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REAL AND VIRTUAL PARETO SET UPPER APPROXIMATIONS

Abstract

This paper deals with the problem of the derivation of lower and upper approximations of an efficient element set.

We consider the case where upper approximations cannot be derived as criteria mapping images of infeasible variants.

Keywords

Multiobjective optimization, evolutionary algorithms, lower Pareto set approximations, upper Pareto set approximations.

Introduction

Under assumption that all criteria are of the type "the more the better" each *outcome* (i.e. the image of an admissible variant under the criteria mapping) lies "below" the Pareto set (the set of efficient outcomes) or is an element of this set. A number of such outcomes form a *lower approximation* of the Pareto set.

By analogy, we consider an *upper approximation* of the Pareto set, i.e. a set of elements of the outcome (criteria) space which lie "above" the Pareto set.

Having pairs of lower and upper approximations is of interest for two reasons. First, provided that elements of a lower and upper approximation are uniformly distributed along the Pareto set, we are in position to assess the maximal error one makes when representing an efficient outcome y by an outcome y' taken from the lower approximation and dominated by y [Kaliszewski 2008; Miroforidis 2008, 2010; Kaliszewski et al. 2011, 2012]. That is important in cases where deriving elements of the Pareto set is computationally costly and working with lower approximations of the Pareto set instead of the Pareto set itself is a rational option. Second, given a pair of lower and upper approximations lower and upper bounds on components of any efficient outcome pointed to by the *Decision Maker* preferences can be easily calculated. How the Decision Maker preferences point to efficient outcomes will be explained below.

In this paper we discuss problems arising when deriving upper approximations and we illustrate our considerations by an illustrative example.

In our earlier papers we have proposed to conduct interactive multiple criteria decision processes with outcome assessments instead of outcomes themselves [Miroforidis 2008, 2010; Kaliszewski, Miroforidis 2010a, b; Kaliszewski et al. 2011]. By an outcome assessment we mean lower and upper bounds on values of outcome components. Such an approach stemmed from a variety of *Multiple Criteria Decision Making* problems where efficient outcomes are given implicitly by a set of constraints and therefore have to be derived by solving optimization problems. To provide for versatility of such an approach we have adapted it to employ *evolutionary calculations* (*Evolutionary Multiobjective Optimization*) driven by Decision Maker preferences revealed in the course of interactive decision processes.

We have founded our approach on two constructs, namely on *lower approximations*, i.e. finite subsets of feasible variants, and *upper approximations*, i.e. finite subsets of infeasible variants with some specific properties. The formal definitions of both constructs are given in the next section.

Of interest are lower and upper approximations which are tight, i.e. their images under the criteria mapping are close, in a sense, to the set of efficient outcomes. Tight approximations provide for tight bounds in outcome assessments mentioned above. Moreover, images of tight lower and upper approximations represent sets of efficient outcomes within measurable accuracy. Except for our earlier papers we are aware of only one paper attempting to exploit a similar concept, namely Legriel et al. [2010].

Evolutionary Multiobjective Optimization (EMO) methods and algorithms are dedicated to deriving tight lower approximations and that subject is represented by numerous publications, see e.g. Deb [2001], Coello Coello et al. [2002]. In contrast to this, deriving lower and upper approximations is a novel concept.

A lower approximation always exists as long as the set of feasible variants is nonempty. However, the existence of an upper approximation is not guaranteed in general. In this work we consider the case where upper approximations cannot be derived as images of infeasible variants under criteria mappings.

The outline of the paper is as follows. In Section 1 we provide basic definitions and notation. In Section 2 we briefly outline the concept of approximating the set of efficient outcomes with the help of *lower shells* and *upper shells*. In Section 3 we address the case where upper shells do not exist and propose how to deal with that case to have our concept of outcome assessments still workable. Section 4 concludes.

1. Definitions and notation

Let x denote a (decision) variant, X a space of variants, X_0 a set of *feasible variants*, $X_0 \subseteq X$. Here we assume that X and X_0 *are infinite.* Then the underlying model for MCDM is formulated as:

$$\text{“}max\text{”} f(x)$$
$$x \in X_0, \tag{1}$$

where $f: X \to R^k$, $f = (f_1,...,f_k)$, $f_i: X \to R$, $i = 1,...,k$, $k \geq 2$, are criteria functions; ”*max*” denotes the operator of deriving all efficient variants (as defined below) in X_0.

Element $\bar{t}$ of T, $T \subseteq R^k$, is:

– *efficient in* T, if $t_i \geq \bar{t}_i$, $i = 1,...,k$, $t \in T$, implies $t = \bar{t}$,
– *weakly efficient in* T, if there is no $t \in T$, such that $t_i > \bar{t}_i$, $i = 1,...,k$.

Variant $\bar{x} \in X_0$ is called efficient (weakly efficient) in X_0 if $\bar{y} = f(\bar{x})$ is efficient (weakly efficient) in $f(X_0)$. Elements of $f(X_0)$ are called outcomes.

We denote the set of efficient variants of X_0 by N. Elements of $f(N)$ are called efficient outcomes for, by the definition, they are efficient in $f(X_0)$.

We define on X – the *dominance relation* $\prec$,

$$x' \prec x \Leftrightarrow f(x') << f(x),$$

where $<<$ denotes $f_i(x') \leq f_i(x)$, $i = 1,...,k$, and $f_i(x') < f_i(x)$ for at least one i. If $x' \prec x$, then we say that x' is dominated by x and x is dominating x'.

2. Lower and upper shells

In this paper we are concerned with specific lower and upper approximations of the set of efficient outcomes stemming from the concept of lower shell and upper shell [cf. also Kaliszewski 2008; Kaliszewski Miroforidis 2010a, b; Kaliszewski et al. 2010, 2011].

The following definitions of lower and upper shells come from [Miroforidis 2008, 2010].

Lower shell is a finite nonempty set $S_L \subseteq X_0$, elements of which satisfy

$$\forall_{x \in S_L} \neg \exists_{x' \in S_L} x \prec x' . \tag{2}$$

Nadir point y^{nad} is defined as

$$y_i^{nad} = \min_{x \in N} f_i(x), \; i = 1,...,k.$$

Upper shell is a finite nonempty set $S_U \subseteq X \setminus X_0$, elements of which satisfy

$$\forall_{x \in S_U} \neg \exists_{x' \in S_U} x' \prec x, \tag{3}$$

$$\forall_{x \in S_U} \neg \exists_{x' \in N} \; x \prec x', \tag{4}$$

$$\forall_{x \in S_U} \; y^{nad} \leq f(x), \tag{5}$$

where the last inequality means $y_i^{nad} \leq f_i(x), \quad i = 1,...,k$ [1].

To illustrate the concept of lower and upper shells, in Figure 1 we present an example of the images of lower and upper shells under criteria mapping derived for a problem described in Kaliszewski, Miroforidis, [2010b]. The problem is as follows

$$\text{“}max\text{”} \; (f_1(x), f_2(x))$$

where $f_1(x) = -x_1^2 + x_2$, $f_2(x) = \frac{1}{2}x_1 + x_2 + 1$,

subject to $x \in X_0$, where X_0 is defined as

[1] Since for *N* is not known (if otherwise, there is no need to approximate *N*) this definition is not operational and in Kaliszewski et al. [2010] we have shown how to overcome this by a somewhat weaker constructs than upper shells, with no direct reference to *N*. But if upper shells do not exist those we weaker constructs exist neither.

$$\frac{1}{6}x_1 + x_2 - \frac{13}{2} \le 0,$$
$$\frac{1}{2}x_1 + x_2 - \frac{15}{2} \le 0,$$
$$\frac{5}{x_1} + x_2 - 30 \le 0,$$
$$0 \le x_i \le 7, \ i = 1, 2.$$

3. The case of nonexistence of S_U

The existence of upper shells is not in general guaranteed. A collection of problems, selected from the EMO literature, for which upper shells do not exist, has been identified in Kaliszewski, Miroforidis [2010a].

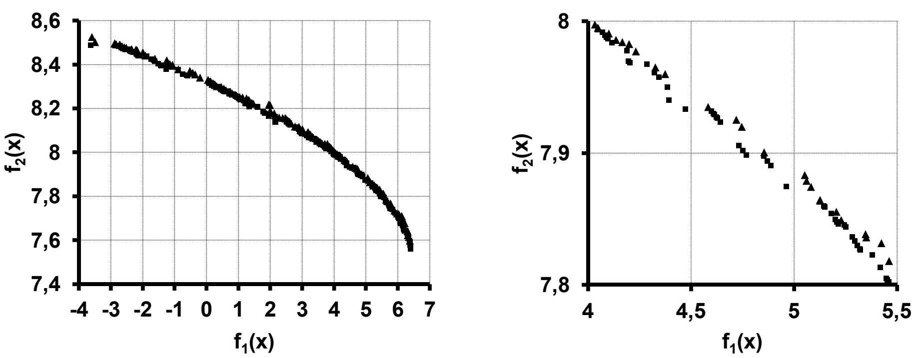

Figure 1. The images of elements of a lower shell (squares) and an upper shell (triangles) under the criteria mapping for the example problem of Section 3: left – full view, right – window $4 \le f_1(x) \le 5.5$, $7.8 \le f_2(x) \le 8.0$

The nonexistence of upper shells means that there is no $x \in X$, $x \notin X_0$, such that $f(x') << f(x)$ for some $x' \in X_0$. However, it does not mean that there does not exists $y \in R^k$, such that $f(x') << y$ for some $x' \in X_0$. In formulas for bounds on outcome components elements x of S_U appear only indirectly, via elements $f(x)$, $x \in S_u$. Therefore we can replace elements $f(x)$, $x \in S_u$, with elements $y \in R^k$ having the same property as $f(x)$, $x \in S_u$, regardless existence of x such that $y = f(x)$.

To implement this concept we are to define an appropriate counterpart of the notion upper shell. We shall call such a construct a *virtual upper shell*.

Virtual upper shell is a finite nonempty set $VS_U \subseteq R^k \setminus f(X_0)$, elements of which satisfy

$$\forall_{y \in VS_U} \neg \exists_{y' \in VS_U} \quad y' << y, \tag{7}$$

$$\forall_{y \in VS_U} \neg \exists_{x \in N} \quad y << f(x), \tag{8}$$

$$\forall_{y \in VS_U} \quad y^{nad} \le y. \tag{9}$$

In the algorithm presented below we operationalize the condition (8) replacing it by

$$\forall_{y \in VS_U} \neg \exists_{x \in S_L} \quad y << f(x). \tag{8'}$$

The following EMO-type algorithm derives virtual upper shells. It builds directly on the logic of algorithm PDAE/M proposed in Miroforidis [2010], cf. also Kaliszewski et al. [2011, 2012].

To limit the domain of searching through the set $R^k \setminus f(X_0)$, we assume existence of bounded set (box)

$$R^k_{DEC} = \{y \in R^k \mid Y_i^L \le y \le Y_i^U,\ i = 1,...,k\},$$

such that $f(X_0) \subseteq \text{int}(R^k_{DEC})$.

Algorithm PDAE/M_VS$_U$

1. 1 $j := 0$, $S_L^j := Ø$, $VS_U^j :=$ Ø.
2. Generate randomly $\eta \ge 1$ elements of X_0 and derive from those elements S_L^0.
3. $S := S_L^j$; for each element $\bar{x}$ of S perform Steps 4-6.
4. Select element $x' \in X_0$ such that $\neg x' \prec \bar{x}$. Select element $y' \in R^k_{DEC}$ such that $f(\bar{x}) << y'$.

5. If $\neg\exists x \in S_L^j : x' \prec x$ then

5.1. $S_L^j := S_L^j \cup \{x'\}$,

5.2. $S_L^j := S_L^j \setminus \{x \in S_L^j \mid x \prec x'\}$.

6. If $\neg\exists x \in X_0 : f(x) = y'$ and $\neg\exists y \in VS_U^j : y << y'$ and $\neg\exists x \in S_L^j : y' << f(x)$ then

6.1. $VS_U^j := VS_U^j \cup \{y'\}$,

6.2. $VS_U^j := VS_U^j \setminus \{y \in VS_U^j \mid y' << y\}$.

7. If $j = j^{\max}$ then STOP, otherwise $j = j + 1$ and go to 3.

Step 1 initializes, whereas in Step 2 an initial lower shell is derived from a number of elements of X_0.

Step 3 specifies that an attempt to modify S_L^j has to be made at each of its elements. It has been found in Miroforids [2010] that such a deterministic strategy, as opposed to random selection of elements to be modified, reduces clustering of elements in S_L^j and thus produces much more uniform lower approximations of N.

The evolutionary multiobjective optimization principle is realized in Step 4 via the mutation operation. In this step element x' is selected in the following process:

4.1. $x' := \bar{x}$.

4.2. $i = rndInt(1, m)$.

4.3. If $i = rndInt(0,1) \le 0.5$ then

$$x_i' := x_i' + (X_i^U - x_i')(1 - rnd(0,1)^{2(1 - \frac{j}{j^{\max}})});$$

otherwise

$$x_i' := x_i' + (x_i' - X_i^L)(1 - rnd(0,1)^{2(1 - \frac{j}{j^{\max}})}).$$

4.4. If $x' \neg\prec \bar{x}$ then go to 5; otherwise go to 4.1.

Function $rndInt(a,b)$ returns an integer number from the range [a,b] with uniform distribution. Function $rnd(a,b)$ returns a random real number from the range [a,b] with uniform distribution. The presented method of mutations and the strategy of decreasing mutation range have been taken form the literature [cf. e.g. Michalewicz 1996].

In Step 5 an attempt is made to modify the current lower shell S_L^j with the newly generated element x'.

Similarly, in Step 6 the same attempt is made with respect to the current virtual upper shell VS_U^j. Here the tricky point is to verify whether for given y' there exists $x \in X_0$ such that $f(x) = y'$. If yes then no amendment of VS_U^j is made[2]. The existence of $x \in X_0$ such that $f(x) = y'$ can be verified by solving the optimization problem $\min \| y' - f(x) \|$ subject to $x \in X_0$ by an evolutionary optimization algorithm.

In Step 7 the stopping rule is checked, where $j^{\max}$ is the limit for the number of iterations of algorithm PDAE/M_VS$_U$.

We illustrate the concept of lower shells and virtual upper shells with Figure 2, where we present the image of a lower shell under the criteria mapping and a virtual upper shell derived for the problem DTLZ1a from Deb et al. [2001], as follows.

$$\text{“}max\text{”}\ (f_1(x, g), f_2(x, g))$$

where $f_1 = 0.5x_1(1 + g)$, $f_2 = 0.5(1 - x_1)(1 + g)$, all objective functions are to be minimized, and

$$g = 100\ [5 + \sum_{i=2}^{6} (x_i - 0.5)^2 - cos(2\pi(x_i - 0.5))]$$

$$X_0 = \{x \mid x_i \in [0, 1],\ i = 1,..., 6\}.$$

Set N is made up of elements in which $x_2,..., x_6 = 0.5$ and $x_1 \in [0, 1]$. Elements of $f(S_L)$ and VS_U were derived in 60 iterations of algorithm PDAE/M_VS$_U$.

[2] If there exists $x \in X_0$ such that $f(x) = y'$ then x is a suitable element for Step 5 for $f(\bar{x}) << y' = f(x')$ entails $x' \neg \prec \bar{x}$. To exploit this fact the order of Step 5 and Step 6 should be reversed. However, in this paper we do not investigate this variant of the algorithm.

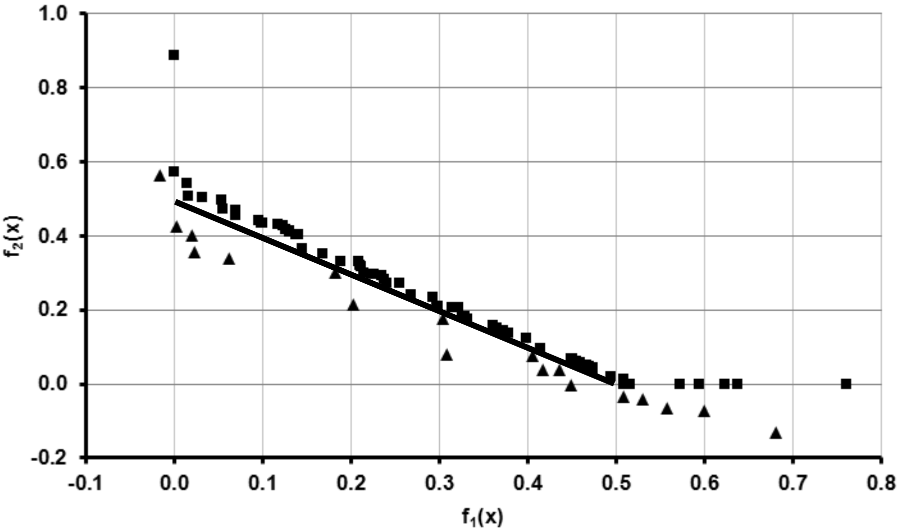

Figure 2. The images of elements of a lower shell (squares) under the criteria mapping and elements of a virtual upper shell (triangles) for the DTLZ1a problem. Set $f(N)$ is represented by the continuous line

4. Concluding remarks and directions for further research

In this paper we have proposed how to derive upper approximations of the Pareto set when upper shells do not exist. To this aim we have introduced the concept of virtual upper shells and we have shown on a numerical example that the idea is perfectly viable.

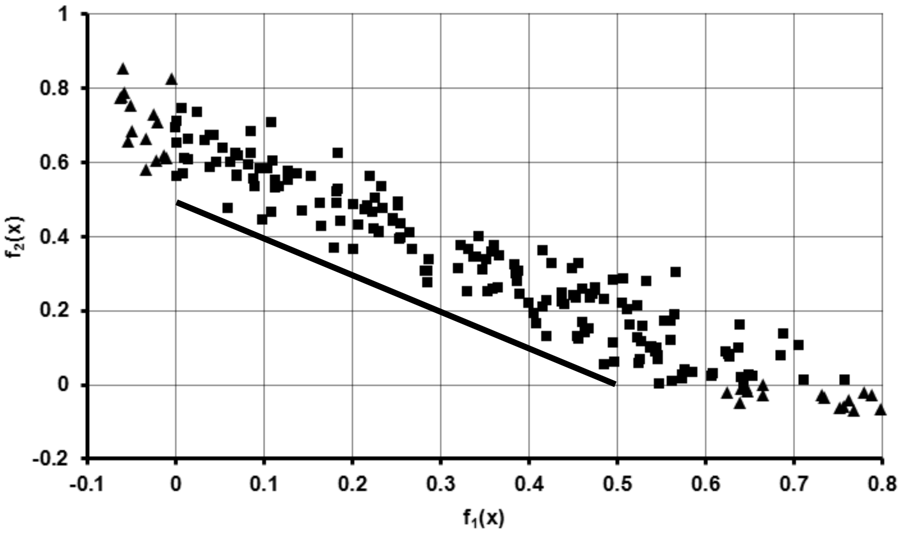

Figure 3. The images of feasible variants (squares) and infeasible variants (triangles) of total 200 variants generated randomly under the criteria mapping for the DTLZ1a problem. Set *f*(*N*) is represented by the continuous line

With virtual upper shells in place we are in the position to derive, for any instance of problem (1), an approximation of the Pareto set in the form of a pair of a lower approximation $f(S_L)$, where S_L is a lower shell, and an upper approximation in the form of VS_U.

In our previous papers we addressed the problem of efficiency of algorithms we proposed to derive S_U. The question of efficiency of the algorithm we proposed in this work to derive VS_U has been left to further research.

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