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DECISION MAKING PROBLEM WITH TWO INCOMPARABLE CRITERIA – GAME THEORY SOLUTION

Abstract

In this paper the solution concept of decision making problem with two incomparable criteria is presented. The incomparability consists in having no premises to aggregate the assessments of the decision variants or to aggregate the relations of partial preferences, which are associated with criteria considered – it is possible to show the best variant with respect of the particular criterion, but there is no information which could enable the definition of relations between the criteria considered.

Construction of the model of multicriteria decision making problem is the starting point of consideration. The problem is presented in literature as a two-person zero--sum game; it is noticed, however, that the multicriteria decision making problem is not antagonistic. Our premise is the construction of model on the ground of nonantagonistic game theory. The proposal is that the criteria play the game with themselves. It is not necessary, but the player as a person can be identified with decision maker who considers the problem from the point of view of one criterion. The set of strategy is defined by the set of decision variants – for each player. The payoffs are defined by assessments of the decision variants if the players use the same strategy (choose the same decision variants). Otherwise, they reach much worse result. Also they know that they must choose the same variant. The characteristic feature of the game is the problem of coordination between equilibria – each equilibrium is equivalent to one decision variant. Thus, the problem is: Which equilibrium should be chosen? Harsanyi and Selten's "A general theory of equilibrium selection in games" and their concept of risk domination is applied to answer this question.

The consequences of the proposal's acceptance are that the solution of the primary bicriterial decision making problem is the decision variant which belongs to a nondominated set of decision variants and for which the sum of the assessments of the decision variants is maximal (the assessments are subtracted for the minimized criteria and added for the maximized ones).

Keywords

Coordination game, incomparable criteria.

INTRODUCTION

The solution of a multiple criteria decision problem usually consists in scalarizing the question [2], i.e. in reducing it to the decision problem with one synthetic criterion or in building a model aggregating the assessments of the decision variants or aggregating the partial preferences [7].

However, it is difficult to obtain information which would allow for the aggregation of the assessments of decision variants or preferences. Therefore, the decision maker must face the problem of making decisions in precarious conditions.

The question that will be considered here is connected with building a multiple criteria decision model on theoretical basis; it is also associated with Harsanyi and Selten's paper [3], in particular, with the concept of equilibrium selection in games presented in it. A multicriteria decision making problem can be presented in the context of game theory as a two-person zero sum game, i.e. as an antagonistic game [4, 5]. However, it has been noticed that such extreme conflict is hardly ever characteristic for multiple criteria problems [1, p. 534], and that is why it is suggested that a model of a multiple criteria problem be constructed, in the form of a multi-person non-zero sum game [8, 9].

The main purpose of this paper is to describe the solution of a multiple criteria decision problem in a precarious situation due to the lack of clear premises that would allow a comparison of criteria.

CONSTRUCTION OF THE MODEL

If X is a set of acceptable decision variants, assessed according to a finite set of criteria $F = \{f_1, f_2\}$ then the assessment of the decision variant $x \in X$ according to its *j*-th criterion is the value $f_j(x)$. Let the decision maker consider each problem separately, from the point of view of every accepted criterion, while there are no premises allowing any kind of aggregation of assessments or preferences. Moreover, the decision maker wants the assessment of each decision variant to be the highest possible (the criteria are maximized).

The problem can be represented in the form of two-person non-cooperating non-zero sum game, with all the information, played between the criteria considered. The set of acceptable strategies is defined by the set *X*, and the payoff of the *j*-th player is defined by the value $f_j(x)$, but only in the situation when the other player also chooses the option *x*. If this is not the case, the player obtains a considerably lower score. The result of the game can be represented as a function $F(x_1, x_2)$ in the following way (assuming that both criteria are maximized):

$$F(x_1, x_2) = \begin{cases} (f_1(x_1), f_2(x_2)) & \text{gdy} \quad x_1 = x_2 \\ (M, M) & \text{gdy} \quad x_1 \neq x_2 \end{cases} \tag{1}$$

where:

x_1 is the decision variant, $x_1 \in X$, chosen by the player representing the first criterion,

x_2 is the decision variant, $x_2 \in X$, chosen by the player representing the second criterion,

M stands for any value fulfilling the following conditions:

$$M << f_j(x_i), \text{ for } j = 1,2, i = 1,2 \tag{2}$$

In the game defined here we come across the problem of coordination of the players' actions performed in order to reach the equilibrium[1]. The players-criteria choose their strategy (decision variant) independently, but the result of this choice depends solely on the other player's choice. It follows from the defined payoff function (1) in the case of both players choosing the same strategy that they will achieve much higher payoffs than if they choose different strategies. Therefore, the players are keen on choosing the same decision variant. To sum up, in the game defined here there are as many equilibria in the strategies set as there are elements in the set *X*. It should also be emphasized that these equilibria can be dominated by another equilibrium[2] only, while each equilibrium in the game represents one decision variant. In other words, the small number M is arbitrary in the sense that situations in the game considered here, where players-criteria choose the same strategy (variant), are equilibria in the game.

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[1] In this paper the equilibrium is understood as Nash equilibrium.

[2] This game is not of the Prisoner's Dillema type.

In this solution of the problem one can suggest choosing the equilibrium which is dominating due to the players' payoffs. It means that one should choose the decision variant corresponding to this equilibrium. Next, in a nontrivial situation, when the equilibrium dominating due to the payoffs does not exists, the equilibrium is chosen (from the non-dominated ones) according to the risk (risk dominating).

The analysis of the game begins with the comparison of pairs of equilibria. Let us consider the payoff matrix A representing the situation when the players have to deal with two equilibria:

$$A = \begin{bmatrix} (f_1(x_1), f_2(x_1)) & (M, M) \\ (M, M) & (f_1(x_2), f_2(x_2)) \end{bmatrix} \tag{3}$$

where:

$f_j(x_1)$ is the payoff of the j-th player when all the players choose the strategy (the decision variant) x_1,
$f_j(x_2)$ is the payoff of the j-th player when all the players choose the strategy (the decision variant) x_2.

The players maximize their expected payoffs. The expected payoff of the first player, due to using the first strategy $E_1(x_1)$, equals:

$$E_1(x_1) = f_1(x_1) \cdot q_1 + M \cdot (1 - q_1) \tag{4}$$

where:

q_1 is the subjective probability (according to the first player's assessment) that the other player will use the strategy x_1
the expected payoff of the first player, due to the use of the second strategy $E_1(x_2)$ – equals:

$$E_1(x_2) = f_1(x_2) \cdot (1 - q_1) + M \cdot q \tag{5}$$

For the second player the expected winnings equal, respectively:

$$E_2(x_1) = f_2(x_1) \cdot (1 - p_2) + M \cdot p_2 \tag{6}$$

$$E_2(x_2) = f_2(x_2) \cdot p_2 + M \cdot (1 - p_2) \tag{7}$$

where:

p_2 is the subjective probability that the other player will use the second strategy.

If the following condition is fulfilled:

$$E_1(x_1) > E_1(x_2) \tag{8}$$

the first player will choose the first strategy. If, in turn, the condition:

$$E_2(x_1) < E_2(x_2) \tag{9}$$

is fulfilled the other player will choose the second strategy.

In the situation when players choose different strategies the equilibrium will not be achieved and both will receive lower payoffs than they would if one of the players "gave up". If the players have the same information about the situation they should use strategies implying the equilibrium, which choice can be based on stronger premises.

The substitution of the expressions (4) and (5) in the condition (8) yields, after consecutive transformations:

$$f_1(x_1) \cdot q_1 + M \cdot (1 - q_1) > f_1(x_2) \cdot (1 - q_1) + M \cdot q_1 \tag{10}$$

$$q_1 \cdot [f_1(x_1) - M + f_1(x_2) - M] > f_1(x_2) - M \tag{11}$$

$$q_1 > \frac{f_1(x_2) - M}{[f_1(x_1) + f_1(x_2) - 2 \cdot M]} = q_0 \tag{12}$$

while it is assumed that the expression in the denominator is positive.

Analogously, the substitution of the expressions (6) and (7) in the condition (9) yields consecutively:

$$f_2(x_1) \cdot (1 - p_2) + M \cdot p_2 < f_2(x_2) \cdot p_2 + M \cdot (1 - p_2) \tag{13}$$

$$p_2 \cdot [f_2(x_2) - M + f_2(x_1) - M] > f_2(x_1) - M \tag{14}$$

$$p_2 > \frac{f_2(x_1) - M}{[f_2(x_1) + f_2(x_2) - 2 \cdot M]} = p_0 \tag{15}$$

while it is also assumed that the value of the expression in the denominator is positive.

The value q_0 is the limit value of the subjective probability that the other player will choose the first strategy, while the value p_0 is the limit value of the subjective probability that the first player will choose the second strategy. One can then acknowledge that if the condition [3, p. 216]:

$$q_0 < p_0 \tag{16}$$

is fulfilled there are stronger premises for all the players to choose the first equilibrium (implying the choice of the variant x_1). It is more likely that the first player will choose the first strategy than that the other will choose the second one. This reasoning presents the idea of dominance associated with the risk (risk-dominance) of the first equilibrium dominating the other – the first equilibrium risk-dominates the second.

The substitution of expressions (12) and (15) in the condition (16) yields, after consecutive transformations:

$$\frac{f_1(x_2) - M}{f_1(x_1) + f_1(x_2) - 2 \cdot M} < \frac{f_2(x_1) - M}{f_2(x_1) + f_2(x_2) - 2 \cdot M} \tag{17}$$

$$\frac{f_1(x_2) \cdot f_2(x_2) - M \cdot f_1(x_2) - M \cdot f_2(x_2)}{f_1(x_1) \cdot f_2(x_1) - M \cdot f_1(x_1) - M \cdot f_2(x_1)} < 1 \tag{18}$$

if the expression in the denominator is positive and, if this expression is negative:

$$\frac{f_1(x_2) \cdot f_2(x_2) - M \cdot f_1(x_2) - M \cdot f_2(x_2)}{f_1(x_1) \cdot f_2(x_1) - M \cdot f_1(x_1) - M \cdot f_2(x_1)} > 1 \tag{19}$$

In the model we assumed that M is any arbitrary number satisfying the condition (2). Therefore, when the assessments of the decision variants are any finite real numbers, one can assume that $M \to -\infty$ and the condition (2) is always fulfilled. In this case the assumption that the denominator in the expressions (12) and (15) is positive is always fulfilled. Thus:

$$\lim_{M \to -\infty} \frac{f_1(x_2) \cdot f_2(x_2) - M \cdot f_1(x_2) - M \cdot f_2(x_2)}{f_1(x_1) \cdot f_2(x_1) - M \cdot f_1(x_1) - M \cdot f_2(x_1)} \leq 1 \tag{20}$$

or

$$\lim_{M \to -\infty} \frac{f_1(x_2) \cdot f_2(x_2) - M \cdot f_1(x_2) - M \cdot f_2(x_2)}{f_1(x_1) \cdot f_2(x_1) - M \cdot f_1(x_1) - M \cdot f_2(x_1)} \geq 1 \tag{21}$$

if the expression in the denominator is negative. Therefore:

$$\frac{f_1(x_2) + f_2(x_2)}{f_1(x_1) + f_2(x_1)} \leq 1 \tag{22}$$

or

$$\frac{f_1(x_2) + f_2(x_2)}{f_1(x_1) + f_2(x_1)} \geq 1 \tag{23}$$

Taking the above reasoning into consideration one can state that in the situation when the condition:

$$f_1(x_2) + f_2(x_2) \le f_1(x_1) + f_2(x_1) \tag{24}$$[3]

is fulfilled, both players will choose the first strategy (the first strategy risk-dominates the other), therefore according to the idea presented here the first variant is better.

CONCLUSIONS

To sum up, one should emphasize that the idea suggested here can be used in the situation when there are neither premises which could allow for the comparison of the assessments of the variants with respect to the criteria nor partial preferences associated with the criteria considered. Moreover, the solution of the problem is obtained by addition (for the maximized criteria) and subtraction (for the minimized) of the assessments of the decision variants. However, due to the apparent incompatibility of the units and the scale in which these assessments are expressed, one should bear in mind the assumptions of usability as well as, above all, the rule (24) which aim is to indicate the risk-dominating decision variant. Pursuing our reasoning one can state that in the case when the third and the next variants are considered, the best variant is the one for which the sum of the assessments (in the case of maximized criteria) is the highest. This reasoning implies the statement that in the model suggested the relation of risk-dominance is transitive.

One can also notice that no conflict between risk-dominance and payoff-dominance exists in this model. Therefore, it is true that if one equilibrium dominates the other due to the payoffs, it also risk-dominates the other[4].

In the case when two equilibria in the set of pure strategies exist for which both sides of the expression (24) have equal value one should regard their respective variants as equivalent[5].

This reasoning and the assumptions of the usability of the method result in the fact that the question of normalization of the decision variants assessments in the model suggested becomes a serious problem, since

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[3] One can reach the same conclusions starting from the reasoning presented in [6, p. 42].

[4] In the general case risk-dominance is not transitive; also, conflict between risk-dominance and payoff-dominance can occur.

[5] In the general case one should also consider the equilibria in the set of mixed strategies, but in the model suggested here the interpretation of mixed equilibrium is ambiguous as it would imply at least two decision variants.

the choice of the normalization method can have a considerable influence on the choice of the best variant. Therefore, taking into account also the fact that the normalization method can be justified, one can assume that the method suggested here can be used in the situation when there are no reasons for normalization or when the normalization of the assessments is acceptable, but one can investigate the influence of the normalization on the solution. One should also emphasize the sensitivity of the condition (24) introduced to the change of the scale caused by the assessments normalization.

To conclude, one can state that when the decision is taken with regard to two incomparable maximized criteria she/he can choose the decision variant from those which are not dominated due to the assessments and for which the sum of the assessments of the decision variants is the highest. This results from the construction of the model of the problem as a coordination game, where an arbitrary small number (payoff) M implies that the players-criteria should choose the same strategy (variant). In this game, the problem of selection of one equilibrium occurs. The selected equilibrium should indicate the best decision variant in the situation under consideration. The procedure of selection is based on the notion of risk dominance presented by Harsanyi and Selten [3].

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