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FUZZY MULTIOBJECTIVE METHODS IN MULTISTAGE DECISION PROBLEMS*

Abstract

In this paper we propose a new approach for solving dynamic multiobjective decision making problems. The decision variants are generated in a discrete multistage model by forward/backward procedure of finding the set of all maximal elements based on Bellman's principle of optimality. As the set of all maximal elements consists of a number of elements – decision variants, our problem is to find among them a compromise element based on decision maker´s preferences with respect to several decision criteria. The evaluation of the weights of the criteria is based on data given by pairwise comparison matrices using triangular fuzzy numbers. Extended arithmetic operations with fuzzy numbers for application of the generalized logarithmic least squares method are defined and six methods for ranking fuzzy numbers to compare fuzzy outcomes are proposed. A numerical example is presented to clarify the methodology.

Keywords

Multicriteria decision making, dynamic programming, multistage decision process, pairwise comparisons, fuzzy numbers, analytic hierarchy process (AHP).

INTRODUCTION

Most of decision making situations consist of sequential decision problems and have multiobjective character. One way of modeling such situations is multiobjective approach. The objectives can be described by means

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of decision criteria given by elements from a partially ordered set. Here we consider a finite process divided into periods and for the decision criteria we use a set of fuzzy values modeling uncertainty of human judgments and evaluations. The goals of the individual periods are appropriately associated with the overall goal of the system and we look for efficient elements – sequences of decisions of the individual periods maximizing the objectives in the sense of Pareto. Forward/backward procedures of finding the set of all maximal elements based on Bellman´s principle of optimality have been investigated in [9, 10, and 11], see also citations therein. In real multistage decision problems the set of all efficient elements is, however, too large to be used as a direct decision support in the decision making process. That is why specific methods are needed for finding a "compromise" solutions, i.e. for narrowing the set of efficient elements, ideally to generate a single element, satisfying additional requirements, and serving as a decision support in the decision process.

In this paper we propose a new method for finding such elements based on the analytical hierarchical process (AHP) which uses uncertain human preferences as input information. Instead of the classical eigenvector prioritization method, employed in the prioritization stage of the AHP, a fuzzy preference method, based on logarithmic least squares method, is applied. The resulting fuzzy AHP enhances the potential of the classical AHP for dealing with imprecise and uncertain human comparison judgments. It allows for multiple representations of uncertain human preferences by crisp, interval, and fuzzy judgments and makes it possible to find out a solution from incomplete sets of pair-wise comparisons.

When applying the classical AHP in decision making process one usually encounters two difficulties:

– evaluating pair-wise comparisons on a nine-point scale one does not deal with uncertainty,
– decision criteria are not independent as is usually required.

Here we deal with the first difficulty by proposing a new method which incorporates uncertainty by adopting pair-wise comparisons by triangular fuzzy numbers. The second difficulty taking into account interdependences between decision criteria is dealt with elsewhere, see [5].

The interface between hierarchies, multiple objectives, and fuzzy sets have been investigated by the author of the AHP, T.L. Saaty, as early as in 1978 in [6]. Later, van Laarhoven and V. Pedrycz extended the AHP to fuzzy pair--wise comparisons [8]. Here we propose a new and relatively simple method based on the original approaches from [1, 3] and extend this method for finding a compromise efficient variant generated by the multistage decision process [9-11]. Finally, we supply an example to demonstrate properties of the proposed method.

1. SEQUENTIAL DECISION PROBLEMS

In this paper we consider a dynamic process which consists of T periods, $t = 1,2,...,T$. At the beginning of each period the system is in one of a finite number of given states. When a decision is made, the system is transformed to another feasible state and the next period begins. The feasibility of states is measured by membership grades and the transformation of the system from one state to another is performed according to decisions based on multiple criteria − objectives. The criteria evaluations are given by fuzzy values. We shall use the following notation based on [11]:

1. X_t is the set of all *states* at the period t. Here we assume $X_t = X$ for all t and X is a finite set of given states.
2. $\widetilde{D}_t = \{\mu^t(d); d = (x,y), x \in X_t, y \in X_{t+1}\}$ is the fuzzy subset of $X \times X$, called the *fuzzy set of feasible decisions at the period t* (see Section 4 below). The feasibility of the decision $d = (x,y)$ transforming the system from state x to y is denoted by the *membership grade* $\mu^t(d)$, a real number from the unit interval [0,1]. If $\mu^t(d) = 0$, then $d = (x,y)$ is infeasible, i.e. it is impossible to transform the system from state x to y. On the other hand, if $\mu^t(d) = 1$, then transforming the system from x to y is fully feasible, i.e. possible. Given a feasibility level $\lambda \in [0,1]$, we define the set of all decisions at the period t with the feasibility at least λ (denoted by $[\widetilde{D}_t]_\lambda$), as follows.
3. $[\widetilde{D}_t]_\lambda = \{d = (x,y); \mu^t(d) \geq \lambda, x, y \in X\}$.
4. $\widetilde{c}_1^t(d), \widetilde{c}_2^t(d), ..., \widetilde{c}_n^t(d)$ are fuzzy values (e.g. triangular fuzzy numbers) of *n decision criteria* $C_1, C_2, ..., C_n$, respectively, for decision $d = (x,y)$ at the period t (see Section 4 below).
5. $V = \{(x_1,x_2); (x_2,x_3);...; (x_T,x_{T+1})\}$ is the sequence of decisions $d_t = (x_t,x_{t+1})$, $x_t \in X$, $t = 1,2,...,T$. Here we call this sequence a *multistage decision variant* (*alternative*). If for each $d_t = (x_t,x_{t+1}) \in V$ it is true that $d_t = (x_t,x_{t+1}) \in [\widetilde{D}_t]_\lambda$, where $\lambda \in [0,1]$ is a given feasibility level, we call V the λ − *feasible multistage decision variant*.

2. ELICITING DECISION VARIANTS

For the case of partially ordered outcomes, e.g. evaluation of variants by fuzzy numbers, the concept of efficiency of decision variants as well as an algorithm for finding such variants has been presented in [10]. The essence of the algorithm is a forward/backward procedure of finding the set of all efficient variants based on Bellman's principle of optimality. This algorithm usually generates a large number of efficient variants which is difficult to use in a decision process. Assume that m efficient multistage decision variants have been generated by the algorithm [10], in particular, we have:

$$V_j = \{(x_1^j, x_2^j);(x_2^j, x_3^j);...;(x_T^j, x_{T+1}^j)\} \tag{1}$$

where:

$x_t^j \in X$,
$j = 1,2,...,m$,
$t = 1,2,...,T$.

Here we propose a method for eliciting a "compromise" multistage decision variant chosen from the set of efficient variants. Moreover, in an environment with uncertainty, we take into consideration multistage decision variants which are also λ – feasible with a sufficiently high λ given in advance by decision maker. Our method is based on pair-wise comparisons of decision criteria and on the logarithmic least squares method for calculating the weights, i.e. the relative importance of the decision criteria. The result will make it possible to choose a single compromise variant corresponding to the decision maker's preferences in the decision process.

3. AHP AND PAIR-WISE COMPARISONS

In the Analytic Hierarchy Process (AHP) we consider a three-level hierarchical decision system: On the first level there is a decision goal G; on the second level, we have n independent decision criteria: C_1, C_2,...,C_n, such that $\sum_{i=1}^{n} w(C_i) = 1$, where $w(C_i) > 0$, $i = 1,2,...,n$, $w(C_i)$ is a positive real number – weight, usually interpreted as a relative importance of the criterion C_i subject to the goal G. On the third level we have m variants of the decision outcomes. We take $V_1, V_2,...,V_m$ such that, again, $\sum_{r=1}^{m} w(V_r, C_i) = 1$, where $w(V_r,C_i)$ is a nonnegative real number − an evaluation (weight) of V_r subject to the criterion C_i, $i = 1,2,...,n$. This system is characterized by the *supermatrix* (see [7]):

$$\mathbf{W} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{W_{21}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{W_{32}} & \mathbf{I} \end{bmatrix}$$

a nonnegative matrix where $\mathbf{W_{21}}$ is the $n\times1$ matrix (weighing vector of the criteria), i.e.:

$$\mathbf{W}_{21} = \begin{bmatrix} w(C_1) \\ \vdots \\ w(C_n) \end{bmatrix}$$

and $\mathbf{W_{32}}$ is the $m\times n$ matrix:

$$\mathbf{W}_{32} = \begin{bmatrix} w(C_1,V_1) & \cdots & w(C_n,V_1) \\ \vdots & \cdots & \vdots \\ w(C_1,V_m) & \cdots & w(C_n,V_m) \end{bmatrix}$$

The columns of this matrix represent evaluations of variants by the criteria. Moreover, $\mathbf{W}$ is a *column-stochastic matrix*, i.e. the sums of columns are equal to one. Then the limit matrix $\mathbf{W}^{\infty} = \lim_{k\to+\infty} \mathbf{W}^k$ (see [4]) exists and has the following form:

$$\mathbf{W}^{\infty} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{W_{32}W_{21}} & \mathbf{W_{32}} & \mathbf{I} \end{bmatrix} \quad (2)$$

Here $\mathbf{Z} = \mathbf{W_{32}W_{21}}$ is the $m\times1$ matrix, i.e. the resulting *priority vector of weights of the variants*, $\mathbf{I}$ is the unit matrix. The variants can be ranked according to these priorities.

4. FUZZY NUMBERS AND FUZZY MATRICES

When applying the AHP in decision making, we usually encounter difficulties in evaluating pair-wise comparisons using the well known Saaty´s 5- (or 9-) point scale. In practice it is sometimes more convenient for the decision maker to express his/her evaluation in "words of natural language" by saying, e.g. "possibly 3", "approximately 4", or "about 5". Similarly, he/she could use evaluations of the form, e.g.: "A is possibly weakly preferable to B". Similarly, when evaluating individual decisions by some criterion, e.g. the level of inventory at a period *t*, we are often uncertain about values of the criterion. It is advantageous to express these evaluations by fuzzy sets of real numbers, in particular, by triangular fuzzy numbers (Figure 1). For the sake of convenience we now shortly refresh some basic concepts of the fuzzy set theory which we shall use in this paper.

A *fuzzy set* $\widetilde{A}$ *of* **R** is given by a *membership function* μ which is a mapping from the set of real numbers **R** into the unit interval [0,1], i.e. $\mu : \mathbf{R} \to [0,1]$. The *membership grade*, $\mu(x)$ of the element $x \in \mathbf{R}$, denotes the *possibility of occurrence (or realization) of x*, or, in other words, how strongly the element *x* belongs to the fuzzy set $\widetilde{A}$. The higher the value, the stronger the membership to $\widetilde{A}$, and vice versa. Full membership is denoted by the membership grade 1, full nonmembership by the grade 0. The evaluation of the membership grades of a fuzzy set may cause serious problems in practice; here we cannot go deeper into details – all what we will say here is that the membership grades of fuzzy sets may be estimated e.g. by experts. In order to distinguish fuzzy and nonfuzzy sets we shall denote the fuzzy sets, fuzzy vectors, and fuzzy matrices by a tilde above the symbol. For more information about fuzzy sets and related topics, see e.g. [12].

Let $\alpha \in [0,1]$ and the set $[\widetilde{A}]_\alpha = \{x \in \mathbf{R};\ \mu(x) \geq \alpha\}$ is called an *alpha cut of* $\widetilde{A}$ (or α-cut). A *triangular fuzzy number* $\widetilde{a}$ is a fuzzy set of **R** defined by a triple of real numbers, i.e. $\widetilde{a} = (\underline{a}; a; \overline{a})$, where $\underline{a}$ is the *lower number*, a is the *middle number* and $\overline{a}$ is the *upper number*, $\underline{a} \leq a \leq \overline{a}$. The membership function $\mu(x)$ is continuous in $[\underline{a}, \overline{a}]$, increasing in $[\underline{a}, a]$, decreasing in $[a, \overline{a}]$, $\mu(a) = 1$ and $\mu(x) = 0$ for $x \notin [\underline{a}, \overline{a}]$. In what follows we shall use the most practical form of the membership function: a piecewise linear one.

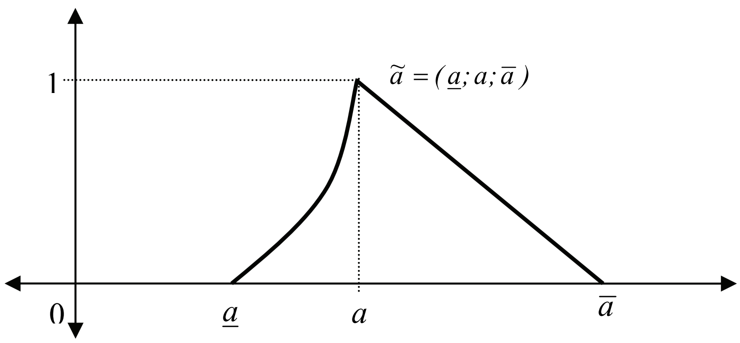

Fig. 1. A triangular fuzzy number

If $\underline{a} = a = \overline{a}$, then $\tilde{a}$ is said to be a *crisp number* (or a *nonfuzzy number*). Evidently, the set of all crisp numbers is isomorphic to the set of real numbers.

It is well known that the arithmetic operations +, –, *, and / on real numbers can be extended to fuzzy numbers by the Extension principle, see e.g. [12]. In the case of triangular fuzzy numbers $\tilde{a} = (\underline{a}; a; \overline{a})$ and $\tilde{b} = (\underline{b}; b; \overline{b})$, $\underline{a} > 0$, $\underline{b} > 0$, we obtain special formulae:

$$\tilde{a} \,\tilde{+}\, \tilde{b} = (\underline{a} + \underline{b}; a + b; \overline{a} + \overline{b})$$

$$\tilde{a} \,\tilde{-}\, \tilde{b} = (\underline{a} - \underline{b}; a - b; \overline{a} - \overline{b})$$

$$\tilde{a} \,\tilde{*}\, \tilde{b} = (\underline{a} * \underline{b}; a * b; \overline{a} * \overline{b})$$

$$\tilde{a} \,\tilde{/}\, \tilde{b} = (\underline{a} / \underline{b}; a / b; \overline{a} / \overline{b})$$

In operations of multiplication and division the form of the membership function of the result of the operation given by the Extension principle is nonlinear, even in the case when the operands are piecewise linear. In that case, piecewise linear membership functions in the above formulae give us good triangular approximations of the exact fuzzy numbers defined by the Extension principle. In what follows we shall use them for their simplicity and versatility.

If all elements in $m \times n$ matrix **A** are triangular fuzzy numbers, we call **A** the *triangular fuzzy matrix* and this matrix is composed of triples in the following way:

$$\widetilde{\mathbf{A}} = \begin{bmatrix} (\underline{a}_{11};a_{11};\overline{a}_{11}) & \cdots & (\underline{a}_{1n};a_{1n};\overline{a}_{1n}) \\ \vdots & \ddots & \vdots \\ (\underline{a}_{m1};a_{m1};\overline{a}_{m1}) & \cdots & (\underline{a}_{mn};a_{mn};\overline{a}_{mn}) \end{bmatrix}$$

Particularly, if $\widetilde{\mathbf{A}}$ is a triangular fuzzy matrix we say that it is *reciprocal*, if $\widetilde{a}_{ij} = (\underline{a}_{ij};a_{ij};\overline{a}_{ij})$ then $\widetilde{a}_{ji} = (\frac{1}{\overline{a}_{ij}};\frac{1}{a_{ij}};\frac{1}{\underline{a}_{ij}})$ for all $i,j = 1,2,...,n$.

Consequently, we have:

$$\widetilde{\mathbf{A}} = \begin{bmatrix} (1;1;1) & (\underline{a}_{12};a_{12};\overline{a}_{12}) & \cdots & (\underline{a}_{1n};a_{1n};\overline{a}_{1n}) \\ (\frac{1}{\overline{a}_{12}};\frac{1}{a_{12}};\frac{1}{\underline{a}_{12}}) & (1;1;1) & \cdots & (\underline{a}_{2n};a_{2n};\overline{a}_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\frac{1}{\overline{a}_{1n}};\frac{1}{a_{1n}};\frac{1}{\underline{a}_{1n}}) & (\frac{1}{\overline{a}_{2n}};\frac{1}{a_{2n}};\frac{1}{\underline{a}_{2n}}) & \cdots & (1;1;1) \end{bmatrix} \tag{3}$$

where $1 \le \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}$, $i,j = 1,2,...,n$. Without loss of generality we can assume that $1 \le a_{ij} \le a_{ik}$ whenever $i \le j \le k$.

5. RANKING FUZZY VARIANTS

Suppose we obtain m triangular fuzzy numbers, which we call here simply *fuzzy variants*:

$$\widetilde{z}_1 = (\underline{z}_1;z_1;\overline{z}_1),\ \widetilde{z}_2 = (\underline{z}_2;z_2;\overline{z}_2),\ \ldots,\widetilde{z}_m = (\underline{z}_m;z_m;\overline{z}_m) \tag{4}$$

Now, the problem is to rank them according to their "magnitude". The simplest way to do it is to rank fuzzy variants according to their middle value, which neglects the lower and upper parts of all fuzzy numbers. This is not a solution we ask.

A better way is the *center gravity method*. This method is based on computing the x-th coordinates x_i^g of the center of gravity of every "triangle" given by the corresponding membership functions $\tilde{z}_i$, $i = 1,2,...,m$. Evidently, the following holds:

$$x_i^g = \frac{\underline{z}_i + z_i + \overline{z}_i}{3} \tag{5}$$

By (5) the variants can be ordered from the best (with the largest value of (5)) to the worst (with the lowest value of (5)). Formula (5) incorporates in some sense the form of the triangular fuzzy number and that is why this method is more appropriate (Figure 2). Notice that in Figure 2, $a < b$, while $x_{\tilde{b}}^g < x_{\tilde{a}}^g$. More sophisticated methods for ranking fuzzy numbers exist, see e.g. [7]. For a comprehensive review of comparison methods see [3].

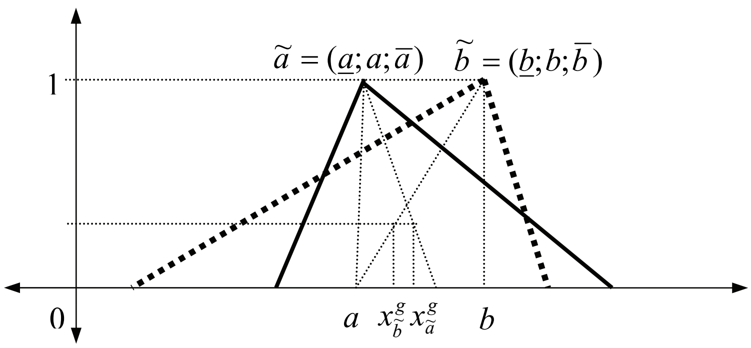

Fig. 2. Ranking fuzzy numbers

The advantage of the center gravity method is that the variants are ranked linearly (the ranking is complete) as each variant is represented by a real number. This representation may, however, become in some situations an unnecessary simplification of the situation. That is why other ranking methods have been proposed [2, 4]. Here we present five ranking methods based on the concept of dominance and aspiration level. The resulting order offered by the new rankings is, however, notcomplete, i.e. some variants remain noncomparable.

Definition 1

Let $\tilde{a}$ and $\tilde{b}$ be fuzzy numbers and $\delta,\gamma,\varepsilon \in [0,1]$ aspiration levels. We say that $\tilde{a}$ *is R-dominated by* $\tilde{b}$ (or $\tilde{b}$ *R-dominates* $\tilde{a}$) *on the aspiration level* δ if:

$$sup[\tilde{a}]_\alpha \le sup[\tilde{b}]_\alpha, \text{ for all } \alpha \in [1-\delta,1]$$

We say that $\tilde{a}$ *is L-dominated by* $\tilde{b}$ (or $\tilde{b}$ *L-dominates* $\tilde{a}$) *on the aspiration level* γ if:

$$inf[\tilde{a}]_\alpha \le inf[\tilde{b}]_\alpha, \text{ for all } \alpha \in [1\text{-}\gamma,1]$$

We say that $\tilde{a}$ *is LR-dominated by* $\tilde{b}$ (or $\tilde{b}$ *LR-dominates* $\tilde{a}$) *on the aspiration level* ε if $\tilde{a}$ is L-dominated and R-dominated by $\tilde{b}$ in the aspiration level ε. Here, L stands for "Left", and R for "Right".

Definition 2

Let $\tilde{a}$ and $\tilde{b}$ be fuzzy numbers and $\rho,\sigma \in [0,1]$ – aspiration levels. We say that $\tilde{a}$ *is P-dominated by* $\tilde{b}$ (or $\tilde{b}$ *P-dominates* $\tilde{a}$) *on the aspiration level* ρ if:

$$sup[\tilde{a}]_\alpha \le inf[\tilde{b}]_\alpha, \text{ for all } \alpha \in [1\text{-}\rho,1]$$

We say that $\tilde{a}$ *is O-dominated by* $\tilde{b}$ (or $\tilde{b}$ *O-dominates* $\tilde{a}$) *on the aspiration level* σ if:

$$inf[\tilde{a}]_\alpha \le sup[\tilde{b}]_\alpha, \text{ for all } \alpha \in [1\text{-}\sigma,1]$$

Here, P stands for "Pessimistic" and O – for "Optimistic". We illustrate these concepts in the following example.

Example

In Figure 3, $\tilde{a}$ is LR-dominated by $\tilde{b}$ (or $\tilde{b}$ LR-dominates $\tilde{a}$) on the aspiration level β, $\beta \le \beta^*$. Moreover, for $\alpha \in [\alpha^*,1]$ $\tilde{a}$ is P-dominated by $\tilde{b}$ (or $\tilde{b}$ P-dominates $\tilde{a}$) on the aspiration level α, $\alpha \le \alpha^*$. At the same time, $\tilde{a}$ is O-dominated by $\tilde{b}$ (or $\tilde{b}$ O-dominates $\tilde{a}$) on the aspiration level $\delta \in [0,1]$.

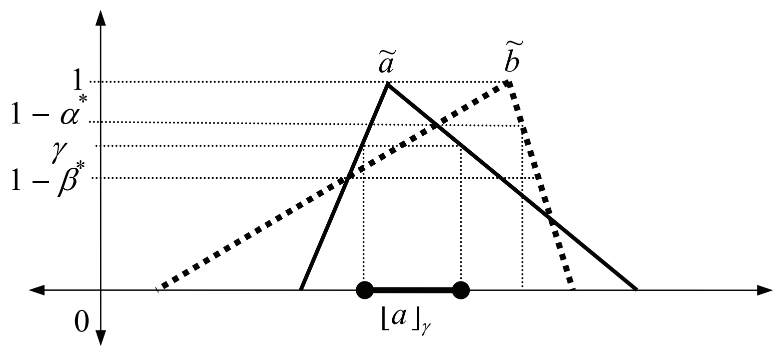

Fig. 3. Dominance of two fuzzy numbers

6. THE ALGORITHM

For each period $t = 1,2,...,T$ and decision $d = (x,y)$ we have n fuzzy values (triangular fuzzy numbers) $\tilde{c}_1^t(d), \tilde{c}_2^t(d),...,\tilde{c}_n^t(d)$ of decision criteria $C_1, C_2, ..., C_n$, respectively. The relative importance of the criteria is given by an $(n \times n)$ pair-wise comparison matrix $\tilde{\mathbf{A}}^t$, a reciprocal fuzzy matrix whose elements are triangular fuzzy numbers $\tilde{a}_{ij}^t = (\underline{a}_{ij}^t; a_{ij}^t; \overline{a}_{ij}^t)$.

The proposed method for finding the "best" multistage decision variant (or for ranking all the variants) can be formulated in an algorithm in the following three steps:

1. Calculate the triangular fuzzy weights from the fuzzy pair-wise comparison matrices and from fuzzy triangular fuzzy numbers.
2. Calculate the aggregating triangular fuzzy evaluations of the multistage decision variants.
3. Find the "best" variant (or rank the variants) defined as triangular fuzzy numbers.

Below we explain in detail each step of this algorithm.

Step 1. Calculate the triangular fuzzy weights from the fuzzy pair-wise comparison matrix.

From now on we assume that the input data are uncertain and that they are given by triangular fuzzy numbers. Our purpose is to calculate the triangular fuzzy numbers – in this context we call them *fuzzy weights* – as evaluations of the relative importance of the criteria at each period.

Let a fuzzy pair-wise comparison matrix $\tilde{\mathbf{A}}$ defined by (3) be given. We assume that there exists a fuzzy vectors of triangular fuzzy weights $\tilde{w}_1, \tilde{w}_2, ..., \tilde{w}_n$, $\tilde{w}_i = (\underline{w}_i ; w_i ; \overline{w}_i)$, $i = 1,2,...,n$ such that the pair-wise comparison matrix (3) is an estimation of the fuzzy matrix:

$$\tilde{\mathbf{W}} = \begin{bmatrix} \frac{\tilde{w}_1}{\tilde{w}_1} & \frac{\tilde{w}_1}{\tilde{w}_2} & \cdots & \frac{\tilde{w}_1}{\tilde{w}_{nr}} \\ \frac{\tilde{w}_2}{\tilde{w}_1} & \frac{\tilde{w}_2}{\tilde{w}_2} & \cdots & \frac{\tilde{w}_2}{\tilde{w}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\tilde{w}_n}{\tilde{w}_1} & \frac{\tilde{w}_n}{\tilde{w}_2} & \cdots & \frac{\tilde{w}_n}{\tilde{w}_n} \end{bmatrix}$$

We shall find the fuzzy weights $\tilde{w}_1, \tilde{w}_2, ..., \tilde{w}_n$ by minimizing the fuzzy functional:

$$\tilde{H} = \sum_{i,j} \left(\tilde{log} \frac{\tilde{w}_i}{\tilde{w}_j} - \tilde{log}\, \tilde{a}_{ij} \right)^2 \tag{6}$$

In (6), minimization of $\tilde{H}$ is understood in the sense of solving the optimization problem:

$$\sum_{i,j} \left\{ \left(\log \frac{\underline{w}_i}{w_j} - \log \underline{a}_{ij} \right)^2 + \left(\log \frac{w_i}{w_j} - \log a_{ij} \right)^2 + \left(\log \frac{\overline{w}_i}{w_j} - \log \overline{a}_{ij} \right)^2 \right\} \longrightarrow \min \tag{7}$$

subject to:

$$\overline{w}_i \geq w_i \geq \underline{w}_i \geq 0, \; i = 1,2,...,n \tag{8}$$

It can be proven by standard methods of calculus that at each period $t = 1,2,...,T$ there exists a unique explicit solution of problem (7), (8) as follows:

$$\tilde{w}_k^t = (\underline{w}_k^t ; w_k^t ; \overline{w}_k^t) \text{ , } k = 1,2,...,n$$

where:

$$\underline{w}_k^t = \frac{\left(\prod_{j=1}^{n} \underline{a}_{kj}^t\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}^t\right)^{1/n}} \text{ , } w_k^t = \frac{\left(\prod_{j=1}^{n} a_{kj}^t\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}^t\right)^{1/n}} \text{ , } \overline{w}_k^t = \frac{\left(\prod_{j=1}^{n} \overline{a}_{kj}^t\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}^t\right)^{1/n}} \tag{9}$$

In [3], the method of calculating triangular fuzzy weights by (9) from the triangular fuzzy pair-wise comparison matrix (3) is called the *logarithmic least squares method*. This method can be applied both for calculating the triangular fuzzy weights of the criteria and for eliciting relative triangular fuzzy values of the criteria for the individual variants. Moreover, it can also be used for calculating feedback impacts of criteria on other criteria.

Step 2. Calculate the aggregating triangular fuzzy evaluations of the multistage variants.

Having calculated triangular fuzzy weights of the criteria for each period $t = 1,2,...,T$, we will calculate, for a decision $d = (x,y)$, its aggregation of evaluations as the weighting sum:

$$\tilde{S}^t(d) = \tilde{w}_1^t \tilde{\cdot} \tilde{c}_1^t(d) \tilde{+} \tilde{w}_2^t \tilde{\cdot} \tilde{c}_2^t(d) \tilde{+} \cdots \tilde{+} \tilde{w}_n^t \tilde{\cdot} \tilde{c}_n^t(d) \tag{10}$$

Here $\tilde{c}_i^t(d) = \left(\underline{c}_i^t(d); c_i^t(d); \overline{c}_i^t(d)\right)$ are fuzzy evaluations of the decisions. We also assume that the *normalization property* is satisfied, namely:

$$\sum_{i=1}^{n} c_i^t(d) = 1 \tag{11}$$

Otherwise, we normalize $\tilde{c}_i^t(d)$ by dividing its three components by $S = \sum_{j=1}^{n} c_j^t(d)$.

Now, let $V = \{d_1; d_2; ...; d_T\}$ be a λ – feasible multistage decision variant, where $d_1 = (x_1,x_2)$, $d_2 = (x_2,x_3)$, ..., $d_T = (x_T,x_{T+1})$, and $\lambda \in [0,1]$ is a given feasibility level.

We define a fuzzy evaluation $\widetilde{Z}(V)$ of the multistage decision variant V as:

$$\widetilde{Z}(V) = \mu^1(d_1)\widetilde{S}^1(d_1)\widetilde{+}\mu^2(d_2)\widetilde{S}^2(d_2)\widetilde{+}\cdots\widetilde{+}\mu^T(d_T)\widetilde{S}^T(d_T) \quad (12)$$

where $\mu^t(d_t)$ is the feasibility of the decision d_t. Notice that $\widetilde{Z}(V)$ given by (10), (11), and (12) is the resulting fuzzy evaluation of the multistage variant. Here, for addition, subtraction and multiplication of triangular fuzzy numbers we use the fuzzy operations defined earlier. That is why $\widetilde{Z}(V)$ is also a triangular fuzzy number, i.e. $\widetilde{Z}(V) = (\underline{z}; z; \overline{z})$. A group of multistage variants can be ranked according to these evaluations.

Step 3. Find the "best" variant, rank the variants.

In Step 2 we have found the fuzzy evaluations of the λ – feasible multistage variants described as triangular fuzzy numbers, i.e. by (11) we calculated the triangular fuzzy vector:

$$\widetilde{Z} = \left(\widetilde{Z}(V_1),...,\widetilde{Z}(V_m)\right) = \left((\underline{z}_1; z_1; \overline{z}_1),...,(\underline{z}_m; z_m; \overline{z}_m)\right)$$

The simplest method for ordering a set of triangular fuzzy numbers is the *center of gravity method*. This method is based on computing the x-th coordinates x_i^g of the center of gravity of every triangle given by the corresponding membership functions $\widetilde{z}_i$, $i = 1,2,...,m$. Evidently, it holds:

$$x_i^g = \frac{\underline{z}_i + z_i + \overline{z}_i}{3} \quad (13)$$

By (13) the variants can be ordered from the best (with the biggest value of (12)) to the worst (with the lowest value of (13)). Naturally, we can use more sophisticated methods for ranking fuzzy numbers [3].

7. EXAMPLE

In this section we analyze an example of decision making situation based on the example from [11]. The discrete process has 2 states, $X = \{x_1, x_2\}$, 3 periods, $t = 1,2,3$, and 2 fuzzy decision criteria C_1 and C_2 defined by the fuzzy functions $\widetilde{c}_1$ and $\widetilde{c}_2$, respectively. The evaluations of individual criteria by triangular fuzzy numbers and fuzzy state transformations are given in the following tables.

Table 1

Fuzzy data

$d = (x_i, x_j)$	$t = 1$	$t = 2$	$t = 3$
	$\mu^t(d_j)$ $\widetilde{c}_1^t(d_j)$ $\widetilde{c}_2^t(d_j)$	$\mu^t(d_j)$ $\widetilde{c}_1^t(d_j)$ $\widetilde{c}_2^t(d_j)$	$\mu^t(d_j)$ $\widetilde{c}_1^t(d_j)$ $\widetilde{c}_2^t(d_j)$
$d_1 = (x_1, x_1)$	0,8 (1;4;5) (1;2;3)	0,9 (4;5;8) (3;4;6)	0,8 (3;5;8) (2;3;4)
$d_2 = (x_1, x_2)$	0,8 (4;6;8) (2;2;2)	0,9 (2;4;5) (1;2;3)	0,8 (3;4;7) (6;7;8)
$d_3 = (x_2, x_1)$	0,9 (1;3;4) (5;7;8)	0,8 (3;6;8) (2;2;2)	0,9 (2;4;6) (1;2;3)
$d_4 = (x_2, x_2)$	1,0 (0;2;4) (3;5;6)	0,7 (2;5;9) (3;7;8)	0,7 (3;6;8) (6;7;8)

Here $\mu^1(d_1) = 0{,}8$, $\widetilde{c}_1^1(d_1) = (1;4;5)$, $\widetilde{c}_2^1(d_1) = (1;2;3)$, etc.

After normalization (11) we obtain the following table.

Table 2

Normalized fuzzy data

$d = (x_i, x_j)$	$t = 1$	$t = 2$	$t = 3$
	$\widetilde{c}_1^t(d_j)$ $\widetilde{c}_2^t(d_j)$	$\widetilde{c}_1^t(d_j)$ $\widetilde{c}_2^t(d_j)$	$\widetilde{c}_1^t(d_j)$ $\widetilde{c}_2^t(d_j)$
$d_1 = (x_1, x_1)$	(0,17; 0,67; 0,83) (0,17; 0,33; 0,50)	(0,44; 0,56; 0,89) (0,33; 0,44; 0,89)	(0,38; 0,63; 0,75) (0,25; 0,38; 0,50)
$d_2 = (x_1, x_2)$	(0,50; 0,75; 1,00) (0,25; 0,25; 0,25)	(0,33; 0,67; 0,83) (0,17; 0,33; 0,50)	(0,27; 0,36; 0,64) (0,55; 0,64; 0,73)
$d_3 = (x_2, x_1)$	(0,10; 0,30; 0,40) (0,50; 0,70; 0,80)	(0,38; 0,75;1,00) (0,25; 0,25; 0,25)	(0,33; 0,67; 1,00) (0,17; 0,33; 0,50)
$d_4 = (x_2, x_2)$	(0,00; 0,29; 0,57) (0,43; 0,71; 0,86)	(0,17; 0,42; 0,75) (0,25; 0,58; 0,67)	(0,23; 0,46; 0,62) (0,46; 0,54; 0,62)

The original multistage decision system is graphically depicted in Figure 4. The goal of this decision situation is to find the "best" multistage decision variant(s) from three preselected ones according to two criteria evaluated by triangular fuzzy numbers.

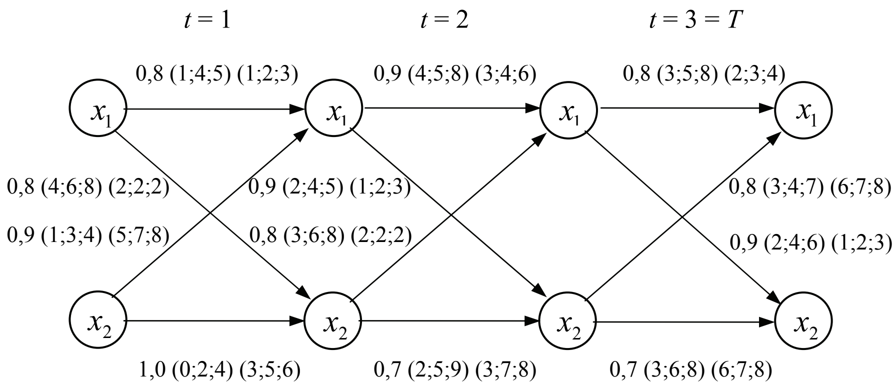

Fig. 4. The graph of the process

The weights of the criteria in the individual periods are given by the following 3 pair-wise comparison matrices:

$$A^1 = \begin{bmatrix} (1;1;1) & (2;3;4) \\ \left(\frac{1}{4};\frac{1}{3};\frac{1}{2}\right) & (1;1;1) \end{bmatrix},\ A^2 = \begin{bmatrix} (1;1;1) & (3;4;5) \\ \left(\frac{1}{5};\frac{1}{4};\frac{1}{3}\right) & (1;1;1) \end{bmatrix},\ A^3 = \begin{bmatrix} (1;1;1) & (4;5;6) \\ \left(\frac{1}{6};\frac{1}{5};\frac{1}{4}\right) & (1;1;1) \end{bmatrix}$$

By (9) we calculate the fuzzy weights of priorities of the criteria in the periods:

$$\tilde{w}_1^1 = (0{,}67;0{,}75;0{,}80),\ \tilde{w}_2^1 = (0{,}20;0{,}25;0{,}33)$$

$$\tilde{w}_1^2 = (0{,}75;0{,}80;0{,}83),\ \tilde{w}_2^2 = (0{,}17;0{,}20;0{,}25)$$

$$\tilde{w}_1^3 = (0{,}80;0{,}83;0{,}86),\ \tilde{w}_2^3 = (0{,}14;0{,}17;0{,}20)$$

In the example the set of 10 efficient realizations (i.e. multistage decision variants) has been generated using the Bellman Principle of optimality [9]. The ordered structure is based on the relation of LR-domination on the aspiration level $\varepsilon = 1$:

$V_1 = \{(x_1,x_2); (x_2,x_1);(x_1,x_1)\}$, $V_2 = \{(x_1,x_2); (x_2,x_2); (x_2,x_2)\}$,
$V_3 = \{(x_2,x_1); (x_1,x_2); (x_2,x_1)\}$,

$V_4 = \{(x_1,x_1); (x_1,x_1);(x_1,x_1)\}$, $V_5 = \{(x_1,x_2); (x_2,x_1); (x_1,x_2)\}$,
$V_6 = \{(x1,x2); (x2,x2); (x2,x1)\}$,

$V_7 = \{(x_2,x_1); (x_1,x_1);(x_1,x_1)\}$, $V_8 = \{(x_2,x_1); (x_1,x_1); (x_1,x_2)\}$,
$V_9 = \{(x_2,x_1); (x_1,x_2); (x_2,x_2)\}$,

$V_{10} = \{(x_2,x_2); (x_2,x_2);(x_2,x_2)\}$

In Figure 4 it is clear that we could generate 16 different variants. We limit ourselves, however, to the efficient realizations.

Using Table 2 we calculate evaluations of the variants according to the criteria from the formula (10), in particular:

$$\tilde{S}^t(d) = \tilde{w}_1^t \tilde{\cdot}\, \tilde{c}_1^t(d) \tilde{+} \tilde{w}_2^t \tilde{\cdot}\, \tilde{c}_2^t(d) = \left(\underline{S}^t(d); S^t(d); \overline{S}^t(d)\right)$$

The results are given in the following table:

Table 3

Normalized fuzzy data

$d = (x_i, x_j)$	$t = 1$	$t = 2$	$t = 3$
	$\left(\underline{S}^1(d); S^1(d); \overline{S}^1(d)\right)$	$\left(\underline{S}^2(d); S^2(d); \overline{S}^2(d)\right)$	$\left(\underline{S}^3(d); S^3(d); \overline{S}^3(d)\right)$
$d_1 = (x_1, x_1)$	(0,17; 0,67; 0,83)	(0,44; 0,56; 0,89)	(0,38; 0,63; 0,75)
$d_2 = (x_1, x_2)$	(0,50; 0,75; 1,00)	(0,33; 0,67; 0,83)	(0,27; 0,36; 0,64)
$d_3 = (x_2, x_1)$	(0,10; 0,30; 0,40)	(0,38; 0,75;1,00)	(0,33; 0,67; 1,00)
$d_4 = (x_2, x_2)$	(0,00; 0,29; 0,57)	(0,17; 0,42; 0,75)	(0,23; 0,46; 0,62)

Finally, applying formula (12), in particular:

$$\tilde{Z}(V) = \mu^1(d)\tilde{S}^1(d) \tilde{+} \mu^2(d)\tilde{S}^2(d) \tilde{+} \mu^3(d)\tilde{S}^3(d)$$

where d is a decision of the corresponding period, we obtain the resulting fuzzy values of the efficient variants and their centers of gravity (Table 4).

Table 4

Ranking of efficient variants based on their centers of gravity

Variants	x_i^g	Ranking
$\tilde{Z}(V_1) = (0,83; 1,49; 2,02)$	1,50	1
$\tilde{Z}(V_2) = (0,60; 1,15; 1,72)$	1,15	9
$\tilde{Z}(V_3) = (0,76; 1,39; 2,21)$	1,41	3
$\tilde{Z}(V_4) = (0,73; 1,41; 2,25)$	1,47	2

Table 4

Variants	x_i^g	ranking
$\widetilde{Z}\,(V_5) = (0{,}80;\ 1{,}35;\ 1{,}98)$	1,38	6
$\widetilde{Z}\,(V_6) = (0{,}68;\ 1{,}37;\ 2{,}12)$	1,39	5
$\widetilde{Z}\,(V_7) = (0{,}77;\ 1{,}31;\ 2{,}11)$	1,40	4
$\widetilde{Z}\,(V_8) = (0{,}74;\ 1{,}17;\ 1{,}90)$	1,27	7
$\widetilde{Z}\,(V_9) = (0{,}58;\ 1{,}23;\ 1{,}72)$	1,18	8
$\widetilde{Z}\,(V_{10}) = (0{,}38;\ 1{,}04;\ 1{,}75)$	1,06	10

The fuzzy values of the arbitrary chosen three variants (*V*1, *V*2, *V*3) are depicted in Figure 5.

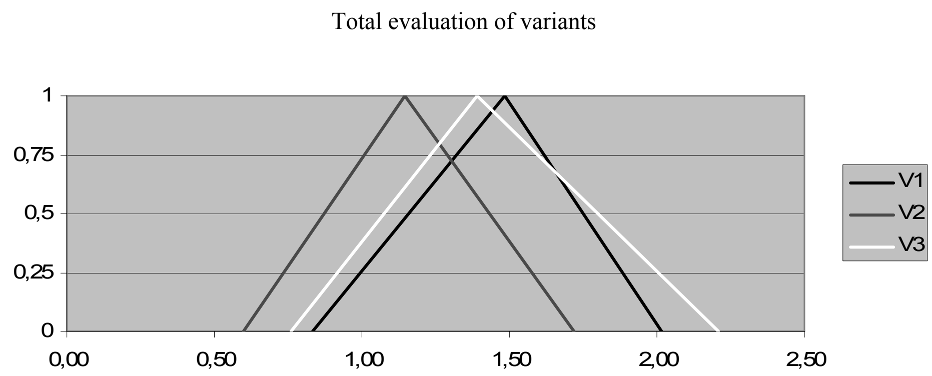

Fig. 5. Total evaluation of variants V1, V2, and V3.

In Figure 5, the rank of the variants is not clear, in particular, V_1 and V_3 are nearly "equally good", while V_2 is evidently the worst. We can confirm this observation using the center gravity ranking method. Applying the formula (13) we calculate $x_1^g = 1{,}50$, $x_3^g = 1{,}41$, $x_2^g = 1{,}15$ which confirms our first observation in Figure 5.

Consider now other rankings based on the concept of domination and on aspiration levels in Definitions 1 and 2. In Figure 5 it is evident that $\widetilde{Z}(V_3)$ is LR-dominated by $\widetilde{Z}(V_1)$ on the aspiration level $\beta = 0{,}25$. Moreover, $\widetilde{Z}(V_3)$ is P-dominated by $\widetilde{Z}(V_1)$ on the aspiration level $\alpha = 0{,}07$. At the same time, $\widetilde{Z}(V_3)$ is O-dominated by $\widetilde{Z}(V_1)$ on the aspiration level $\delta = 1{,}0$. With all rankings in mind, the multistage decision variant V_1 should be considered the best.

CONCLUSION

In this paper we proposed a new approach for solving the dynamic multiobjective decision making problems. The decision variants are generated in a discrete multistage model by forward/backward procedure of finding the set of all maximal elements based on Bellman's principle of optimality. As the set of all maximal elements consists of a number of elements – decision variants, our problem is to find among them a compromise variant based on decision maker´s preferences with respect to several decision criteria. The evaluation of the weights of the criteria is based on the data given by pair-wise comparison matrices using triangular fuzzy numbers. Extended arithmetic operations with fuzzy numbers for application of the generalized logarithmic least squares method are defined and six methods for ranking fuzzy numbers to compare fuzzy outcomes are proposed. A numerical example is presented to clarify the methodology.

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