Dmitry Podkopaev

REPRESENTING PARTIAL INFORMATION ON PREFERENCES WITH THE HELP OF LINEAR TRANSFORMATION OF OBJECTIVE SPACE

Abstract

We consider the case in which information about preferences in a multiobjective optimization problem is represented in the form of upper bounds on trade-off coefficients, defined for any pair of objective functions. We introduce some requirements for this preference system and derive a system of constraints which should be put on the upper bounds. We use a matrix which elements are numbers inverse to upper bounds on trade-offs as the linear operator for transforming the objective space. The presented constraint system allows for proving that if a given evaluation is weakly Pareto optimal in the transformed space, then in the initial space it is Pareto optimal and satisfies the corresponding bounds on trade-off coefficients.

Keywords

Multiple objective optimization, incomplete preference information, bounds on trade-off coefficients, linear transformation of objective space.

INTRODUCTION AND PROBLEM SETTING

Incomplete information about preferences is typical for practical multiobjective problems. Therefore much attention in multiobjective optimization is paid to methods of solving problems exploiting partial information on preferences. One of approaches to representing and treating such information is to put bounds on global trade-offs. This approach is elaborated in a series of works by Kaliszewski, Michalowski, and Zionts (see for example [1-3]) and is applied in the construction of interactive methods of solving multiobjective optimization problems [3].

In the present work we show that a linear constraint system should be put on the bounds mentioned above. Using these constraints we elaborate a new approach to setting bounds on trade-offs based on linear transformation of the objective space.

Below we give the problem setting and the definition of trade-off solutions known from the literature. In the first Section we introduce a strict preference relation which corresponds to bounding trade-offs. We require for this preference relation to be asymmetric and weakly transitive. From these requirements we derive a set of constraints which should satisfy the given bounds on trade-offs. In the second Section we present a new approach to constructing a preference relation based on the linear transformation of the objective space. Using the constraints mentioned above we establish a connection between the "classical" bounds on trade-offs and the new approach.

Let us consider the multiobjective optimization problem of the following form:

$$\max_{x \in X} f(x) \qquad (1)$$

where X is the set of feasible solutions, $f(x) = (f_1(x), f_2(x), \ldots, f_k(x))$ is the vector objective, $f_i: X \rightarrow \mathbf{R}$, $i \in N_k$ are objective functions, $N_k = \{1,2,\ldots,k\}$; $k \geq 2$. To each feasible solution x we assign its (vector) evaluation $y = f(x)$.

Solving this problem means finding an element of X, which is the most preferred one from the point of view of Decision Maker (DM). When choosing the "most preferred" solution, we take into account only those vector evaluations which components should be "as large as possible". Therefore we go from the problem (1) to the problem of finding the most preferred evaluation:

$$\max_{y \in Y} y$$

where

$$Y = \{f(x): x \in X\}$$

is the set of all evaluations, $Y \subset \mathbf{R}^k$.

Given Y, the set of Pareto optimal evaluations $P(Y)$ and the set of weakly Pareto optimal evaluations $W(Y)$ are defined as:

$$P(Y) = \{y \in Y: \nexists\, y' \in Y\, (y' \geq y\ \&\ y' \neq y)\}$$

$$W(Y) = \{y \in Y: \forall\, y' \in Y \setminus \{y\}\ \exists\, p \in N_k\ (y_p \geq y'_p)\}$$

A feasible solution x is called a Pareto optimal (weakly Pareto optimal) solution of the problem (1), if its evaluation is Pareto optimal (weakly Pareto optimal).

We use the notion of global trade-off coefficient introduced in Kaliszewski and Michalowski (1997).

For any $y^* \in \mathbf{R}^k$ and any $j \in N_k$, we define the set:

$$Z_j(y^*) = \{y \in \mathbf{R}^k:\ y_j < y_j^* \ \&\ \forall\ s \in N_k \setminus \{j\}\ (y_s \geq y_s^*)\}$$

Definition 1 [1]

Let $i, j \in N_k$, $i \neq j$. If $Z_j(y^*) \cap Y \neq \emptyset$, then the number

$$T_{ij}(y^*, Y) = \sup_{y \in Z_j(y^*) \cap Y} \frac{y_i - y_i^*}{y_j^* - y_j} \tag{2}$$

is called the (global) trade-off between i-th and j-th objective functions for the evaluation y^*.

If $Z_j(y^*) \cap Y = \emptyset$, then by definition, we assume $T_{ij}(y^*, Y) = -\infty$.

The trade-off $T_{ij}(y^*, Y)$ indicates by how much, at most, the evaluation y∗ can be improved in its i-th component with respect to its worsening in its j-th component when passing to any other evaluation from Y, under the condition that the remaining components do not become worse.

For any $i,j \in N_k$, $i \neq j$, let the number $\alpha ij > 0$, which represents the needed upper bound on the trade-off between i-th and j-th objective functions, be given.

Definition 2

Let $y^* \in Y$. The evaluation y∗ is called the trade-off evaluation of the problem (1), if it is Pareto optimal and the following inequalities hold:

$$T_{ij}(y^*, Y) \leq \alpha_{ij} \ \text{ for all } \ i,j \in N_k,\ i \neq j \tag{3}$$

Let $x \in X$. We call x the trade-off solution of the problem (1), if its evaluation is a trade-off evaluation.

The number α_{ij} can be interpreted as the minimal gain in the i-th objective function which in DM's opinion overweighs the unitary loss in j-th objective. If $T_{ij}(y^*, Y) > \alpha_{ij}$ for some $i,j \in N_k$, then there exists another evaluation y more preferable than y^* because when going from y^* to y, the DM gains (improving the i-th objective) relatively more than he/she looses (worsening the j-th objective).

1. TRADE-OFFS AND PREFERENCE RELATIONS

Let us introduce a strict preference relation on $\mathbf{R}^k$ such that its maximal elements among Pareto optimal evaluations (and only among them) are trade--off evaluations.

Definition 3

Let $y, y' \in \boldsymbol{R}^k$. We say that vector y ***t-dominates*** *vector y' and write*:

$$y \succ y'$$

if and only if there exist $s,p \in N_k$, $s \neq p$ such that:

$$y \in Z_p(y') \text{ and } \frac{y_s - y'_s}{y'_p - y_p} > \alpha_{sp}$$

This relation between evaluations y and y' means, that there exist $i,j \in N_k$, $i \neq j$ such that by passing from y to y' we get an improvement in the i-th component more than α_{ij} times as large as is the deterioration in the j-th component, at that all the other components do not deteriorate[1].

Note that the preference relation $\succ$ is $k(k\text{-}1)$-parametric because it depends on the numbers α_{ij}, $i,j \in N_k$, $i \neq j$. But we do not include these numbers in its designation, because changing values of the parameters is not assumed in this work.

The negation of relation ϕ is denoted by $\not\phi$.

We call our preference relation *t-dominance* where letter "t" is the abbreviation for "trade-off" because this relation is closely connected with the definition of trade-off evaluations. This connection is established by the following evident proposition.

Proposition 1

Let $y^* \in P(Y)$. The evaluation y^* is a trade-off evaluation if and only if there does not exist another evaluation $y \in Y$ such that $y \succ y^*$.

Indeed, the existence of such evaluation, and only it, ensures that (3) is violated.

Proposition 1 implies that the set of trade-off evaluations is the intersection of Pareto set with the set of maximal elements of preference relation $\succ$ on Y.

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[1] We use terms "better" and "worse" as synonyms for words "greater" and "smaller" according to the principle "better if more" for all components of the vector objective.

Consider an example showing one peculiarity of t-dominance preference relation.

Example 1

Let $k = 3$, $\alpha_{ij} = 3$ *for all* $i,j \in N_k$, $i \neq j$. *Let* $y=(2,6,4)$, $y'=(2,5,7)$, $y''=(3,2,4)$. *Then we have* $y-y'' = (-1,4,0)$, $y'-y'' = (-1,3,3)$, $\{y,y'\} \subseteq Z_1(y'')$. Assuming that all three criteria are of similar worth, y′ in comparison to y″ seems more advantageous than y in comparison to y″. But we have

$$\frac{y_2 - y_2''}{y_1'' - y_1} = 4 > \alpha_{21}, \quad \frac{y_2' - y_2''}{y_1'' - y_1'} = 3 \leq \alpha_{21}, \quad \frac{y_3' - y_3''}{y_1'' - y_1'} = 3 \leq \alpha_{31}$$

which means that $y \phi y''$ *and* $y' \not\phi y''$ *according to Definition 3.*

We see in this example that t-dominance preference relation does not have a cumulative effect when gain of passing from one solution to another is calculated. It takes into account only relation between gain and loss in a pair of objectives. As a result, y' does not dominate y'' and y does, while in sum y' has more gains in comparison with y.

The fact that evaluation y is t-dominated by evaluation y' means that in DM's opinion the evaluation y' is *better than* the evaluation y. It follows that t-dominance is a strict preference relation. There are some properties that strict preference relations are expected to satisfy. They have to be irreflexive and asymmetric. Recall that relation $\succ$ is:

– irreflexive, if $y \phi y$ for any $y \in \mathbf{R}^k$,
– asymmetric, if $y \phi y'$ implies $y' \phi y$ for any $y, y' \in \mathbf{R}^k$,
– transitive, if $y \phi y' \phi y''$ implies $y \phi y''$ for any $y, y', y'' \in \mathbf{R}^k$.

Note that transitivity is a desirable, but not necessary property of a strict preference relation.

It is evident that t-dominance preference relation is irreflexive since $y \notin Z_j(y)$ for any $y \in \mathbf{R}^k$.

The next proposition gives the necessary and sufficient condition for this relation to be asymmetric.

Proposition 2

Preference relation $\succ$ *is asymmetric if and only if the bounds on trade--offs satisfy the inequalities:*

$$1/\alpha_{ij} \leq \alpha_{ji} \text{ for any } i, j \in N_k,\ i \neq j \tag{4}$$

Proof. Sufficiency

Let inequalities (4) hold. Suppose $y \phi y'$ and prove that $y' \not\phi\ y$. By definition, there exist $s, p \in N_k$, $s \neq p$ such that $y \in Z_p(y')$ and

$$\frac{y_s - y'_s}{y'_p - y_p} > \alpha_{sp} \tag{5}$$

There are two possible cases.

Case 1

There exists $q \in N_k$ such that $y' \in Z_q(y)$. From (5) we have $y'_s < y_s$, which implies $q = s$. Observe that from $y' \in Z_s(y)$ we have $y_i \leq y'_i$ for any $i \in N_k \backslash \{s\}$ and from $y \in Z_p(y')$ we have $y_i \geq y'_i$ for any $i \in N_k \backslash \{p\}$. It follows that:

$$y_i = y'_i \text{ for any } i \in N_k \backslash \{s,p\} \tag{6}$$

Combining (5) and (4) we get:

$$\frac{y'_p - y_p}{y_s - y'_s} < \frac{1}{\alpha_{sp}} \leq \alpha_{ps}$$

Taking into account (6) we conclude that there does not exist an $i \in N_k \backslash \{s\}$ satisfying

$$\frac{y'_i - y_i}{y_s - y'_s} > \alpha_{is}$$

Hence $y' \not\phi\ y$ in this case.

Case 2

There does not exist $q \in N_k$ such that $y' \in Z_q(y)$. Then $y' \not\phi\ y$ by definition.

The sufficiency has been proved.

Necessity

Suppose that (4) violates, i.e. for some s, $p \in N_k$, $s \neq p$, the inequality $1/\alpha_{sp} > \alpha_{ps}$ holds. Let $y \in \mathbf{R}^k$. Consider the vector $y' \in \mathbf{R}^k$ with elements defined as follows: $y'_i = y_i$, $i \in N_k \backslash \{s,p\}$, $y'_s = y_s - u$, $y'_p = y_p + v$, where u and v are positive numbers such that:

$$1/\alpha_{sp} > v/u > \alpha_{ps}$$

Then we have $y \in Z_p(y')$, $\dfrac{y_s - y'_s}{y'_p - y_p} > \alpha_{sp}$ and $y' \in Z_s(y)$, $\dfrac{y'_p - y_p}{y_s - y'_s} > \alpha_{ps}$

It follows that $y \succ y'$ and $y' \succ y$ which means that the preference relation ϕ is not asymmetric.

Now we will study the question of transitivity. It is evident that the t-dominance preference relation is not transitive due to the following technical reason. Relation y ϕ y$''$ requires that the inclusion $y \in Z_j(y'')$ hold for some $j \in N_k$. But it is evident that the relation $y \phi y' \phi y''$ in general case does not imply this inclusion, hence it does not imply $y \phi y''$.

What if we weaken the condition of transitivity to eliminate the technical problem described above? Namely, let's require that the implication $y \phi y' \phi y'' \Rightarrow$ y ϕ y$''$ hold only when $y \in Z_j(y'')$ for some $j \in N_k$. The following example shows that even in this case the implication does not hold.

Example 2

Let $i,j,s \in N_k$, $i \neq j \neq s \neq i$; v_i, v_j, v_s *be positive numbers such that*

$$\alpha_{ij} v_j < v_i < 2\alpha_{ij} v_j \tag{7}$$

$$\alpha_{sj} v_j < v_s < 2\alpha_{sj} v_j \tag{8}$$

Let $y \in \boldsymbol{R}^k$. *Consider vectors* y' *and* y'' *with components defined as follows:*

$$y_i' = y_i - v_i, \quad y_j' = y_j + v_j, \quad y_l' = y_l \text{ for all } l \in N_k \setminus \{i,j\}$$

$$y_s'' = y_s' - v_s, \quad y_j'' = y_j' + v_j, \quad y_l'' = y_l' \text{ for all } l \in N_k \setminus \{s,j\}$$

Observe that:

$$y_i'' = y_i - v_i, \quad y_s'' = y_s - v_s, \quad y_j'' = y_j + 2v_j, \quad y_l'' = y_l \text{ for all } l \in N_k \setminus \{i,s,j\}$$

Then we have $y \in Z_j(y')$, $y' \in Z_j(y'')$ *and* $y \in Z_j(y'')$.

Using (7) we obtain $\dfrac{y_i - y_i'}{y_j' - y_j} = \dfrac{v_i}{v_j} > \alpha_{ij}$ *which implies* $y \phi y'$.

Using (8) and obtain $\dfrac{y_s' - y_s''}{y_j'' - y_j'} = \dfrac{v_s}{v_j} > \alpha_{sj}$ *which implies* $y' \phi y''$.

From (7) and (8) we obtain $\dfrac{y_i - y_i''}{y_j'' - y_j} = \dfrac{v_i}{2v_j} < \alpha_{ij}$, $\dfrac{y_s - y_s''}{y_j'' - y_j} = \dfrac{v_s}{2v_j} < \alpha_{sj}$.

It follows that $y \phi y' \phi y''$, *but* $y \not\phi y''$ *in spite of* $y \in Z_j(y'')$.

The explanation of non-transitivity in this example is as follows. When passing from y'' to y' we improve the s-th objective at the expense of worsening the j-th objective; when passing from y' to y we improve the i-th objective at the expense of worsening the j-th objective. So when passing from y'' to y we improve the i-th and s-th criteria and get double worsening in the j-th objective. This double worsening is not formally compensated by the individual improvement of either the i-th or the s-th objective. While DM could be satisfied with these two improvements for the exchange of double worsening, this satisfaction is not taken into account by the preference relation because of the improvements are not cumulative.

To eliminate the described effect we weaken the condition of transitivity even more. We consider only those cases where y improves y'' in a single objective.

For any $y^* \in \mathbf{R}^k$ and any $i,j \in N_k$, $i \neq j$, we define the set:

$$Z_{ij}(y^*) = \{y \in \mathbf{R}^k : y_i \geq y^*_i,\ y_j < y^*_j,\ \forall\ s \in N_k \setminus \{j,j\}\ (y_s = y^*_s)\}$$

This is the set of vectors which improve y^* in the i-th objective and worsen y^* in the j-th objective while the values of other objectives do not change. Observe that $Z_{ij}(y^*) \subseteq Z_j(y^*)$.

Definition 4

The preference relation $\succ$ *is called weakly transitive if for any* $y, y', y'' \in \boldsymbol{R}^k$ *the following implication holds:*

$$(y \phi y' \phi y'' \text{ and } y \in Z_{ij}(y'') \text{ for some } i,j \in N_k) \Rightarrow y \phi y''$$

The following proposition shows that t-dominance preference relation satisfies this kind of weakened transitivity under some conditions imposed on trade-off bounds.

Proposition 3

The preference relation $\succ$ *is weakly transitive if and only if the bounds on trade-offs satisfy the inequalities:*

$$\alpha_{is}\alpha_{sj} \geq \alpha_{ij} \quad \text{for any } i,j,s \in N_k,\ i \neq s,\ j \neq s \tag{9}$$

The proof of Proposition 3 is given in Appendix.

Note that inequalities (4) are particular cases of (9) (when $i = j$).

2. LINEAR TRANSFORMATION AND TRADE-OFFS

We define the elements of a transformation matrix $\mathrm{B} = (\beta_{ij})_{k \times k} \in \mathbf{R}^{k \times k}$ by:

$$\beta_{ij} = \frac{1}{\alpha_{ji}},\ i,j \in N_k \tag{10}$$

The objective space is transformed by multiplying its vectors by the matrix. The transformed vector evaluations are compared componentwise:

$$\mathrm{B}y > \mathrm{B}y' \;\Leftrightarrow\; \sum_{j \in N_k} \beta_{ij} y_j > \sum_{j \in N_k} \beta_{ij} y'_j \quad \text{for all } i \in N_k$$

A connection between such comparison and t-dominance relation is established by the following proposition.

Proposition 4

Let bounds on trade-offs satisfy inequalities (9). Then for any evaluations $y, y' \in \boldsymbol{R}^k$ the following implication holds:

$$y \phi y' \Rightarrow By > By'$$

Proof

Let $y \phi y'$. By Definition 3 there exist $s, p \in N_k$, $s \neq p$, such that $y_p < y'_p$,

$$y_l \geq y'_l \text{ for any } l \in N_k \setminus \{p\} \tag{11}$$

$$y_s - y'_s > \alpha_{sp}\left(y'_p - y_p\right) \tag{12}$$

To prove the proposition it is enough to show that $\sum_{j \in N_k} \beta_{ij}\left(y_j - y'_j\right) > 0$ for any $i \in N_k$.

Let $i \in N_k$. Observe that due to $\beta_{ii} = 1$ we have:

$$\sum_{j \in N_k} \beta_{ij}\left(y_j - y'_j\right) = y_i - y'_i + \sum_{j \in N_k \setminus \{i\}} \beta_{ij}\left(y_j - y'_j\right)$$

Consider three possible cases.

Case 1

$i=p$. Applying (10), (11), and (12) we obtain:

$$y_p - y'_p + \sum_{j \in N_k \setminus \{p\}} \beta_{pj}\left(y_j - y'_j\right) \geq y_p - y'_p + \beta_{ps}\left(y_s - y'_s\right) >$$

$$y_p - y'_p + \alpha_{sp}\beta_{ps}\left(y'_p - y_p\right) = 0$$

Case 2

$i = s$. From (9), (10), (11), and (12) we have:

$$y_s - y'_s + \sum_{j \in N_k \setminus \{s\}} \beta_{sj}\left(y_j - y'_j\right) \geq y_s - y'_s + \beta_{sp}\left(y_p - y'_p\right) >$$

$$\alpha_{sp}\left(y'_p - y_p\right) + \beta_{sp}\left(y_p - y'_p\right) = \left(\alpha_{sp} - \frac{1}{\alpha_{ps}}\right)\left(y'_p - y_p\right) \geq 0$$

Case 3

$i \notin \{s,p\}$. From (11) we have:

$$y_i - y_i' + \sum_{j \in N_k \setminus \{i\}} \beta_{ij}\left(y_j - y_j'\right) \geq \beta_{is}\left(y_s - y_s'\right) + \beta_{ip}\left(y_p - y_p'\right) =$$

$$\beta_{is}\left(y_s - y_s'\right) - \beta_{ip}\left(y_p' - y_p\right)$$

Applying (9), (10), and (12) we obtain:

$$\frac{\beta_{is}\left(y_s - y_s'\right)}{\beta_{ip}\left(y_p' - y_p\right)} > \frac{\alpha_{sp}\alpha_{pi}}{\alpha_{si}} \geq 1$$

It follows that $\beta_{is}\left(y_s - y_s'\right) - \beta_{ip}\left(y_p' - y_p\right) > 0$.

In each of three cases we obtained $\sum_{j \in N_k} \beta_{ij}\left(y_j - y_j'\right) > 0$.

■

The next proposition follows from the fact that all the elements of matrix B are positive.

Proposition 5

Let the evaluations $y, y' \in \mathbf{R}^k$ satisfy the inequalities $y \geq y'$, $y \neq y'$. Then $By > By'$.

By definition, put:

$$Y_{\mathrm{B}} = \{\mathrm{B}y\colon\ y \in Y\}$$

Corollary 1 [4]

If $By^* \in W(Y_B)$, then y^* is a trade-off evaluation of problem (1).

Proof

Let $By^* \in W(Y_{\mathrm{B}})$. Then there does not exist $y \in Y$ such that $\mathrm{B}y > \mathrm{B}y^*$. It follows from Proposition 5 that $y^* \in P(Y)$. It follows from Proposition 4 that there does not exist $y \in Y$ such that $y \phi y^*$. Then according to Proposition 1 y^* is a trade-off evaluation of problem (1).

■

According to Corollary 1, all weakly Pareto optimal solutions of the problem

$$\max_{x \in X} \mathrm{B}f(x) \tag{13}$$

are trade-off solutions of problem (1). But the converse is not true: not all the trade-off solutions of problem (1) can be found among weakly Pareto optimal solutions of problem (13) because the relations $y \phi y'$ and $By > By'$ are not equivalent. To illustrate the difference between them, we introduce an alternative interpretation of t-dominance preference relation.

By definition, let $\frac{a}{0} = +\infty$ for any $a > 0$. From (10) we obtain the following evident proposition which is actually a reformulation of Definition 3.

Proposition 6

Let $y, y' \in \mathbf{R}^k$. The evaluations y and y' satisfy the relation $y \succ y'$ if and only if there exist $i,j \in N_k$, $i \neq j$, such that:

$$y \in Z_i(y') \text{ and } \frac{y_i' - y_i}{y_j - y_j'} < \beta_{sp}$$

We see that t-dominance preference relation can be defined using numbers β_{ij} as parameters instead of α_{ij}. We propose the following interpretation of these numbers: β_{ij} is the maximum loss in the i-th objective which DM agrees to pay for unitary gain in the j-th objective under the condition that all the other objectives do not worsen. If during passing from y' to $y \in Z_i(y')$ the relation between loss and gain is less than β_{ij} for some $j \in N_k$, then y is more preferable than y' because DM loses (worsening the i-th objective) relatively less than the maximum he/she agreed to pay for the obtained gain (of improving the j-th objective).

Lemma 1

Let $y' \in \mathbf{R}^k$, $i \in N_k$, $y \in Z_i(y')$ be bounds on trade-offs satisfying (9). Then $By > By'$ if and only if

$$y_i' - y_i < \sum_{j \in N_k \setminus \{i\}} \beta_{ij} \left(y_j - y_j' \right) \tag{14}$$

Proof

The necessity is evident. Let (14) be valid and let's prove that an analogous inequality holds for any $s \in N_k \setminus \{i\}$. Observe that (9) implies:

$$\beta_{si} \beta_{ij} \leq \beta_{sj} \quad \text{for any} \quad i,j,s \in N_k,\ i \neq s,\ i \neq j \tag{15}$$

and $y \in Z_i(y')$ implies:

$$y_i < y_i' \tag{16}$$

$$y_l \geq y_l' \quad \text{for any} \quad l \in N_k \setminus \{i\} \tag{17}$$

Let $s \in N_k \backslash \{i\}$. Multiplying both sides of (14) by β_{si} and taking into account (15)-(17) we obtain:

$$\beta_{si}\left(y_i' - y_i\right) < \beta_{is}\beta_{si}\left(y_s - y_s'\right) + \sum_{j \in N_k \backslash \{i,s\}} \beta_{si}\beta_{ij}\left(y_j - y_j'\right) \le$$

$$\left(y_s - y_s'\right) + \sum_{j \in N_k \backslash \{i,s\}} \beta_{sj}\left(y_j - y_j'\right)$$

Thus, we get $\left(y_s' - y_s\right) < \sum_{j \in N_k \backslash \{s\}} \beta_{sj}\left(y_j - y_j'\right)$ for any $s \in N_k \backslash \{i\}$ which in combination with (14) yields $By > By'$.

■

Now compare two approaches to representing preferences in multi-objective problem (1): t-dominance preference relation and the relation of weak Pareto domination in the transformed objective space. Let $y \in Z_i(y')$. From Proposition 6 we have:

$$y \succ y' \Leftrightarrow y_i' - y_i < \beta_{ij}\left(y_j - y_j'\right) \text{ for some } j \in N_k \backslash \{i\}$$

From Lemma 1 we have:

$$\mathrm{B}y > \mathrm{B}y' \Leftrightarrow y_i' - y_i < \sum_{j \in N_k \backslash \{i\}} \beta_{ij}\left(y_j - y_j'\right)$$

In the first approach we check if the price of decreasing the i-th objective is small enough to agree to pay it for increasing one of the other objectives. In the second approach we "include in the bill" all the gains in other objectives and feel satisfied if we have paid by decreasing the i-th objective less than sum of the prices.

Let us return to Example 1 to see how the new approach functions in comparison with t-dominance relation. The following example is an extension of Example 1.

Example 3

Let $k = 3$, $\alpha_{ij} = 3$ for all $i,j \in N_k$, $i \neq j$. Let $y = (2,6,4)$, $y' = (2,5,7)$, $y'' = (3,2,4)$. Then we have $y - y'' = (-1,4,0)$, $y' - y'' = (-1,3,3)$, $\{y, y'\} \subseteq Z_1(y'')$

and $\dfrac{y_2 - y_2''}{y_1'' - y_1} = 4 > \alpha_{21}$, $\dfrac{y_2' - y_2''}{y_1'' - y_1'} = 3 \le \alpha_{21}$, $\dfrac{y_3' - y_3''}{y_1'' - y_1'} = 3 \le \alpha_{31}$

This means that $y \succ y''$ and $y' \not\succ y''$.

Using (10) we obtain β_{ij} = 1/3 for all $i,j \in N_k$, $i \neq j$ and calculate By = (5 1/3, 8, 6 2/3), By' = (6, 8, 9 1/3), By'' = (5, 4 1/3, 5 2/3). Thus, we have $By' \geq By > By''$.

In this example the new approach due to cumulativeness of calculating gains has revealed that y' is better than y'' and even "almost better" than y.

CONCLUSION

In this paper we presented two mutually connected results. In the first Section we showed that bounds on trade-offs should satisfy certain constraints (Propositions 2 and 3). These constrains ensure that the preference relation represented by these bounds meets some rational requirements. In the second Section we proposed a new approach to the representation of preferences based on linear transformation of the objective space. The connection between these two results is the following. To prove Proposition 4 which establishes a relation between linear transformation and trade-offs we use the constraints obtained in the first Section.

Appendix

Proof of Proposition 3. ***Sufficiency***

Let (9) hold. Suppose that the vectors $y, y', y'' \in \mathbf{R}^k$ satisfy the conditions:

$$y \succ y' \succ y'' \text{ and } y \in Z_{qr}(y'') \text{ for some } q, r \in N_k$$

Then by definition there exist numbers $i,j,s,p \in N_k$ such that:

$$y_i > y_i',\ y_j < y_j',\ \ y_l \geq y_l' \text{ for any } l \in N_k \setminus \{i,j\} \tag{18}$$

$$\frac{y_i - y_i'}{y_j' - y_j} > \alpha_{ij} \tag{19}$$

$$y_s' > y_s'',\ y_p' < y_p'',\ \ y_l' \geq y_l'' \text{ for any } l \in N_k \setminus \{s,p\} \tag{20}$$

$$\frac{y_s' - y_s''}{y_p'' - y_p'} > \alpha_{sp} \tag{21}$$

Observe that $y \in Z_{qr}(y'')$ implies:

$$y_q \geq y_q'',\ y_r < y_r'',\ y_l = y_l'' \text{ for any } l \in N_k \setminus \{r,q\} \tag{22}$$

To prove sufficiency it is enough to show that $y \succ y''$, i.e. that:

$$\frac{y_q - y''_q}{y''_r - y_r} > \alpha_{qr} \tag{23}$$

Observe that for any $l \in N_k \backslash \{j,p,r,q\}$ from (18), (20) and (22) we have $y_l \geq y'_l \geq y''_l$ and $y_l = y''_l$. It follows that:

$$y_l = y'_l = y''_l \quad \text{for any} \quad l \in N_k \backslash \{j,p,r,q\} \tag{24}$$

Since $y_i > y'_i$ and $i \neq j$, from (24) we have

$$i \in \{p,r,q\}. \tag{25}$$

Consider three possible cases.

Case 1

$r \neq s$, $r \neq p$. From (20) and (22) we have:

$$y_r < y''_r \leq y'_r \tag{26}$$

Using (18) we obtain:

$$r = j \tag{27}$$

Then taking into account $r \neq s$, (18), and (20) we have:

$$y_s \geq y'_s > y''_s \tag{28}$$

From (22) we get:

$$q = s \tag{29}$$

Since $i \neq j$, from (25) and (27) we have only two possibilities: $i = p$ or $i = q$. Consider these two subcases.

Case 1.1

$i = p$. Since $p \neq r$ and $p \neq s = q$ (see (20), (29)), from (22) we have $y_p = y''_p$. Using (26) we have:

$$\frac{y''_p - y'_p}{y''_r - y_r} \geq \frac{y_p - y'_p}{y'_r - y_r}$$

Taking into account (27) and applying (19) and we get:

$$\frac{y''_p - y'_p}{y''_r - y_r} \geq \frac{y_p - y'_p}{y'_r - y_r} > \alpha_{pr} \tag{30}$$

Taking into account (28), (29), and applying (21) we get:

$$\frac{y_q - y_q''}{y_p'' - y_p'} \geq \frac{y_q' - y_q''}{y_p'' - y_p'} > \alpha_{qp}$$

Combining the last inequality with (30) and applying (9) we obtain:

$$\frac{y_q - y_q''}{y_r'' - y_r} = \frac{y_q - y_q''}{y_p'' - y_p'} \frac{y_p'' - y_p'}{y_r'' - y_r} > \alpha_{qp}\alpha_{pr} \geq \alpha_{qr}$$

Case 1.2

$i = q$. From (28) and (29) we have $y_q' > y_q''$. From (26) we have $y_r' \geq y_r''$. Then:

$$\frac{y_q - y_q''}{y_r'' - y_r} > \frac{y_q - y_q'}{y_r' - y_r}$$

Taking into account $q = i$ and (27) we apply (19) and obtain:

$$\frac{y_q - y_q''}{y_r'' - y_r} > \frac{y_q - y_q'}{y_r' - y_r} > \alpha_{qr}$$

Case 2

$r = p$. Consider two possible subcases.

Case 2.1

$q \neq s$. Then taking into account $s \neq p = r$ we apply (20) and (22) to get:

$$y_s = y_s'' < y_s' \tag{31}$$

From (18) we obtain $j = s$. Thus, we have:

$$r = p \neq s = j \tag{32}$$

Then taking into account (18) and (22) we get:

$$y_p'' > y_p \geq y_p' \tag{33}$$

Since $q \neq r = p$, from (20) we have:

$$y_q' \geq y_q'' \tag{34}$$

From (25) and (32) we have $i \in \{p,q\}$. But i cannot be equal to p. Indeed, under assumption $i = p$ taking into account $j = s$ we apply (19) to have:

$$\frac{y_p - y_p'}{y_s' - y_s} > \alpha_{ps} \tag{35}$$

but on the other hand taking into account (31), (33) we have:

$$\frac{y'_s - y_s}{y_p - y'_p} > \frac{y'_s - y''_s}{y''_p - y'_p} \tag{36}$$

and applying (9), (21) we obtain:

$$\frac{y'_s - y_s}{y_p - y'_p} > \frac{y'_s - y''_s}{y''_p - y'_p} > \alpha_{sp} \geq \frac{1}{a_{ps}} \tag{37}$$

which contradicts to (35).

Hence $i = q$. Then taking into account $j = s$ (see (32)) and applying (19), (31), and (34) we obtain:

$$\frac{y_q - y''_q}{y'_s - y''_s} = \frac{y_q - y''_q}{y'_s - y_s} \geq \frac{y_q - y'_q}{y'_s - y_s} > \alpha_{qs}$$

Taking into account $r = p$ and applying consequently (33) and (21) we obtain:

$$\frac{y'_s - y''_s}{y''_r - y_r} \geq \frac{y'_s - y''_s}{y''_r - y'_r} > \alpha_{sr}$$

Combining the last two inequalities and applying (9) we obtain:

$$\frac{y_q - y''_q}{y''_r - y_r} = \frac{y_q - y''_q}{y'_s - y''_s}\frac{y'_s - y''_s}{y''_r - y'_r} > \alpha_{qs}\alpha_{sr} \geq \alpha_{qr}$$

Case 2.2

$q = s$. Consider three subcases.

a) $j \notin \{q,r\}$. Then from (18) we have $y_q \geq y'_q$ and $y_r \geq y'_r$. Taking into account $q = s$, $r = p$ and applying (21) we obtain:

$$\frac{y_q - y''_q}{y''_r - y_r} \geq \frac{y'_q - y''_q}{y''_r - y'_r} > \alpha_{qr}$$

b) $j = r$. Then taking into account $r = p$ and (25) we have $i = q$. From (19) we obtain:

$$\frac{y_q - y'_q}{y'_r - y_r} > \alpha_{qr}$$

Since $q = s$ and $r = p$, from (21) we have:

$$\frac{y'_q - y''_q}{y''_r - y'_r} > \alpha_{qr}$$

Using the last two inequalities we obtain:

$$\frac{y_q - y_q''}{y_r'' - y_r} = \frac{\left(y_q - y_q'\right) + \left(y_q' - y_q''\right)}{y_r'' - y_r} > \frac{\alpha_{qr}\left(y_r' - y_r\right) + \alpha_{qr}\left(y_r'' - y_r'\right)}{y_r'' - y_r} = \alpha_{qr}$$

c) $j = q$. Then taking into account $r = p$ and (25) we have $i = r$. From (9) and (19) we have:

$$\frac{y_r - y_r'}{y_q' - y_q} > \alpha_{rq} \quad \Rightarrow \quad \frac{y_q' - y_q}{y_r - y_r'} < \frac{1}{\alpha_{rq}} \le \alpha_{qr} \quad \Rightarrow y_q - y_q' > \alpha_{qr}(y_r' - y_r) \tag{38}$$

Since $q = s$ and $r = p$, from (21) we have:

$$\frac{y_q' - y_q''}{y_r'' - y_r'} > \alpha_{qr}$$

Combining this with (38) we obtain:

$$\frac{y_q - y_q''}{y_r'' - y_r} = \frac{\left(y_q - y_q'\right) + \left(y_q' - y_q''\right)}{y_r'' - y_r} > \frac{\alpha_{qr}\left(y_r' - y_r\right) + \alpha_{qr}\left(y_r'' - y_r'\right)}{y_r'' - y_r} = \alpha_{qr}$$

Case 3

$r = s$. From (20) and (22) we have:

$$y_r' > y_r'' > y_r \tag{39}$$

which due to (18) implies:

$$j = r = s \tag{40}$$

Consider two possible subcases.

Case 3.1

$q = p$. Taking into account (25) and (40) we have:

$$i = q = p \tag{41}$$

From (19), (40), and (41) we have:

$$\frac{y_q - y_q'}{y_r' - y_r} > \alpha_{qr} \tag{42}$$

From (9), (21), (40), and (41) we have:

$$\frac{y_r' - y_r''}{y_q'' - y_q'} > \alpha_{rq} \quad \Rightarrow \quad \frac{y_q'' - y_q'}{y_r' - y_r''} < \frac{1}{\alpha_{rq}} \le \alpha_{qr} \quad \Rightarrow y_q' - y_q'' > \alpha_{qr}(y_r'' - y_r')$$

Applying the last inequality and (42) we obtain:

$$\frac{y_q - y_q''}{y_r'' - y_r} = \frac{\left(y_q - y_q'\right) + \left(y_q' - y_q''\right)}{y_r'' - y_r} > \frac{\alpha_{qr}\left(y_r' - y_r\right) + \alpha_{qr}\left(y_r'' - y_r'\right)}{y_r'' - y_r} = \alpha_{qr}$$

Case 3.2

$q \neq p$. Then from (22) we have

$$y_p'' \geq y_p \tag{43}$$

and from (20) we have:

$$y_q' \geq y_q'' \tag{44}$$

Taking into account (25) and (40) we have $i \in \{p,q\}$. But i cannot be equal to p. Indeed, under assumption $i = p$ taking into account $j = s$ (see (40)) we apply (19) to have (35). But on the other hand, from (39), (40), and (43) we have (36), then applying (9) and (21) we obtain (37) which contradicts (35).

Hence $i = q$. Then using (39), (44), and applying (19) we obtain:

$$\frac{y_q - y_q''}{y_r'' - y_r} > \frac{y_q - y_q'}{y_r' - y_r} > \alpha_{qr}$$

In each of the cases considered, inequality (23) holds. Sufficiency has been proved.

Necessity

Suppose that (9) does not hold, i.e. there exist $i,j,s \in N_k$, $i \neq s, j \neq s$, such that:

$$\alpha_{is}\alpha_{sj} < \alpha_{ij} \tag{45}$$

Consider two possible cases.

Case 1

$i \neq j$. Let $y \in \mathbf{R}^k$. Take three positive numbers v_i, v_j and v_s satisfying the inequalities

$$\alpha_{is}\alpha_{sj}v_j < v_i < \alpha_{ij}v_j \tag{46}$$

$$\alpha_{is}\alpha_{sj}v_j < \alpha_{is}v_s < v_i \tag{47}$$

and put:

$$y' \in \mathbf{R}^k,\quad y_i' = y_i - v_i,\quad y_s' = y_s + v_s,\quad y_l' = y_l \text{ for all } l \in N_k \backslash \{i,s\}$$

$$y'' \in \mathbf{R}^k,\quad y_j'' = y_j' + v_j,\quad y_s'' = y_s' - v_s,\quad y_l'' = y_l' \text{ for all } l \in N_k \backslash \{j,s\}$$

Observe that:

$y_i'' = y_i' = y_i - v_i, \quad y_j'' = y_j' + v_j = y_j + v_j, \quad y_l'' = y_l$ for all $l \in N_k \setminus \{i,j\}$

Then we have:

$$y \in Z_{is}(y') \subseteq Z_s(y'), \quad y' \in Z_{sj}(y'') \subseteq Z_j(y''), \quad y \in Z_{ij}(y'')$$

Using (47) we obtain $\dfrac{y_i - y_i'}{y_s' - y_s} = \dfrac{v_i}{v_s} > \alpha_{is}$, which implies $y \succ y'$, and $\dfrac{y_s' - y_s''}{y_j'' - y_j'} = \dfrac{v_s}{v_j} > \alpha_{sj}$, which implies $y' \succ y''$. Using (46) we obtain $\dfrac{y_i - y_i''}{y_j'' - y_j} = \dfrac{v_i}{v_j} < \alpha_{ij}$, which implies $y \not\succ y''$. Thus, we have $y \succ y' \succ y''$, $y \in Z_{ij}(y'')$ and $y \not\succ y''$ which means that t-dominance preference relation is not weakly-transitive in Case 1.

Case 2

$i = j$. From (45) we have:

$$\alpha_{si} < \frac{1}{\alpha_{is}}$$

Take two positive numbers v_i and v_s satisfying the inequalities:

$$\alpha_{si} v_i < v_s < \frac{v_i}{\alpha_{is}}$$

Then we have:

$$\frac{v_s}{v_i} > \alpha_{si} \tag{48}$$

and

$$v_i - \alpha_{is} v_s > 0$$

It follows that there exist positive numbers v_i' and v_s' satisfying the inequalities:

$$v_i' > v_i,\ v_s' > v_s \tag{49}$$

and

$$\alpha_{is} v_s' < v_i' < \alpha_{is} v_s' + (v_i - \alpha_{is} v_s)$$

Then we have:

$$\frac{v_i'}{v_s'} > \alpha_{is} \tag{50}$$

and $v_i' - v_i < \alpha_{is}(v_s' - v_s)$ which implies:

Let $y \in \mathbf{R}^k$, then:

$y' \in \mathbf{R}^k$, $y'_i = y_i + v_i$, $y'_s = y_s - v_s$, $y'_l = y_l$ for all $l \in N_k \backslash \{i,s\}$

$y'' \in \mathbf{R}^k$, $y''_i = y'_i - v'_i$, $y''_s = y'_s + v'_s$, $y''_l = y'_l$ for all $l \in N_k \backslash \{i,s\}$

Observe that (49) implies:

$y''_i = y_i + v_i - v'_i < y_i$, $y''_s = y_s - v_s + v'_s > y_s$, $y''_l = y_l$ for all $l \in N_k \backslash \{i,s\}$

Then we have:

$$y \in Z_{si}(y') \subseteq Z_i(y'),\quad y' \in Z_{is}(y'') \subseteq Z_j(y''),\quad y \in Z_{is}(y'')$$

Using (48) we obtain $\dfrac{y_s - y'_s}{y'_i - y_i} = \dfrac{v_s}{v_i} > \alpha_{si}$ which implies $y \phi y'$. Using (50) we obtain $\dfrac{y'_i - y''_i}{y''_s - y'_s} = \dfrac{v'_i}{v'_s} > \alpha_{is}$ which implies $y' \phi y''$. Using (51) we obtain $\dfrac{y_i - y''_i}{y''_j - y_j} = \dfrac{v'_i - v_i}{v'_s - v_s} < \alpha_{is}$ which implies $y \not\phi y''$. Thus, we have $y \phi y' \phi y''$, $y \in Z_{is}(y'')$ and $y \not\phi y''$ which means that t-dominance preference relation is not weakly-transitive in Case 2.

REFERENCES

1. Kaliszewski I.: Soft Computing for Complex Multiple Criteria Decision Making. International Series in Operations Research & Management Science. Springer--Verlag, Berlin 2006.
2. Kaliszewski I., Michalowski W.: Efficient Solutions and Bounds on Tradeoffs. “Journal of Optimization Theory and Applications” 1997, 94, pp. 381-394.
3. Kaliszewski I., Zionts S.: A Generalization of the Zionts-Wallenius Multiple Criteria Decision Making Algorithm. “Control and Cybernetics” 2004, 33, pp. 477-500.
4. Podkopaev D.: An Approach to Finding Trade-Off Solutions by a Linear Transformation of Objective Functions. “Control and Cybernetics” 2007, 36, No. 2.