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METRICS IN THE COMPROMISE HYPERSPHERE METHOD

Abstract

Compromise programming is one of the most often applied methods of multi-criteria optimization, both discrete and continuous. This paper deals with decision making in multicriteria linear programming problems. The approach presented here is based on finding a hypersphere (in the criteria space), which minimalizes the distance from the set of all nondominated extreme points. Next, we look for the nondominated extreme point closest to the hypersphere found previously. This point, called the best compromise nondominated solution, depends on the chosen metric. We consider the method of compromise hypersphere with different metrics and analyze their influence on the best compromise nondominated solution.

Keywords

Multicriteria linear programming, compromise programming.

INTRODUCTION

Compromise programming is one of the most often applied methods of multicriteria optimization, both discrete and continuous. Steuer and Choo [10] present an interactive weighted Tchebycheff procedure for multiple objective programming. A problem of weight choice in compromise programming is considered by Ballestro and Romero [3]. Other similar approaches are presented in the work of Carrizosa et al. [6] who consider the so-called AS norms in Ideal-Point methods. Ballestro [4] studies a problem of selection of a compromise programming metric and the risk aversion. For operative applications of compromise programming, the following works are worth mentioning: Opricovic and Tzeng [9], who discuss comparative analysis of compromise solution by the multicriteria decision making methods as well as Abdelaziz et al. [1], who discuss multiobjective programming technics with goal programming and compromise programming used to choose the portfolio which best satisfies the decision maker.

This paper deals with decision making in multicriteria linear programming problems. The concept of the method follows from the work of Gass and Roy [8]. The approach presented here is based on finding a hypersphere (in the criteria space) which minimizes the distance from the set of all nondominated extreme points. Next, we look for the nondominated extreme point which is closest to a hypersphere. This point, called the best compromise nondominated solution, depends on the chosen metric. We consider the method of compromise hypersphere with different metrics and analyze their influence on the best compromise nondominated solution.

The paper consists of three sections. Section 1 presents the general description of the compromise hypersphere for multiobjective linear programming. Section 2 describes methods of chosing distance functions. Section 3 contains an example. At the end, there are concluding remarks and further research.

1. DESCRIPTION OF THE COMPROMISE HYPERSPHERE METHOD

Let us consider the following multicriteria linear programme:

$$\text{VMax } \{\mathbf{Cx}: \mathbf{x} \in X\} \tag{1}$$

where:

$X = \{\mathbf{x} \in \mathcal{R}^N: \mathbf{Ax} \le \mathbf{b}, \mathbf{x} \ge 0\}$ or $X = \{\mathbf{x} \in \mathcal{R}^N: \mathbf{Ax} = \mathbf{b}, \mathbf{x} \ge \mathbf{0}\}$ – feasible region in decision space,

$\mathbf{x} \in \mathcal{R}^N$ – vector of decision variables,

$\mathbf{C} \in \mathcal{R}^{k \times N}$ – matrix of objective function coefficients,

$\mathbf{A} \in \mathcal{R}^{m \times N}$ – full row rank matrix of constraint coefficients,

$\mathbf{b} \in \mathcal{R}^m$ – right hand side vector.

We call $y^* \in \mathcal{R}^k$ a nondominated solution of (1) if:

$$\underset{\mathbf{x}^* \in X}{\exists} \mathbf{y}^* = \mathbf{Cx}^*$$

and

$$\sim \underset{\mathbf{x}' \in X}{\exists} \mathbf{Cx}^* \le \mathbf{Cx}' \wedge \mathbf{Cx}^* \ne \mathbf{Cx}'$$

The corresponding point $\mathbf{x}^* \in \mathcal{R}^n$ is called an efficient solution.

The aim of the method presented here is to rank the nondominated extreme points of the problem (1). The details of the method are presented below:

Step 1

Determination of the set of all nondominated extreme points (efficient solutions) of the problem (1). We will denote the nondominated extreme points as:

$$\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^n$$

Step 2

Solution of the programme:

$$\min_{\mathbf{y}^0, r_0} D\Big(r_0, \ \Big(d\big(\mathbf{y}^1, \mathbf{y}^0\big), \ldots, d\big(\mathbf{y}^n, \mathbf{y}^0\big)\Big)\Big) \tag{2}$$

where:

$\mathbf{y}^0 \in R^k$, $r_0 \in R$ denote the decision variables of (2),
d: $R^k \times R^k \to R$ denotes the distance between two vectors,
D: $R \times R^n \to R$ denotes the distance between one number and the set of n numbers (we will identify the set of n numbers as n dimensional vector).

We will denote the optimal solution of (2) as ${}^*y^0$, ${}^*r_0$ and the minimal value of the cost function as ${}^*\min(2)$.

Interpretation of problem (2). The problem is to find a hypersphere with the centre $\mathbf{y}^0 \in R^k$ and the radius $r_0 \in R$ such that its distance from the set $\{\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^n\}$ is minimal.

Step 3

Solution of the programme:

$$\min_{i=1,\ldots,k} \left| {}^*r_0 - d({}^*y^0, \mathbf{y}^i) \right| \tag{3}$$

We will denote the optimal solution of (3) as ${}^*i$, the optimal extreme point as ${}^*\mathbf{y}^i$, and the minimal value of the cost function as ${}^*\min(3)$.

Interpretation of problem (3). The problem is to find the extreme point which is closest to hypersphere found in Step 2.

Remark 1

We can find the set of all nondominated extreme points (efficient solutions) with the help of ADBASE [11].

Remark 2

The method of solving the programme (2) depends on the choice of D and d. In general, this programme is a complicated optimization problem (examples will be presented later).

Remark 3

Problem (3) is trivial, it suffices to compare n numbers which were used in Step 2.

2. MEASURE OF DISTANCE

As we have seen in Section 1, the compromise hypersphere method uses functions d and D as measures of distance. The following sections describe the methods of building these functions.

2.1 Choosing function *d*

We will use the well known family of metrics $l^p : R^k \times R^k \to R$ to measure the distance between two vectors:

$$l^p(\mathbf{y},\mathbf{z}) = \begin{cases} \left(\sum_{i=1}^{k} |y_i - z_i|^p \right)^{\frac{1}{p}}, & p \in [1,\infty) \\ \max_{i=1,\dots,k} |y_i - z_i|, & p = \infty \end{cases}$$

where:

$\mathbf{y} = (y_1, \dots, y_k) \in R^k$, $\mathbf{z} = (z_1, \dots, z_k) \in R^k$.

Therefore, in the problem (2) we will use the function l^p as function d with parameter $p \in [1,\infty]$.

2.2. Choosing function D

We will use a modification of metrics l^q to measure the distance between a number and the set of n numbers (identified as an n dimensional vector). The modification function $l^q : R \times R^k \to R$ is defined as follows:

$$L^q(r,\mathbf{w}) = l^q((r,r,...,r),\mathbf{w})$$

where:

$r \in R$, $(r,r, \ldots, r) \in R^k$, $\mathbf{w} \in R^k$ and l^q is defined in Subsection 2.1.

Therefore, in the problem (2) we will use the function l^q as function D with parameter $q \in [1,\infty]$.

2.3. Problem $H(p, q)$

By using the function l^p as d and l^q as D in the problem (2) we obtain the following problem $H(p,q)$:

$$\min_{y^0, r_0} l^q(r_0,(l^p(\mathbf{y}^1,\mathbf{y}^0),...,l^p(\mathbf{y}^n,\mathbf{y}^0))) \qquad H(p,q)$$

The problem $H(2, \infty)$ is considered in the papers by Anthony et al. [2] and by Butler et al. [5]. However, Gass and Roy [8] consider an approximation of the problem $H(2, \infty)$ and it's quality can be found in Gass et al. [7].

3. EXAMPLE

Consider the following two-criteria problem (Figure 1):

$$\text{VMax } [x_1, x_2]$$

$$\begin{aligned} 3x_1 + 2x_2 &\le 51 \\ x_1 + 2x_2 &\le 21 \\ x_1 + 3x_2 &\le 25 \\ x_1 + 4x_2 &\le 30 \\ x_1 + 6x_2 &\le 42 \\ x_1, x_2 &\ge 0 \end{aligned}$$

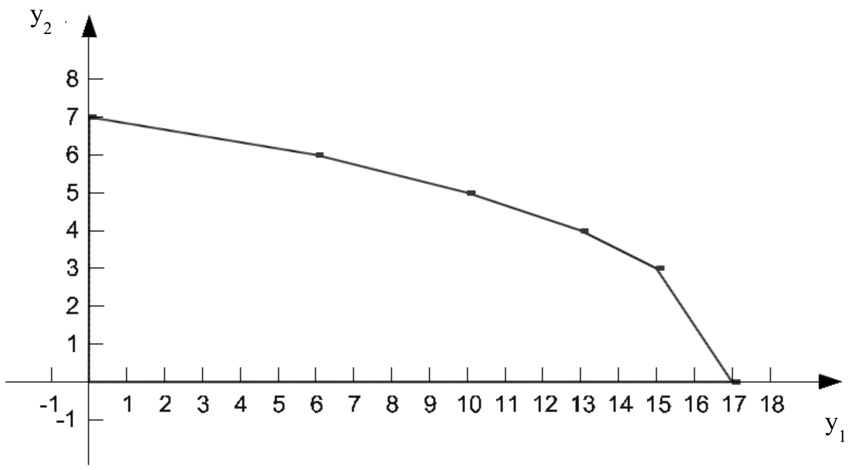

Fig. 1. Illustration of the example

We apply the hypersphere compromise method described in Section 1.

Step 1

We have the following six nondominated efficient extreme points:

$$\mathbf{y}^1 = (0, 7),\quad \mathbf{y}^2 = (6, 6),\quad \mathbf{y}^3 = (10, 5),\quad \mathbf{y}^4 = (13, 4),\quad \mathbf{y}^5 = (15, 3),\quad \mathbf{y}^6 = (17, 0)$$

Steps 2 and 3

In Step 2 we consider the problems $H(p, q)$ for $p = 1, 2, \infty$ and $q = 1, 2, \infty$. To solve these problems we use metaheuristic methods (genetic algorithms). The aim of the paper is not to analyze the numerical aspects of the optimization problems presented here, but to analyze the solutions obtained. Thus, we will not discuss numerical methods in details. The numerical analysis of the methods used will be the topic of future research.

In Table 1 there are optimal solutions of the problems (2) and (3); the optimal value of cost function of (2) is also presented. The symbols in the cells of Table 1 denote:

$^*y^0, {}^*r_0$ – the centre and the radius of minimal hypersphere,

$^*y^i$ – the optimal extreme point in problem 3,

$^*\min(2)$ – the minimal value of the cost function in problem 2.

Table 1

Optimal solutions of the problems (2) and (3) and the optimal value of cost function of the problem (2)

$d = l^p$				
$p = 1$	$p = 2$	$p = \infty$		
(5, 0), 12 $\mathbf{y}^1, \mathbf{y}^4, \mathbf{y}^6$ 8	(0.8683, −15.0536), 22.0703 $\mathbf{y}^6$ 1.8139	(5.2700, −4.6828), 10.3157 $\mathbf{y}^2$ 6	q=1	$D=l^q$
(4.4, −0.3), 11.7 $\mathbf{y}^1$ 4.7749	(1.6262, −15.0776), 21.9415 $\mathbf{y}^3$ 0.8844	(7.2222, -3.2778), 8.7778 $\mathbf{y}^2, \mathbf{y}^3$ 2.6457	q=2	
(4.3333, −0.6667), 11.3333 $\mathbf{y}^3$ 3	(1.5906, −15.1667), 22.1007 $\mathbf{y}^1$ 0.4792	(7, −1), 8 $\mathbf{y}^5$ 2	q=∞	

Case analysis of $p = 2$ and $q = \infty$

We discuss the case of $p = 2$ and $q = \infty$ in detail. An example of minimal hypersphere with the centre ${}^*y^0 = (1.5906, -15.1667)$ and the radius ${}^*r_0 = 22.1007$ is presented in Figure 2.

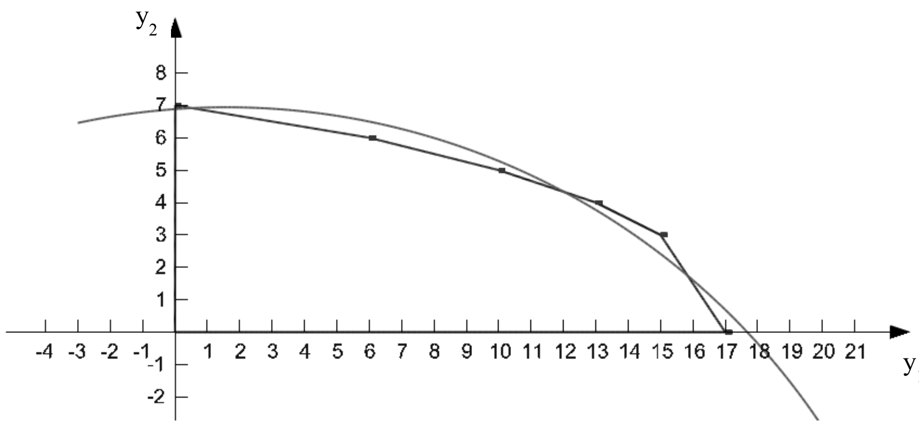

Fig. 2. Minimal hypersphere in the case of the problem $H(2, \infty)$

Table 2 presents distances between nondominated efficient extreme points and minimal hypersphere and a ranking of these points. In Table 2 an optimal solution of the problem (3) is also presented. The solution $\mathbf{y}^1$ is connected with the minimal distance which is equal to 0.1234.

Table 2

Distances between nondominated efficient extreme points and minimal hypersphere for $p = 2$ and $q = \infty$

	$\left| {}^*r_0 - d({}^*y^0, \mathbf{y}^i) \right|$	Ranking
$\mathbf{y}^1$	0.1234	1
$\mathbf{y}^2$	0.4792	4
$\mathbf{y}^3$	0.2506	3
$\mathbf{y}^4$	0.2051	2
$\mathbf{y}^5$	0.4792	4
$\mathbf{y}^6$	0.4792	4

Moreover, in the case analyzed here, the formula for the function D has the following form:

$$l^q\left({}^*r_0, \mathbf{w}\right) = l^\infty\left(\left({}^*r_0, {}^*r_0, \ldots, {}^*r_0\right), \mathbf{w}\right) = \max_{i=1,\ldots,6} \left| {}^*r_0 - w_i \right|$$

where:

$$\mathbf{w} = (d({}^*y^0, \mathbf{y}^1) d({}^*y^0, \mathbf{y}^2) d({}^*y^0, \mathbf{y}^3) d({}^*y^0, \mathbf{y}^4) d({}^*y^0, \mathbf{y}^5) d({}^*y^0, \mathbf{y}^6)) \in R^6$$

and

$$({}^*r_0, {}^*r_0, \ldots, {}^*r_0) = (22.1007, 22.1007, 22.1007, 22.1007, 22.1007, 22.1007) \in R^6$$

Using values presented in Table 2 we obtain:

$$l^q\left({}^*r_0, \mathbf{w}\right) = \max\{0.1234, 0.4792, 0.2506, 0.2051, 0.4792, 0.4792\} = 0.4792$$

Therefore, only vectors lying farthest from minimal hypersphere influence the value of cost function D. There are three $\mathbf{y}^2$, $\mathbf{y}^5$, $\mathbf{y}^6$ vectors with the maximum distance 0.4792 (see Table 2).

CONCLUDING REMARKS AND FURTHER RESEARCH

We have presented a method of decision supporting in problems of multicriteria linear programming. The method is based on finding a hypersphere which is closest to the set of the efficient extreme points. The method presented by Gass and Roy [8] has been developed using different methods of measuring the distance. We have presented an example with nine possible variants of hypersphere compromise programming. In the example we considered six nondominated extreme points. As we have shown (Table 1), each of the nondominated extreme points (depending on the assumed variant of measuring) turned out to be a optimal solution of the programmes $H(p, q)$. An extension of the presented method could be constructed by means of the augmented Tchebycheff metric [10].

The author suggests the following problems as the subject of further research:
- constructing a method of choosing functions d and D using interaction with decision maker,
- finding mathematical properties of the presented problem $H(p, q)$,
- numerical analysis of algorithms searching for optimal solutions of the problem $H(p, q)$.

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