_{32}	S_1	S_2	S_3
E_1	0.70	0.30	0.15
E_2	0.20	0.40	0.20
E_3	0.07	0.20	0.25
E_4	0.03	0.10	0.40

W_{23}^{*}	E_1	E_2	E_3	E_4
S_1	0.29	0.22	0.03	0.12
S_2	0.62	0.77	0.78	0.60
S_3	0.09	0.01	0.18	0.28

By Theorem 1 and 2 the limiting matrices to the following matrices **W** and **W***:

$$\mathbf{W} = \begin{bmatrix} 0 & 0 & 0 \\ W_{21} & 0 & W_{23} \\ 0 & W_{32} & 0 \end{bmatrix}, \mathbf{W^*} = \begin{bmatrix} 0 & 0 & 0 \\ W_{21} & 0 & W_{23}{}^* \\ 0 & W_{32} & 0 \end{bmatrix}$$

are identical, in spite of $Q \neq W_{32}$.

CONCLUSION

In this paper we have investigated an economic diagnostic system in the situation of lack of data. We have proposed a diagnostic model working both with statistical and expert data. In case there exists statistical data, the diagnostic model should supply the results based on the well known Bayes´ approach, otherwise, the model should combine statistical and expert data by a generalized approach. Hence, this model is a generalization both the classical statistical approach and also expert one, which is allowed by Analytic Network Process.

REFERENCES

1. Berger J.: Statistical decision theory and Bayesian analysis. Springer-Verlag, Berlin 1985.
2. Blair A., Nachtmann, R. Olson J., Saaty T.L.: Forecasting foreign exchange rates: An expert judgement approach. Socio-Economic Planning Sci. 21, pp. 363-369.
3. Charniak E., McDermott D.: Introduction to artificial intelligence. Addison Wesley, New York 1995.
4. Gantmacher F.R.: Teorija matric (in Russian). Nauka, Moskva, 1966.
5. Saaty T.L.: Multicriteria decision making - the Analytical Hierarchy Process. RWS Publications, Pittsburgh 1991, Vol. I, pp. 502.
6. Saaty T.L.: Decision Making with Dependence and Feedback – The Analytic Network Process. RWS Publications, Pittsburgh 1996, pp. 370.
7. Saaty T.L., Vargas T.L.: Diagnosis with dependent symptoms: Bayes theorem and the Analytic Hierarchy Process. Operations Research 1998, Vol. 46, No 4, pp. 491-501.