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THE USE OF THE REFERENCE MCDM METHODS TO DEFINE THE SECOND STOCHASTIC DOMINANCE EFFECTIVE PORTFOLIOS*

Abstract

The paper is devoted to the application of stochastic dominance rules to portfolio selection problem with diversification possibilities. The approach based on multi-criteria decision making methodology, proposed by W. Ogryczak, is considered. The paper describes the application of the reference methods to define the set of the SSD effective portfolios and to choose the portfolio according to the general model of preference under risk.

Keywords

Second stochastic dominance, effective portfolios, compromise programming, bi-reference procedure of multi-criteria optimization.

Introduction

The portfolio selection problem is one of the classical problems of decision theory under risk. The selection of portfolio can be done according to the investor's risk preferences described by the certainty equivalent function – the utility function in the expected utility theory or the distortion function in the dual theory of the decision under risk. These certainty equivalent functions are implicit and not available before the decision process. That is why the stochastic dominance (SD) concept has been widely applied to portfolio selection problems in the last decades. The theoretical attractiveness of SD lies in its non-parametric orientation. SD criteria do not require the full specification of decision-maker's risk preferences, but rather rely on general preference assumptions [4].

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But practical application of SD rules to portfolio problems with diversification possibilities is difficult, because these rules are based on pairwise comparison of distribution functions of linear combinations of random variables. This problem can be modeled using the multi-criteria optimization problem. Such approach, proposed by Ogryczak [7], is considered in this paper.

First, the motivation for the use of the multi-criteria optimization methods to define the SSD effective portfolios is presented. We analyze the consistence of the preference structure among the criteria of the multi-criteria problem generating the SSD effective portfolios with the preferences under risk. We consider different models of the risk preferences.

Then the definition of the set of the SSD effective portfolios by the methods of the compromise programming, proposed by M. Zeleny [13], is described.

The selection of the SSD effective portfolio according to the models of the risk preferences is also possible. For this aim, we use the bi-reference procedure of multi-criteria optimization, proposed by W. Michalowski and T. Szapiro [5].

1. The models of preference under risk

The portfolio selection problem is considered, as follows. Let us denote the returns of n assets, comprising the investment universe, by $\mathbf{r} = (r_1, r_2, ..., r_n)$. The returns are the random variables with cumulative distributions functions $G_{r_i}(t) = \Pr\{r_i \le t\}, t \in \mathbf{R}, (i = 1,...,n)$. The investor may diversify between the assets and the decision vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ is a vector of portfolio weights. Portfolio weights belong to the portfolio possibilities set $\mathbf{X} = \{\mathbf{x} \in \mathbf{R}_+^\mathbf{n} : \sum_{i=1}^{n} x_i = 1\}$. The return of portfolio $R = \sum_{i=1}^{n} r_i x_i$ as a linear combination of random variables is a random variable too with cumulative distribution function $G_R(t) = \Pr\{R \le t\}, t \in \mathbf{R}$.

The random returns of portfolios are compared using the investor's risk preferences described by the function of the certainty equivalent – the utility function u(t) in the expected utility theory or the distortion function w(t) in the dual theory of the decision under risk.

The utility function u(t) assigns to the value $t \in [a, b]$ the probability u(t) with which the lottery, where the gain b is given with probability u(t) or gain a is given with probability (1-u(t)), is equivalent to the receiving the certainty value t. The utility function u(t) is non-decreasing on t and for a risk-averse decision-maker this function is concave on t.

If the investor's preferences are described by the utility function u(t), the optimal portfolio is the one maximizing the Neumann-Morgenstern's expected utility function of the portfolio return:

$$U(R) = Eu(R) = \int_a^b u(t)dG_R(t) \to \max \tag{1}$$

In the dual theory of decision under risk [12] the certainty equivalent is the distortion function w(t). Distortion function w(p) assign to the probability value $p \in [0,1]$ the part of the lottery gain, which received certainly is equivalent to the participation in lottery where gain is given with probability p. The distortion function w(p) is non-decreasing on p and $w(0) = 0$, $w(1) = 1$. The risk-averter's distortion function is convex on p.

If the distortion function describes the investor's preferences, the optimal portfolio is the one which maximizes the Yaari's functional:

$$W(R) = \int_a^b w(G_R^*(t))\,dt = \int_0^1 (G_R^*)^{-1}(p)\,dw(p) \tag{2}$$

where:

$G^*_R(t) = \Pr\{R > t\} = 1 - G_R(t)$ $t \in \boldsymbol{R}$ survival function of R,

$(G_R^*)^{-1}(p) = \inf\{t \mid G^*_R(t) \le p\}$ $0 \le p \le 1$ – inverted function of $G^*_v(t)$.

The utility function and the distortion function are complementary descriptions of the decision-maker's attitude to the risk and are explored together in the modern theories of decision under risk. Rank-dependent Expected Utility Theory (RDUT) [8] and Cumulative Prospect Theory (CPT) [11], using the combination of the utility function and the distortion function in the model of decision-maker's preferences, allow to describe the preferences under risk more flexibly.

The most characteristic feature of this theories is the consideration of the rank-dependence and reference-dependence of the preferences under risk [9, 1]. The rank-dependence of the decision-maker's preferences is the non--linear perception of the probabilities – the overweighting of large probabilities and underweighting of small probabilities. The reference-dependence of the decision-maker's preferences means that the losses (the negative deviations from the status-quo) are perceived differently than gains. The aversion to the loss weights significantly more than the attraction of a corresponding gain, this feature called loss aversion.

2. Stochastic dominance concept and portfolio selection problem

In practical applications full information about the preference function is not usually available and this is the reason for using the stochastic dominance criteria that rely on a set of general assumptions rather than a full specification of the preference function.

Stochastic dominance criteria allows to divide the set of feasible decisions into efficient and inefficient sets depending on general assumption about attitude to risk. Uncertain returns are compared by pointwise comparison for some performance function constructed from distributions functions [4].

The first stochastic dominance (FSD) criterion assumes that the decision-maker prefers more to less. The return of portfolio **x′** dominates the return of portfolio **x″** in the sense of first stochastic dominance if and only if

$$G_{R'}(t) \le G_{R''}(t), \forall t \in R$$

or $$G_{R'}^{(-1)}(p) \ge G_{R''}^{(-1)}, \forall p \in [0;1]$$

where $G_R^{(-1)}(p) = \inf\{t \mid G_R(t) \ge p\}$ is a p-quantile function of random variable R.

The second stochastic dominance (SSD) criterion assumes risk aversion. The return of portfolio **x′** dominates the return of portfolio **x″** in the sense of second stochastic dominance if and only if

$$G_{R'}^{(2)}(t) = \int_{-\infty}^{t} G_{R'}(\alpha)d\alpha \le \int_{-\infty}^{t} G_{R''}(\alpha)d\alpha = G_{R''}^{(2)}(t), \forall t \in R$$

or $$G_{R'}^{(-2)}(p) = \int_{0}^{p} G_{R'}^{(-1)}(\alpha)d\alpha \ge \int_{0}^{p} G_{R''}^{(-1)}(\alpha)d\alpha = G_{R''}^{(-2)}(p), \forall 0 \le p$$

A portfolio is efficient if its return is nondominated.

An SSD efficient portfolio is preferred to an inefficient portfolio within all risk-averse preference models where larger returns are preferred.

3. SSD consistent portfolio diversification as a multi-criteria optimization problem

The diversification of portfolio makes an infinite number of choice alternatives.

Stochastic dominance consistent diversification is possible within multiple-criteria optimization methodology [7]. This approach based on point-wise approximation of the stochastic dominance conditions to a set of criteria for multi-criteria optimization problem.

This approach, based on the quantile stochastic dominance conditions, allows for taking into consideration non-expected utility theories of choice under risk.

In this approach the finite set of tolerance levels of probability $0 < \lambda_1 < \lambda_2, ..., \lambda_K = 1$ is selected and the criteria $G_R^{(-2)}(\lambda_k)$ are maximized for k=1,…,K. If the joint probability function of returns: $p_j = \Pr\{r_1 = r_{1j}, r_2 = r_{2j}, ..., r_n = r_{nj}\} \forall j = 1,...,m$, is known, this multiple criteria problem can be modeled as a problem of maximization of the worst conditional means of portfolio return m_{λ_k} for k=1,…, K [7]:

$$
\begin{aligned}
&\max\{m_{\lambda_1}, m_{\lambda_2}, ..m_{\lambda_K}\} \\
&m_{\lambda_k} = \lambda_k q_k - \sum_{j=1}^{m} d_{kj}^{-} p_j, \mathrm{k} = 1,...,\mathrm{K} \\
&d_{kj}^{-} \geq q_k - \sum_{i=1}^{n} r_{ij} x_i, \mathrm{k} = 1,...,\mathrm{K}, \ \mathrm{j} = 1,...,\mathrm{m} \\
&d_{kj}^{-} \geq 0, \mathrm{k} = 1,...,\mathrm{K}, \mathrm{j} = 1,...,\mathrm{m} \\
&\mathrm{x} \in X
\end{aligned}
\tag{3}
$$

This multiple criteria model is consistent with the SSD relation in the sense that the set of efficient solutions of this multi-criteria problem is the set of the portfolios with nondominated returns in the sense of second stochastic dominance.

Using the multi-criteria optimization methods we can not only generate the SSD efficient portfolios, but also to choose the best portfolio according to the decision-maker's preferences. When choosing the solution of the multi--criteria problem (3) we consider the preferences among the criteria of the problem (3). These preferences are consistent with the Yaari (2) model of preferences under risk.

The Yaari's functional (2) can be rewrite in the form [1]:

$$W(v)=\int_0^1 (G_R^*)^{-1}(p)dw(p)=w'(0)G_R^{(-2)}(1)+\int_0^1 G_R^{(-2)}(1-p)dw'(p) \qquad (4)$$

From (4) we can see that the Yaari's functional can be approximated by the linear combination of the worst conditional means of portfolio returns m_{λ_k} with the positive coefficients $dw'(1-\lambda_k)$ and expected value of portfolio returns with coefficient $w'(0)$. The coefficients $dw'(p)$ are positive, because the function $w(p)$ is convex, if it presents risk aversion.

The coefficients representing the preferences among the criteria of the problem (3) characterize the form of the Yaari's functional, describing the preferences under risk. That is why the choice of the final solution of the problem (3) using the multi-criteria methods is consistent with the decision-maker's preferences under risk.

To define the set of efficient portfolios and choose the best portfolio according to the decision-maker's preferences the multi-criteria optimization methods based on the idea of the reference point are useful. These methods generate the efficient solutions, in which criteria values vector is closest to the vector of the desired (reference) values of criteria.

It is possible to define the set of the SSD efficient portfolios by applying to problem (3) the method of compromise programming proposed by M. Zeleny [13]. This method is based on the idea of the reference point.

The best portfolio according to the decision-maker's preferences can be found using the interactive methods to solve the problem (3). One of the interactive methods is a bi-reference procedure of multi-criteria optimization [5].

4. The definition of the SSD efficient portfolios using the reference multi-criteria optimization methods

To define the set of the SSD efficient portfolios we apply to the multi--criteria problem (3) the method of compromise programming, proposed by M. Zeleny [13] (described in [2, 3]). This method allows to define the set of effective solutions of the multi-criteria problem, in which criteria values vectors are closest to the vector of the reference (desired) values of criteria according to the set of metrics. This set of solutions is called the set of compromise solution.

This set of efficient solutions can be defined by solving the two-criteria problem:

$$\begin{aligned}
&\min \sum_{k=1}^{n} v_k (a_k - f_k(\mathbf{x})) \\
&\min(\max\{v_k (a_k - f_k(\mathbf{x})) : k = 1,\dots,n\}) \\
&\mathbf{x} \in \mathbf{X}
\end{aligned} \tag{5}$$

where:

$f_k(\mathbf{x})$ – k-th criterion of the multi-criteria problem (k=1,…, n),

a_k – reference value of the k-th criterion (k=1,…,n),

v_k – weight of the k-th criterion (k=1,…,n).

Using the parametric method to solve the two-criteria problem (5), we can define the corner points of the set of compromise solutions.

Applying this methods to problem (3) we have to choose the set of reference values of criteria. It is reasonable to assume that a reference point is a certain portfolio return of the desired value. Then, if the desired value of portfolio return is y^*, than the reference value of the k-th criterion in the problem (3) is $\lambda_k y^*$.

The two-criteria problem defining the set of compromise solutions of the problem (3) is the problem (6).

The weights v_k used in the problem (5) to normalize the criteria values are not necessary to use in the problem (6).

Solving the problem (6) by the parametric method we can define corner points of the set of the SSD efficient portfolios, whose returns are closest to the reference return y^*.

By varying the value of the reference return we can define the corner points of the set of the SSD efficient portfolios in the area of the reference return.

$$
\begin{aligned}
&\max \sum_{i=1}^{m} (m_{\lambda_k} - \lambda_k y^*) \\
&\max(\min\{(m_{\lambda_k} - \lambda_i y^*) : k = 1,..., K\}) \\
&m_{\lambda_k} = \lambda_k q_k - \sum_{j=1}^{m} d_{kj}^{-} \mathrm{p}_{\mathrm{j}},\ \mathrm{k} = 1,..., \mathrm{K} \\
&d_{kj}^{-} \geq q_k - \sum_{\mathrm{i}=1}^{n} r_{ij} x_i,\ \mathrm{k} = 1,..., \mathrm{K},\ \mathrm{j} = 1,..., \mathrm{m} \\
&d_{kj}^{-} \geq 0,\ \mathrm{k} = 1,..., \mathrm{K},\ \mathrm{j} = 1,..., \mathrm{m} \\
&\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{i}} = 1; \\
&\mathrm{x}_{\mathrm{i}} \geq 0,\ \mathrm{i} = 1,..., n
\end{aligned}
\tag{6}
$$

To define the best portfolio with respect to the decision-maker's risk preferences we can use the interactive multi-criteria decision making methods. Using the interactive methods we do not need to define the preference function in explicit form, but investigate the decision-maker's preferences trying to generate the most acceptable effective solution.

One of the interactive methods based on the idea of reference point is a bi-reference procedure of multi-criteria optimization [5]. In this method the structure of preference is specified by two sets of reference points (worst and ideal values of criteria), that is why it can be used to search for the best portfolio according to the decision-maker's risk preferences.

When the worst and the ideal values of the criteria are identified, the improvement direction from the worst to the ideal points is constructed. A trial solution is found by moving from a current solution along the improvement direction, while maximizing the step size. For a trial solution the decision-maker is requested to divide the set of criteria to three categories: those to be improved, those to be unchanged, and those which may be relaxed. Based on this partition new sets of worst and ideal values of criteria are constructed, a new improvement direction is calculated and a new trial solution is found. The method terminates when two trial solutions are reasonably similar.

By varying the set of the ideal values of portfolio and the partition of the set of criteria we can realize different strategies of the best solution search, modeling the rank-depending and the reference-depending risk preferences.

5. Illustrative example

Consider the diversification among the 4 investment projects. The joint probabilities of the project's returns are estimated and given in Table 1.

Table 1

Probability	Returns (%)			
	project A	project B	project C	project D
p_j	r_{1j}	r_{2j}	r_{3j}	r_{4j}
0,1	–200	90	50	350
0,1	–105	90	10	150
0,3	25	85	10	–100
0,3	75	–39	0	–120
0,2	100	–150	0	50

We have to define the portfolio weights $x_i \geq 0, i = 1,..4, \sum_{i=1}^{4} x_i = 1$ which maximize the random return of the portfolio: $R = \sum_{1}^{4} r_i x_i \xrightarrow[\mathbf{x}]{} \max$

Assuming risk-aversion of the decision-maker's preferences, we applied the second stochastic dominance rule to define the effective portfolios.

We formulated the multi-criteria problem for maximizing the worst conditional means of portfolio return (3) for the set of the probability levels $\lambda = \{0{,}1;\ 0{,}4;\ 0{,}6;\ 0{,}9;\ 1\}$ and solved it using the M. Zeleny's method of the compromise programming (6).

By warying the value of the reference return y^* from 6 to 10, we defined the corner SSD effective portfolios with expected portfolio return around 10. This corner effective portfolio is presented in Table 2.

Table 2

Expected value of portfolio return (%)	SSD effective portfolios (decisions weights)			
	X_1	X_2	X_3	X_4
1	2	3	4	5
9,80	0,09	0,02	0,89	0
10,21	0,14	0,03	0,83	0

Table 2 contd.

1	2	3	4	5
10,48	0,15	0,01	0,84	0
10,34	0,15	0,03	0,82	0
10,54	0,16	0,01	0,83	0
10,19	0,15	0,05	0,79	0,01
10,37	0,16	0,03	0,8	0,01
9,72	0,2	0,1	0,66	0,04
9,79	0,25	0,11	0,57	0,07

As a final decision we can select any portfolio presented in Table 2.

To generate the portfolio in accordance with the decision-maker's risk preferences we used the bi-reference procedure, proposed by W. Michalowski and T. Szapiro.

We selected the set of the worst values of the criteria of the multi-criteria problem maximizing the worst conditional means of the portfolio return:

λ	0,1	0,4	0,6	0,9	1
$m_{\lambda}^{W}(0)$	-6	0	0	5,4	8

Selecting the portfolio by the bi-references procedure, we performed three search strategies, modeling reference-depending and rank-depending risk preferences.

The first strategy is to model the aversion to worst returns. We improved the values of the worst conditional means for levels from 0,1 to 0,4, relaxing the values for the other levels.

The second strategy modeled the aversion to the worst returns and the aversion to not receiving the best returns. We improved the values of the worst conditional means for the levels 0,1 and 1, relaxing the values for the other levels.

The third strategy modeled loss-aversion, when the losses were the returns less than 0. We looked for the portfolio with the positive value of the worst conditional mean for the level 0,1 by improving the value for the level 0,1 and relaxing the values for other levels. Then this solution was improved by fixing the achieved value for level 0,1 and improving the value for other levels.

The portfolios selected by the strategies are presented in Table 3.

Table 3

	x_1	x_2	x_3	x_4
Strategy 1	0,10	0,03	0,87	0
Strategy 2	0,21	0,14	0,65	0
Strategy 3	0,14	0,08	0,78	0

The values of the worst conditional means of the selected portfolios are presented in Figure 1.

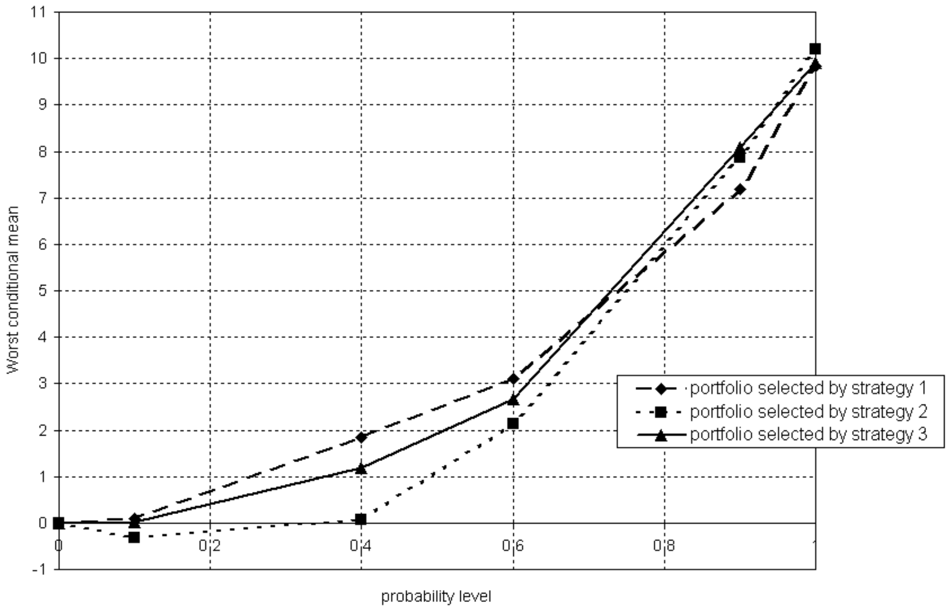

Figure 1. The worst conditional means of the returns of the portfolios selected by the three strategies

Conclusions

The SSD-consistent portfolio selection can be modeled using the multi--criteria decision making approach. The consistency of the preferences for the criteria of the multi-criteria problem generating the SSD-effective portfolios with the preferences under risk allows to select the SSD-effective portfolio using the multi-criteria optimization methods. The reference methods are useful for this problem.

By applying the method of the compromise programming, proposed by M. Zeleny, we defined the corner SSD-effective portfolios with returns around the reference value. By using the interactive bi-references multi-criteria optimization method, proposed by W. Michalowski and T. Szapiro, we selected the best portfolios with respect to the decision-maker's preferences under risk, modeling the rank-dependence and the reference-dependence of the preferences.

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