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ALGORITHM FOR DERIVING PRIORITIES FROM INCONSISTENT PAIRWISE COMPARISON MATRICES

Abstract

In several multiobjective decision problems Pairwise Comparison Matrices (PCM) are applied to evaluate the decision variants. The problem that arises very often is inconsistence of given PCM. In such a situation it is important to approximate the PCM with a consistent one. The most common way is to minimize the Euclidean distance between the matrices. In the paper we consider minimization of the maximum distance.

Keywords

Heuristics, nonlinear programming, decision making, pairwise comparison.

Introduction

One of the popular tools of multiobjective decision making is the Analytic Hierarchy Process, introduced by Saaty [see e.g. Saaty 1980; Erkut and Tarimcilar 1991] and studied by numerous authors. During the process, the Decision Maker compares pairwise n given decision variants. Usually the comparisons are represented by the *pairwise comparison matrix* $A = [a_{ij}]$, where the number a_{ij} says how many times the variant i is preferred to the variant j.

The values of a_{ij}, $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$ should fulfill the following conditions:

$$a_{ji} = \frac{1}{a_{ij}} \text{ for } i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n, \tag{1}$$

$$a_{ij}a_{jk} = a_{ik} \text{ for } i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n, k = 1, 2, \ldots, n. \tag{2}$$

If the above conditions are satisfied, the pairwise comparison matrix A is called *consistent*. The condition (1) is rather easy to fulfill in practice (the decision maker may e.g. fill only the elements of A above the diagonal and then the remaining ones are easily calculated). The condition (2) is much more difficult to satisfy and is the main source of the inconsistency.

It is easy to prove that the matrix A is consistent if and only if there exist positive weights $w_1, w_2, \ldots, w_n$ (forming the vector w) such that

$$a_{ij} = \frac{w_i}{w_j}, i = 1,2, \ldots, n, j = 1, 2, \ldots, n \tag{3}$$

The elements of w are interpreted as the explicit values representing the priorities of the decision variants. Finding their values is thus essential. Note that if some vector w defines the matrix A then also the vector λw for every $\lambda > 0$.

1. Problem formulation

As in real-life problems the matrix A is very often not consistent, it is impossible to find the vector w (in fact, it does not exist). In such a situation the goal is to find the vector w that defines the matrix B which is as close as possible to the original pairwise comparison matrix A.

The distance between matrices A and B may be calculated in various ways. One of the methods is to calculate Saaty's inconsistency index using the eigenvalues of the (relative) estimation error matrix, which can be approximated by the row-wise geometric means [see e.g. Saaty 1980; Mogi and Shinohara 2009]. Estimation errors are calculated as the quotients or differences of the respective elements of A and B. Another approach, based on the additive PCM (a formulation equivalent to the one discussed in this paper), may be found e.g. in Fedrizzi, Giove [2007].

Another approach is to calculate some kind of average of errors. The most popular measure is the square mean calculated according to the formula

$$G_2(A, v) = \left(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij} - \frac{v_i}{v_j}\right)^2\right)^{\frac{1}{2}}. \tag{4}$$

This method of the inconsistency measurement (called least square method, LSM) was introduced in this context by Chu et al. [1979] and used e.g. by Anholcer et al. [2011], Bozóki [2008], Fülöp, Koczkodaj and Szarek [2010], Fülöp [2008], Bozóki and Rapcsàk [2008], Mogi and Shinohara [2009].

In the last two papers other inconsistency measures were also considered. Mogi and Shinohara analyzed the general mean which can be defined as

$$G_p(A, v) = \left(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij} - \frac{v_i}{v_j}\right|^p\right)^{\frac{1}{p}}. \tag{5}$$

If $p = 2$, we obtain the LSM. Other special cases, also considered in the paper, are $p = -\infty$ (minimum), $p = -1$ (harmonic mean), $p = 0$ (geometric mean), $p = 1$ (arithmetic mean) and $p = \infty$ (maximum). In the remainder of this paper we will be interested in the last measure. To be more precise, we want to solve the following problem:

$$\min\left\{G_\infty(A, v) = \max_{1\leq i,j\leq n}\left\{\left|a_{ij} - \frac{v_i}{v_j}\right|\right\}\right\}, \tag{6}$$

s.t.

$$v_1 = 1, \tag{7}$$

$$v_j > 0,\ j = 1, 2, \ldots, n. \tag{8}$$

The condition $v_1 = 1$ has been introduced to normalize the vector v (if some vector v is the solution to the above problem, then also every vector λv for every $\lambda > 0$). Of course other normalizing conditions can be used [compare e.g. Anholcer et al. 2011; Bozóki 2008; Fülöp 2008].

The problem under consideration is a difficult optimization problem, as the objective function is neither convex nor concave and thus no local search algorithm may be applied to find the global optimum.

The LSM problem (with G_2 instead of G_∞) was studied e.g. in Anholcer et al. [2011] – heuristic approach, Bozóki [2008] – systems of nonlinear equations and Fülöp [2008] – branch and bound algorithm. The statistical approach was used by Hovanov, Kolari and Sokolov [2008], while Mogi and Shinohara [2009] used simulation. Our goal is to give an effective method to derive the weights minimizing the value of function G_∞ as the inconsistency measure.

2. New algorithm

The problem (6)-(8) may be reformulated as follows. Let us introduce additional variable $z = G_\infty(A, v)$. Then we can rewrite the problem as

$$\min\{z\}, \tag{9}$$

s.t.

$$\left|a_{ij} - \frac{v_i}{v_j}\right| \leq z, \qquad i,j = 1,2,\dots,\mathrm{n} \tag{10}$$

$$v_1 = 1, \tag{11}$$

$$v_j > 0, j = 1,2,\dots,n. \tag{12}$$

Note that the problems are not identical (the sets of feasible solutions are distinct), but they are equivalent (the optimal solutions to both problems are the same and they always exist). The problem (9)-(12) is a difficult mathematical programming problem – the constraints (10) are nonlinear, neither convex nor concave. Moreover, the set of feasible solutions is not closed and thus not compact (although the optimum exists). In order to find its approximate solution we are going to treat z as a parameter.

If we assume that the value of z is given, the problem (9)-(12) reduces to the following system of linear equations and inequalities:

$$\left(a_{ij} - z\right)v_j \leq v_i \leq \left(a_{ij} + z\right)v_j \qquad i,j = 1,2,\dots,\mathrm{n} \tag{13}$$

$$v_1 = 1, \tag{14}$$

$$v_j \geq 0, j = 1,2,\dots,n. \tag{15}$$

Note that the constraint (12) may be replaced with (15) as none of v_j can be equal to 0 – otherwise all of them would be equal to 0 according to the constraints (13). That would in turn contradict the constraint (14).

The number of inequalities (13) may be reduced. First, for every i, the inequalities $(a_{ii} - z)v_i \leq v_i \leq (a_{ii} + z)v_i$ are always satisfied as $a_{ii} = 1$ and $z \geq 0$.

Another operation lets us remove half of the remaining inequalities. Let us consider the two inequalities in which the variables v_i and v_j occur for some $i \neq j$. They can be rewritten in the following form:

$$-v_i + \left(a_{ij} - z\right)v_j \leq 0, \tag{16}$$

$$-v_i + \frac{1}{\left(a_{ji} + z\right)}v_j \leq 0, \tag{17}$$

$$\frac{1}{\left(a_{ij} + z\right)}v_i - v_j \leq 0, \tag{18}$$

$$\left(a_{ji} - z\right)v_i - v_j \leq 0. \tag{19}$$

Exactly one of the inequalities (16) and (17) implies the other one, so one of them can be removed. More precisely, we leave the inequality

$$-v_i + \left(\max\left\{\left(a_{ij} - z\right), \frac{1}{\left(a_{ji} + z\right)}\right\}\right) v_j \leq 0. \tag{20}$$

Analogously, we can eliminate one of the inequalities (18) and (19), by choosing the following one

$$\left(\max\left\{\left(a_{ji} - z\right), \frac{1}{\left(a_{ij} + z\right)}\right\}\right) v_i - v_j \leq 0. \tag{21}$$

Note that in both cases the chosen maxima have positive values. To solve the resulting system of linear inequalities and equations, we formulate the following auxiliary linear programming problem.

$$\min\left\{z_0\right\} \tag{22}$$

s.t.

$$-v_i + \left(\max\left\{\left(a_{ij} - z\right), \frac{1}{\left(a_{ji} + z\right)}\right\}\right) v_j + z_{ij}^1 = 0,\ \ 1 \leq i < j \leq \text{n}, \tag{23}$$

$$\left(\max\left\{\left(a_{ji} - z\right), \frac{1}{\left(a_{ij} + z\right)}\right\}\right) v_i - v_j + z_{ij}^2 = 0,\ \ 1 \leq i < j \leq \text{n}, \tag{24}$$

$$v_1 + z_0 = 1, \tag{25}$$

$$v_j \geq 0,\ \ j = 1, 2, \dots, n, \tag{26}$$

$$z_0 \geq 0,\ \ z_{ij}^k \geq 0,\ \ 1 \leq i < j \leq n,\ \ k = 1,2. \tag{27}$$

We solve the above problem using the adapted version of the simplex method. The initial feasible base solution is formed by the variables included in constraint (27): $z_0 = 1$ and $z_{ij}^k = 0,\ 1 \leq i < j \leq n,\ k = 1,2$. The reduced costs are equal to the coefficients in the constraint (25). Also, we use additional stopping criterion: $z_0 = 0$. If this criterion is used, the initial system of inequalities has feasible solution where the values of v_j are equal to those in the optimal solution of the problem (22)-(27). On the other hand, if the standard optimality condition is in use, that means that $z_0 = 1$ and the problem (22)-(27) is inconsistent.

Note also that if the feasible solution exists for some value of $z = z^\star$, then it is also the solution for every value $z \geq z^\star$. That means also that if the system (13)-(15) is inconsistent for some value of $z = z^\star$, then it is also inconsistent for every $z \leq z^\star$. This leads us to the following algorithm, where the starting point is generated by the geometric means of rows of A.

Algorithm 1

1. Assume the accuracy level $\varepsilon > 0$. Let $v_i^\star = \left(\prod_{j=1}^{n} a_{ij}\right)^{\frac{1}{n}}$ and $v_i = \frac{v_i^\star}{v_1^\star}$ for $i = 1, 2, \dots, n$. Let $z = z_{max} = G_\infty(A, v)$ and $z_{min} = 0$. Proceed to step 2.
2. If $z - z_{min} < \varepsilon$ then STOP. The vector v is the desired approximation of the weight vector w. Otherwise go to step 3.
3. Set $z := \frac{(z_{max} - z_{min})}{2}$. Solve the problem (22)-(27). If $z_0 = 0$, save the new value of v and set $z_{max} := z$. Otherwise do not change the value of v and set $z_{min} := z$, $z := z_{max}$. Go back to step 2.

In every step of the algorithm the value of $z_{max} - z_{min}$ decreases twice, so in the finite number of iterations we obtain the approximation of the optimal solution (more precisely, if $z_{max}^\star$ denotes the initial value of z_{max}, then the algorithm stops after $\left\lceil \log_2 \left(\frac{z_{max}^\star}{\varepsilon}\right) \right\rceil$ steps).

3. Numerical example

Let us present a small illustrative example. Assume that

$$A = \begin{bmatrix} 1 & 2 & 1 & 5 & 2 \\ 0.5 & 1 & 0.8 & 2.5 & 0.4 \\ 1 & 1.25 & 1 & 2.5 & 1 \\ 0.2 & 0.4 & 0.4 & 1 & 0.8 \\ 0.5 & 2.5 & 1 & 1.25 & 1 \end{bmatrix}$$

and $\varepsilon = 0.1$.

Step 1. We derive the initial solution as the geometric means of the rows and divide all of them by v_1, so $v_1 = 1.000$, $v_2 = 0.457$, $v_3 = 0.690$, $v_4 = 0.264$, $v_5 = 0.821$. The matrix derived with the values v_j has the form

$$B = \begin{bmatrix} 1.000 & 2.187 & 1.450 & 3.789 & 1.665 \\ 0.457 & 1.000 & 0.663 & 1.733 & 0.761 \\ 0.690 & 1.509 & 1.000 & 2.614 & 1.149 \\ 0.264 & 0.577 & 0.383 & 1.000 & 0.439 \\ 0.601 & 1.313 & 0.871 & 2.276 & 1.000 \end{bmatrix}.$$

As one can easily check, the inconsistency measure equals 1.211. Thus $z_{min} = 0$ and $z = z_{max} = 1.211$.
Step 2. $z_{max} - z_{min} > \varepsilon$, we proceed to step 3.
Step 3. $z = 0.605$. In the optimal solution of the problem (22)-(27), $z_0 = 1$. Thus $z_{min} = 0.605$, $z = z_{max} = 1.211$. We go back to step 2.
Step 2. $z_{max} - z_{min} > \varepsilon$, we proceed to step 3.
Step 3. $z = 0.908$. In the optimal solution of the problem (22)-(27), $z_0 = 1$. Thus $z_{min} = 0.908$, $z = z_{max} = 1.211$. We go back to step 2.
Step 2. $z_{max} - z_{min} > \varepsilon$, we proceed to step 3.
Step 3. $z = 1.059$. In the optimal solution of the problem (22)-(27), $z_0 = 0$ and $v_1 = 1.000$, $v_2 = 0.366$, $v_3 = 0.486$, $v_4 = 0.254$, $v_5 = 0.527$. We save this solution. Moreover, $z_{min} = 0.908$, $z = z_{max} = 1.059$. We go back to step 2.
Step 2. $z_{max} - z_{min} > \varepsilon$, we proceed to step 3.
Step 3. $z = 0.984$. In the optimal solution of the problem (22)-(27), $z_0 = 1$ Thus $z_{min} = 0.984$, $z = z_{max} = 1.059$. We go back to step 2.
Step 2. $z_{max} - z_{min} < \varepsilon$, STOP. The optimal weights are equal to $v_1 = 1.000$, $v_2 = 0.366$, $v_3 = 0.486$, $v_4 = 0.254$, $v_5 = 0.527$. They define consistent PCM of the form

$$B = \begin{bmatrix} 1.000 & 2.735 & 2.059 & 3.941 & 1.899 \\ 0.366 & 1.000 & 0.753 & 1.441 & 0.694 \\ 0.486 & 1.328 & 1.000 & 1.914 & 0.922 \\ 0.254 & 0.694 & 0.523 & 1.000 & 0.482 \\ 0.527 & 1.441 & 1.085 & 2.075 & 1.000 \end{bmatrix}.$$

4. Computational experiments

The algorithm has been implemented in Java and tested for a number of randomly generated problems. The assumed accuracy level was $\varepsilon = 0.001$. The application has been tested on the PC with Intel Core2 Duo CPU (2.20 GHz). For every value of $n = 3,4,\ldots,10$ (in real-life problems, the size of the comparison matrix rarely exceeds 10) the elements of A were chosen uniformly at random from the interval $< 1, a_{\max} >$, where $a_{max} \in \{3,5,10\}$. All PC matrices obtained were inconsistent. In every case 100 problems have been solved (which gives the total number of 2400 test problems). The average running times (in milliseconds) are given in the Table 1.

Table 1

Average running times

n	$a_{max} = 3$	$a_{max} = 5$	$a_{max} = 10$
3	0,0117	0,0127	0,0158
4	0,0237	0,0284	0,0297
5	0,0528	0,0550	0,0605
6	0,0979	0,1240	0,1191
7	0,1720	0,1998	0,2047
8	0,2686	0,2934	0,3288
9	0,4215	0,4546	0,5110
10	0,6271	0,6607	0,7507

As we can see, in all the cases the running times are much less than one second, which is acceptable time in real life applications.

Conclusions

The algorithm presented guarantees obtaining the solution for which the objective value is arbitrarily close to the optimal one. Of course this does not mean that the coordinates of vector v are arbitrarily close to their optimal values (distinct local optima may be far from each other even if the objective values are very close). However it is more that gives the heuristic for LSM given by Anholcer et al. [2011], which does not guarantee obtaining the objective value close to the optimal one. On the other hand the algorithm presented is fast and therefore very useful for finding the best consistent approximate of an inconsistent pairwise comparison matrix.

As far as the author knows the method presented here is the first one for the inconsistency measured using the maximum distance G_∞. Further research should focus on looking for the exact method of solving this problem and any methods for other measures (e.g. G_p distance for arbitrary p, including Manhattan distance G_1).

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