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ROBUST ORDINAL REGRESSION APPLIED TO *TOPSIS*

Abstract

This paper proposes a new method for ranking a finite set of alternatives evaluated on multiple criteria. The presented method combines the robust ordinal regression (ROR) approach and the ranking score based on the aggregate distance measure function coming from the $TOPSIS$ method. In our method, the preference model is a set of additive value functions compatible with a non-complete set of pairwise comparisons of some reference alternatives given by the decision maker (DM). Based on this set of compatible value functions, we define an aggregate function representing relative closeness to the reference point (ideal solution) in the value space. The ranking score determined by this distance measure is then used to rank all alternatives. Calculating the distance in the value space permits to avoid normalization used in $TOPSIS$ to transform original evaluations on different criteria scales into a common scale. This normalization is perceived as a weakness of $TOPSIS$ and other methods based on a distance measure, because the ranking of alternatives depends on the normalization technique and the distance measure. Thus, ROR applied to $TOPSIS$ does not only facilitate the preference elicitation but also solves the problem of non-meaningfulness of $TOPSIS$. Finally, an instructive example is given to illustrate the proposed method.

Keywords: Multiple criteria decision aiding, robust ordinal regression, $TOPSIS$ method, UTA^{GMS} and $GRIP$ methods.

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1 Introduction

Multiple criteria decision aiding (MCDA) helps in constructing an aggregation model on the basis of preference information provided by the DM. Such an aggregation model is called preference model. It induces a preference structure in the set of considered alternatives (for a recent state-of-the-art, see Figueira et al. (2005)). The preference information may be either direct or indirect, depending on whether it specifies directly values of some parameters used in the preference model (e.g., trade-off weights, aspiration levels, discrimination thresholds, etc.), or else some examples of holistic judgments from which compatible values of the preference model parameters are induced. Direct preference information is used in the traditional aggregation paradigm, according to which the aggregation model is constructed first and then applied to the whole set of alternatives to get information about the comprehensive preference relation.

Eliciting direct preference information from the DM can be counterproductive in real-world decision making situations because of a high cognitive effort required. Very often this information is not easily definable. Consequently, asking the DM directly to provide values of the parameters makes the DM rather uncomfortable. For example, this is the case of the price or the interest rates in cost-benefit analysis, or the case of the coefficients in objectives and constraints of mathematical programming models, or the case of attribute weights and several thresholds in outranking methods.

Eliciting indirect preferences in the form of holistic pairwise comparisons of some reference or training alternatives is much less demanding of cognitive effort. This kind of preference information is given as decision examples. Such a reverse search of a preference model from decision examples is done by so-called *ordinal regression* (also called disaggregation-aggregation approach). The preference model found by ordinal regression is *compatible* with the given preference information, i.e., it restores the holistic pairwise comparisons made by the DM. Finally, it is used on the whole set of alternatives to recommend the best choice, classification, or ranking. In this paper we will use the preference model to recommend a ranking only.

The ordinal regression paradigm has been known for at least fifty years in the field of multidimensional analysis (see March, 1978). This paradigm has been applied within the two main MCDA approaches: those using a value function as preference model (see Srinivasan and Shocker, 1973; Pekelman and Sen, 1974; Jacquet-Lagrèze and Siskos, 1982), and those using an outranking relation as preference model (see Kiss et al., 1994; Mousseau and Słowiński, 1998; Mousseau et al., 2000). This paradigm has also been used since the mid nineties in MCDA methods involving a new, third family of preference models - a set of dominance decision rules induced from rough approximations of holistic preference relations (see Greco et al., 2001).

Usually, among the many sets of parameters of a preference model representing the preference information, only one specific set is used to give a recommendation on a set of alternatives. For example, among many value functions representing pairwise comparisons of some alternatives made by the DM, only one value function is finally used to recommend the best choice, or sorting, or ranking of alternatives. Since the choice of one among many sets of parameters compatible with the preference information is rather arbitrary, *robust ordinal regression* (ROR) has been recently proposed with the aim of taking into account all the sets of parameters compatible with the preference information given by the DM (see Greco et al., 2008; Figueira et al., 2009; Greco et al., 2010).

The robust ordinal regression builds a set of additive value functions compatible with preference information provided by the DM and results in two rankings: the necessary ranking and the possible ranking. Such rankings answer to robustness concerns, since they provide, in general, "more robust" conclusions than a ranking made by an arbitrarily chosen compatible value function. However, in some decision-making situations, it may be desirable to give a score to different alternatives (solutions), and despite the interest of the rankings provided, some users would like to see, and they indeed need, to know the "most representative" value function among all the compatible ones. This allows assigning a score to each alternative. Recently, a methodology to identify the "most representative" function in ROR without losing the advantage of taking into account all compatible value functions has been proposed in Greco et al. (2011); Kadziński et al. (2012).

In this paper we will adopt the similar idea of providing robust conclusion by applying ROR to the $TOPSIS$ method. $TOPSIS$ ranks alternatives according to their closeness to two reference points: ideal and anti-ideal solutions in the normalized and weighted criteria space. The best alternative should have simultaneously the shortest distance to the ideal solution and the farthest distance to the anti-ideal solution (see Hwang and Yoon, 1981; Chen and Hwang, 1992).

To eliminate the impact of different physical scales on the final recommendation, a method like $TOPSIS$, and other methods based on the distance measure, need to normalize the multi-criteria evaluations before introducing the distance measure. This normalization, that transforms original evaluations into a common scale, is perceived as a weakness of such methods, because it is responsible for their non-meaningfulness (see Martel and Roy, 2006).

In the proposed extension of the $TOPSIS$ method we will consider the preference model in the form of a set of additive value functions compatible with the preference information given by the DM. We will propose a new way of calculating relative closeness score in the value space that takes into account all compatible value functions and provides a robust conclusion.

The paper is organized in the following way. Section 2 recalls some concepts of robust ordinal regression, as well as some elements of the GRIP method which is presently the most general of all UTA-like methods. Section 3 recalls basic concepts of the $TOPSIS$ method. Section 4 presents a new method that combines the robust ordinal regression and the ranking score based on the aggregate distance measure coming from the $TOPSIS$ method. Section 5 presents a didactic example. Last section contains conclusions.

2 Robust Ordinal Regression

Let us consider a multiple criteria decision problem where a finite set of m alternatives $A = \{a_1, \ldots, a_m\}$, is evaluated on a finite family $F = \{g_1, \ldots, g_n\}$ of n criteria. Let $I = \{1, \ldots, n\}$ denote the set of criteria indices, and assume, without loss of generality, that the greater $g_i(a)$, the better alternative a on criterion g_i, for all $i \in I$, $a \in A$, i.e. g_i are all gain-type criteria $(i = 1, \ldots, n)$.

A DM is willing to rank the alternatives of A from the best to the worst, according to his/her preferences. The ranking can be complete or partial, depending on the preference information provided by the DM and on the way of exploiting this information. The family of criteria F is supposed to satisfy some consistency conditions, i.e. completeness (all relevant criteria are considered), monotonicity (the better the evaluation of an alternative on considered criteria, the more it is preferable to another), and non-redundancy (no superfluous criteria are considered) (see Roy and Bouyssou, 1993). Such a decision-making problem statement is called a *multiple criteria ranking problem*.

It is well known that the only objective information coming out from the above problem statement is a *dominance relation* in the set A. In fact, alternative a_k dominates alternative a_l if a_k is at least as good as a_l on all criteria from F, and there is at least one criterion from F such that a_k is better than a_l. Moreover, there is no doubt that then a_k should be comprehensively considered at least as good as a_l, independently of the specific preferences of the DM. Instead, when a_k is not dominating a_l, the statement that a_k is at least as good as a_l depends on the preferences of the DM. According to the dominance relation, alternative $a_k \in A$ is preferred to alternative $a_l \in A$ (denoted as $a_k \succ a_l$) if and only if $g_i(a_k) \geq g_i(a_l)$ for all $i \in I$, with at least one strict inequality; a_k is indifferent to a_l (denoted as $a_k \sim a_l$) if and only if $g_i(a_k) = g_i(a_l)$ for all $i \in I$; finally, a_k is incomparable with a_l (denoted as $a_k \ ? \ a_l$) otherwise, i.e. if $g_i(a_k) > g_i(a_l)$ for at least one criterion $i \in I$ and $g_j(a_k) < g_j(a_l)$ for at least one other criterion $j \in I$. Since incomparability is the most frequent situation, the dominance relation is usually very poor.

To enrich this relation, the DM has to provide preference information which is used to construct a preference model (also called an aggregation model) making the alternatives more comparable. This preference model induces a preference structure on the set A, whose proper exploitation permits to work out a ranking proposed to the DM.

The robust ordinal regression approach (ROR) extends the simple ordinal regression by taking into account not a single instance of the preference model compatible with the DM's preference information, but the whole set of compatible instances of the preference model. As a result of considering the whole set of compatible instances of the preference model, one gets two kinds of results with respect to each pair of alternatives $a_k, a_l \in A$:

- *necessary preference relation* $a_k \succsim^N a_l$, if and only if a_k is at least as good as a_l according to all instances of the preference model compatible with the preference information,
- *possible preference relation* $a_k \succsim^P a_l$, if and only if a_k is at least as good as a_l according to at least one instance of the preference model compatible with the preference information.

The necessary preference relation can be considered as robust with respect to the preference information. The robustness of the necessary preference relation refers to the fact that a given pair of alternatives compares in the same way whatever the instance of the preference model compatible with the preference information. Indeed, when no preference information is given, the necessary preference relation boils down to the dominance relation, and the possible preference relation is a complete relation. Every new item of the preference information, e.g., a pairwise comparison of some reference alternatives for which the dominance relation does not hold, is enriching the necessary preference relation and it is impoverishing the possible preference relation, so that they converge with the growth of the preference information. Such an approach gives also space for interactivity with the DM.

In what follows, the evaluation of each alternative $a \in A$ on each criterion $g_i \in F$ will be denoted by $g_i(a)$. Let G_i denote the value set (scale) of the criterion $g_i, i \in I$. Consequently, $G = \prod_{i \in I} G_i$ represents the evaluation space, and $a \in G$ denotes a profile of an alternative in such a space. We consider a *weak preference relation* $\succsim$ on A which means, for each pair of vectors, $a_k, a_l \in G,\ a_k \succsim a_l \Leftrightarrow$ "$a_k$is at least as good as a_l". This weak preference relation can be decomposed into an asymmetric and a symmetric part, as follows:

1) $a_k \succ a_l \equiv [a_k \succsim a_l$ and not $a_l \succsim a_k] \Leftrightarrow$ "a_k is preferred to a_l",

2) $a_k \sim a_l \equiv [a_k \succsim a_l$ and $a_l \succsim a_k] \Leftrightarrow$ "a_k is indifferent to a_l".

From a pragmatic point of view, it is reasonable to assume that $G_i = [\alpha_i, \beta_i]$, i.e. the evaluation scale on each criterion g_i is bounded, such that $\alpha_i < \beta_i$ are the worst and the best (finite) evaluations, respectively. Thus, $g_i : A \to G_i, i \in I$, therefore, each alternative $a \in A$ is associated with an evaluation vector $[g_1(a), \ldots, g_n(a)] \in G$.

The idea of considering the whole set of value functions compatible with the preference information provided by the DM was originally introduced in UTA^{GMS} (see Greco et al. (2008)). In this method, the preference information is given in the form of a partial preorder $\succsim$ on a subset of reference alternatives $A^R \subseteq A$ (i.e., a set of pairwise comparisons of reference alternatives), called a *reference preorder*. The reference alternatives are usually those contained in the set A for which the DM is able to express holistic preferences.

The method $GRIP$ (Generalized Regression with Intensities of Preference) proposed by Figueira et al. (2009) can be seen as an extension of UTA^{GMS} permitting to take into account additional preference information in form of comparisons of intensities of preference between some pairs of reference alternatives. These comparisons are expressed in two possible ways (not exclusive):

- comprehensively, on all criteria, that defines a partial preorder $\succsim^*$ on $A^R \times A^R$, such that, given $a_k, a_l, a_p, a_q \in A^R$, $(a_k, a_l) \succsim^* (a_p, a_q)$ means "a_k is preferred to a_l at least as much as a_p is preferred to a_q",

- partially, on any criterion, that defines a partial preorder $\succsim_i^*$ on $A^R \times A^R$, such that, given $a_k, a_l, a_p, a_q \in A^R$, $(a_k, a_l) \succsim_i^* (a_p, a_q)$ means "a_k is preferred to a_l at least as much as a_p is preferred to a_q, on criterion $g_i,\ i \in I$".

In what follows, after Figueira et al. (2009), we also consider the weak preference relation $\succsim_i$ on A being a complete preorder whose meaning is: for all $a_k, a_l \in A$, $a_k \succsim_i a_l \Leftrightarrow$ "a_k is at least as good as a_l on criterion g_i, $i \in I$." This relation is not provided by the DM, but it is obtained directly from the evaluation of alternatives a_k and a_l on criterion g_i, i.e., $a_k \succsim_i a_l \Leftrightarrow g_i(a_k) \geq g_i(a_l)$.

In ROR involving additive value functions as a preference model, it has the form $U(a) = \sum_{i \in I} u_i(a)$, where u_i are marginal value functions which are either:

(a) *piecewise-linear*,

(b) *general monotone, non-decreasing.*

In case (a), each range $[\alpha_i, \beta_i]$ is divided into $k_i \geq 1$ equal sub-intervals $[x_i^0, x_i^1]$, $[x_i^1, x_i^2]$, ..., $[x_i^{k_i-1}, x_i^{k_i}]$, where $x_i^j = \alpha_i + \frac{j}{k_i}(\beta_i - \alpha_i)$, $j = 0, \ldots, k_i$, and $i \in I$. The marginal value of an alternative $a \in A$ is obtained by linear interpolation:

$$u_i(a) = u_i(x_i^j) + \frac{g_i(a) - x_i^j}{x_i^{j+1} - x_i^j}(u_i(x_i^{j+1}) - u_i(x_i^j)), \quad g_i(a) \in [x_i^j, x_i^{j+1}]$$

The piecewise-linear additive model is completely defined by the marginal values at the breakpoints, i.e., $u_i(x_i^0) = u_i(\alpha_i)$,$u_i(x_i^1)$,$u_i(x_i^2)$,..., $u_i(x_i^{k_i}) = u_i(\beta_i)$, $i \in I$. The number of linear pieces k_i is fixed a priori for each marginal value function u_i, $i \in I$.

In case (b), the characteristic points of the marginal value functions u_i, $i \in I$ are fixed in evaluation points of considered alternatives. Let l_i be the permutation on the set of indices of alternatives from A^R that reorders them according to the increasing evaluation on criterion i, i.e.:

$$\alpha_i \leq x_{l_i(1)} \leq x_{l_i(2)} \leq \ldots \leq x_{l_i(m-1)} \leq x_{l_i(m)} \leq \beta_i, \; i \in I$$

The general, non-decreasing additive model is completely defined by the marginal values at the characteristic points, i.e., $u_i(\alpha_i)$,$u_i(x_{k_i(1)})$, $u_i(x_{k_i(2)})$, ..., $u_i(x_{k_i(m)}), u_i(\beta_i)$. Note that in this case, no linear interpolation is required to express the marginal value of any reference alternative.

A value function is called *compatible* if it is capable of restoring the partial preorder $\succsim$ on A^R, as well as the given relation of intensity of preference among ordered pairs of reference alternatives. Moreover, each compatible value function induces a complete preorder (ranking) on the whole set A.

In particular, for any two alternatives $a_k, a_l \in A$, a compatible value function U ranks a_k and a_l in one of the following ways:

- a_k is preferred to a_l because $U(a_k) > U(a_l)$,

- a_l is preferred to a_k because $U(a_k) < U(a_l)$,
- a_k is indifferent to a_l because $U(a_k) = U(a_l)$.

With respect to $a_k, a_l \in A$, it is thus reasonable to ask the following two questions:

- are a_k and a_l ranked in the same way by *all* compatible value functions?
- is there *at least one* compatible value function ranking a_k at least as good as a_l (or a_l at least as good as a_k)?

Having answers to these questions for all pairs of alternatives $(a_k, a_l) \in A \times A$, one gets a *necessary weak preference relation* $\succsim^N$, if $U(a_k) \geq U(a_l)$ for all compatible value functions, and a *possible weak preference relation* $\succsim^P$ in A, if $U(a_k) \geq U(a_l)$ for at least one compatible value function.

Let us remark that preference relations $\succsim^N$ and $\succsim^P$ are meaningful only if there exists at least one compatible value function. Therefore, whenever the contrary is not explicitly stated, we suppose that there exists at least one compatible value function. Observe also that in this case, for any $a_k, a_l \in A^R$:

$$a_k \succsim a_l \Rightarrow a_k \succsim^N a_l, \text{ and } a_k \succ a_l \Rightarrow \text{ not } (a_l \succsim^P a_k).$$

In fact, if $a_k \succsim a_l$, then for any compatible value function, $U(a_k) \geq U(a_l)$ and, therefore, $a_k \succsim^N a_l$. Moreover, if $a_k \succ a_l$, then for any compatible value function, $U(a_k) > U(a_l)$ and, consequently, there is no compatible value function such that $U(a_l) \geq U(a_k)$, which means that not $(a_l \succsim^P a_k)$.

Formally, an additive compatible value function is an additive value function $U(a) = \sum_{i \in I} u_i(a)$ satisfying the following set of constraints corresponding to the DM's preference information:

a) $U(a_k) \geq U(a_l) + \varepsilon$ if $a_k \succ a_l$,
b) $U(a_k) = U(a_l)$ if $a_k \sim a_l$,
c) $U(a_k) - U(a_l) \geq U(a_p) - U(a_q) + \varepsilon$ if $(a_k, a_l) \succ^* (a_p, a_q)$,
d) $U(a_k) - U(a_l) = U(a_p) - U(a_q)$ if $(a_k, a_l) \sim^* (a_p, a_q)$,
e) $u_i(a_k) - u_i(a_l) \geq u_i(a_p) - u_i(a_q) + \varepsilon$ if $(a_k, a_l) \succ_i^* (a_p, a_q),\ i \in I$,
f) $u_i(a_k) - u_i(a_l) = u_i(a_p) - u_i(a_q)$ if $(a_k, a_l) \sim_i^* (a_p, a_q),\ i \in I$,

where $a_k, a_l, a_p, a_q \in A^R$, and $\varepsilon > 0$. Moreover, the following monotonicity and normalization constraints are also taken into account:

g) $u_i(a_k) \geq u_i(a_l)$ if $a_k \succsim_i a_l,\ \forall a_k, a_l \in A,\ i \in I$,
h) $u_i(\alpha_i) = 0$, where $\alpha_i = \min\{g_i(a) \colon a \in A\}$,

i) $\sum_{i\in I} u_i(\beta_i) = 1$, where $\beta_i = \max\{g_i(a)\colon a \in A\}$.

If the constraints from a) to i) are satisfied, then there exists at least one compatible value function, and the partial preorders $\succsim$ and $\succsim^*$ on A^R and $A^R \times A^R$ can be extended on A and $A \times A$, respectively.

To avoid the use of an arbitrary value of ε, we consider it as an auxiliary variable, and we test the feasibility of constraints a), c) and e). In this way, we take into account all possible value functions, even those for which the threshold ε is very small. In fact, the value of ε is not meaningful by itself and it is useful only because it permits to discriminate preference from indifference.

Therefore, to conclude about the truth or falsity of binary relations $\succsim^N$ and $\succsim^P$, for all $a_k, a_l \in A$ and $i \in I$, one has to solve a series of linear programming problems with ε as the objective function to be maximized, as explained below (Greco et al., 2008):

1) $a_k \succsim^P a_l \Leftrightarrow \varepsilon^* > 0$,
where $\varepsilon^* = \max\,\varepsilon$, subject to the constraints a), b), c), d), e), f), g), h), i), plus the constraint: $U(a_k) - U(a_l) \geq 0$;

2) $a_k \succsim^N a_l \Leftrightarrow \varepsilon^* \leq 0$,
where $\varepsilon^* = \max\,\varepsilon$, subject to the constraints a), b), c), d), e), f), g), h), i), plus the constraint: $U(a_l) - U(a_k) \geq \varepsilon$;

Analogously, one can test relations $\succsim^{*^N}$, $\succsim^{*^P}$, $\succsim_i^{*^N}$ and $\succsim_i^{*^P}$, for all $a, b, c, d \in A$ and $i \in I$ (see Figueira et al., 2009).

3 *TOPSIS* method

TOPSIS (Technique for Order Preference by Similarity to an Ideal Solution) is a multiple criteria decision method to rank alternatives or to select the best alternative from a finite set of alternatives. It was initially proposed by Hwang and Yoon (1981) for solving a multiple attribute decision making problem with no articulation of preference information. The original *TOPSIS* concept and its various extensions have been widely used in the literature (Lai et al., 1994; Chen, 2000; Deng et al., 2000; Chu, 2002; Braglia et al., 2003; Liao, 2003; Chen and Tzeng, 2004; Olson, 2004; Opricovic and Tzeng, 2004; Abo-Sinna and Amer, 2005).

The basic principle of *TOPSIS* is that the best alternative should have simultaneously the shortest distance from the ideal alternative and the farthest distance from the anti-ideal alternative (see Chen and Hwang, 1992). Assuming, without loss of generality, that all criteria are of gain-type, the ideal alternative (called a positive ideal solution) is an alternative that maximizes each individual criterion, whereas the anti-ideal alternative (called a negative ideal solution) minimizes each individual criterion.

Here, we are adopting the same notation as in Section 2, with the exception that alternatives are numbered and the identifying index is j, i.e. $g_i(a_j) = g_{ji},\ i = 1,\ldots,n;\ j = 1,\ldots,m$. All evaluations of the alternatives on particular criteria are presented in a decision matrix denoted by $\left[g_{ji}\right]_{m\times n}$. It is also assumed that the DM has determined the relative weights of criteria (denoted as w_i, for $i = 1,\ldots,n$), satisfying $\sum_{i=1}^{n} w_i = 1$.

Then, the steps of the *TOPSIS* method can be expressed in the following way:

1. Construct the normalized decision matrix $\left[r_{ji}\right]_{m\times n}$. The normalized value r_{ji} is calculated as:

$$r_{ji} = \frac{g_{ji}}{\sqrt{\sum_{j=1}^{m} g_{ji}^2}}, \quad i = 1,\ldots,n;\ j = 1,\ldots,m$$

   In this step the various criteria dimensions are transformed into non-dimensional attributes, which allows comparison across the criteria. It should be noted that different kinds of normalization procedure usually produce different rankings of alternatives. More information about the normalization can be found in (Opricovic and Tzeng, 2004) and in (Martel and Roy, 2006).

2. Construct the weighted normalized decision matrix $\left[v_{ji}\right]_{m\times n}$. The weighted normalized value v_{ji} is calculated as:

$$\left[v_{ji}\right] = \left[w_i\right]\left[r_{ji}\right], \quad i = 1,\ldots,n;\ j = 1,\ldots,m$$

3. Determine the ideal and anti-ideal solutions (alternatives):

$$a^+ = \left\{v_1^+,\ldots,v_n^+\right\} = \left\{\max_j v_{ji},\ j = 1,\ldots,m\right\},$$

$$a^- = \left\{v_1^-,\ldots,v_n^-\right\} = \left\{\min_j v_{ji},\ j = 1,\ldots,m\right\}.$$

4. Calculate the distance measures of each alternative, using the n-dimensional Euclidean distance. The distance of the alternative $a_j \in A$ to the ideal solution (denoted as $d^+(a_j)$) and the anti-ideal solution (denoted as $d^-(a_j)$) is given as:

$$d^+(a_j) = \sqrt{\sum_{i=1}^{n}(v_{ji} - v_i^+)^2}, \quad j = 1,\ldots,m,$$

$$d^-(a_j) = \sqrt{\sum_{i=1}^{n}(v_{ji} - v_i^-)^2}, \quad j = 1, \ldots, m.$$

5. Calculate the relative closeness of each alternative to the ideal solution. The relative closeness of the alternative $a_j \in A$ (denoted as $c^*(a_j)$) with respect to a^+ is defined as:

   $$c^*(a_j) = \frac{d^-(a_j)}{d^+(a_j) + d^-(a_j)}, \quad j = 1, \ldots, m$$

   Since $d^+(a_j) \geqslant 0$ and $d^-(a_j) \geqslant 0$, then $c^*(a_j) \in [0, 1]$ for each $a_j \in A$. Moreover, $c^*(a_j) = 1$ if $a_j = a^+$ and $c^*(a_j) = 0$ if $a_j = a^-$.

6. Rank the alternatives in the descending order of relative closeness c^*. The best alternative is the one with the greatest relative closeness to the ideal solution.

Note that the *TOPSIS* method introduces two reference points, but the way of calculating relative closeness does not take into account the relative importance of the distances from these points.

4 ROR applied to *TOPSIS*

As announced in the Introduction, in this section we present a new method for ranking a finite set of alternatives that combines the robust ordinal regression approach and the relative closeness ranking score based on the aggregate distance measure function coming from the *TOPSIS* method.

In the proposed approach, the preference model is composed of a set of additive value functions compatible with the preference information given by the DM. We consider marginal value functions having one of two forms: *piecewise-linear*, or *general monotone, non-decreasing*. The preference information is composed of a non-complete set of pairwise comparisons of reference alternatives, and of comparisons of intensities of preference between some pairs of reference alternatives. These comparisons are expressed in two possible, not exclusive ways: comprehensively, i.e. with respect to all criteria, and partly, i.e. with respect to particular criteria. Based on this set of compatible value functions, we solve a series of non-linear programming problems to introduce an aggregate function representing closeness in the *value space* of each alternative to the two reference points: the ideal, and the anti-ideal solutions. The ranking score determined by this distance measure function is then used to rank all alternatives.

Below we adopt notation introduced in previous sections. The detailed procedure of the proposed approach can be described in the following steps:

1. Determine the feasibility of the preference information provided by the DM, by solving the linear programming problem $\varepsilon^*= \max \varepsilon$, subject to the constraints from a) to i) (see Section 2), and testing the positive value of ε^*. Moreover, we want to use an additive value function compatible with preference information given by the DM that uses marginal value functions having as simple form as possible. To achieve this, we test the representability of the preference information in the simplest, linear form of the additive value function. If this test fails, we change the preference model to more complex, i.e. increase the number of linear pieces in piecewise-linear marginal value functions, or use the general monotone, non-decreasing marginal value functions.

2. Calculate the distance measures of each alternative, using the n-dimensional Euclidean distance in the value space with respect to the ideal and anti-ideal points. From among all compatible value functions we choose one that minimizes the Euclidean distance from the alternative considered to the ideal point a^+ in the value space. We assume that the ideal point in the value space has all co-ordinates equal to 1, i.e., $a^+ = \big[1, \ldots, 1\big]$. Such a point is certainly non-attainable for any preference information. It is a stable reference point for all alternatives throughout the entire procedure. Analogously, from among all compatible value functions we choose one that maximizes the Euclidean distance from the alternative considered to the anti-ideal point a^- in the value space. The anti-ideal point is defined as $a^- = \big[0, \ldots, 0\big]$ in the value space. It is also constant for the whole procedure. Therefore, the distance of the alternative $a_j \in A$ to the ideal point (denoted as $d^+(a_j)$) in the value space can be obtained as the result of the following non-linear programming problem:

$$\text{Minimize: } d^+(a_j) = \sqrt{\sum_{i=1}^{n} \big(u_i(a_j) - 1\big)^2}$$

subject to the following set of constraints corresponding to the DM's preference information:

a') $c^*(a_k) \geq c^*(a_l) + \varepsilon$ if $a_k \succ a_l$,

b') $c^*(a_k) = c^*(a_l)$ if $a_k \sim a_l$,

c') $c^*(a_k) - c^*(a_l) \geq c^*(a_p) - c^*(a_q) + \varepsilon$ if $(a_k, a_l) \succ^* (a_p, a_q)$, d') $c^*(a_k) - c^*(a_l) = c^*(a_p) - c^*(a_q)$ if $(a_k, a_l) \sim^* (a_p, a_q)$,

e') $u_i(a_k) - u_i(a_l) \geq u_i(a_p) - u_i(a_q) + \varepsilon$ if $(a_k, a_l) \succ_i^* (a_p, a_q), i \in I$,

f') $u_i(a_k) - u_i(a_l) = u_i(a_p) - u_i(a_q)$ if $(a_k, a_l) \sim_i^* (a_p, a_q), i \in I$,

where $a_k, a_l, a_p, a_q \in A^R$, ε is a small positive value, and $c^*(a_h)$ is the relative closeness of the alternative $a_h \in A^R$ to the ideal solution in the value space, given as:

$$c^*(a_h) = \frac{\sqrt{\sum_{i=1}^{n} \big(u_i(a_h)\big)^2}}{\sqrt{\sum_{i=1}^{n} \big(u_i(a_h)\big)^2} + \sqrt{\sum_{i=1}^{n} \big(u_i(a_h) - 1\big)^2}}$$

Moreover, the monotonicity and normalization constraints g), h) and i) (see Section 2) are also taken into account. Analogously, the distance of the alternative $a_j \in A$ to the anti-ideal point (denoted as $d^-(a_j)$) in the value space can be obtained as the result of the following non-linear programming problem:

$$\text{Maximize: } d^-(a_j) = \sqrt{\sum_{i=1}^{n} \big(u_i(a_j)\big)^2}$$

subject to constraints a'), b'), c'), d'), e'), f'), g), h) and i).

3. Calculate the ranking score of each alternative which is the relative closeness to the ideal solution in the value space. The relative closeness of the alternative $a_j \in A$ (denoted by $c^*(a_j)$) with respect to the ideal solution a^+ is defined as:

$$c^*(a_j) = \frac{d_*^-(a_j)}{d_*^+(a_j) + d_*^-(a_j)}, \quad j = 1, \ldots, m,$$

where $d_*^+(a_j)$ and $d_*^-(a_j)$ are optimal distances resulting from step 2.

4. Rank the alternatives in the descending order of ranking score c^*. The best alternative is the one with the greatest relative closeness to the ideal solution in the value space.

It should be noted that the proposed way of calculating distance in the value space takes into account all value functions compatible with the given preference information. Moreover, it permits to avoid the normalization procedure that is used in *TOPSIS* to transform original evaluations on different scales into a common scale.

5 A didactic example

To illustrate the method proposed in the previous Section, we present the following problem inspired from practice.

Suppose that a DM in the public transport company has to rank order 12 buses taking into account the following criteria considered during the periodical technical inspection of the buses:

- MaxSpeed [gain] – maximum speed [km/h],
- Torque [gain] – torque [Nm],
- FuelCons [cost] – fuel consumption [l/100km],
- OilCons [cost] – oil consumption [l/100km],
- HorsePow [gain] – maximum horsepower of the engine [KM].

Criteria “MaxSpeed”, “FuelCons” evaluate the overall performance of the bus, while others concentrate on the characteristics of the engine. Evaluations of all buses considered on the above criteria are given in Table 1.

BusId	MaxSpeed	Torque	FuelCons	OilCons	HorsePow
b01	90	426	27	2	112
b02	90	425	27	1	112
b03	87	400	23	4	96
b04	86	448	26	1	120
b05	83	402	26	2	128
b06	82	428	33	2	121
b07	80	445	26	1	122
b08	71	480	23	0	119
b09	75	449	26	1	120
b10	74	430	25	2	115
b11	72	479	35	1	145
b12	68	440	26	2	126

Table 1
Evaluations of 12 buses on 5 criteria

Suppose the DM provided the following preference information:

1. Pairwise comparisons of some buses:
   - bus b06 is preferred to bus b03: b06 $\succ$ b03,
   - bus b01 is indifferent to bus b02: b01 $\sim$ b02,

2. Overall intensity of preference:
   - bus b04 is preferred to bus b08 stronger than bus b07 is preferred to bus b06: (b04, b08) $\succ$ (b07, b06),
3. Intensity of preference on criterion "Torque":
   - with respect to "Torque", bus b02 is preferred to bus b03 stronger than bus b04 is preferred to bus b06:
     (b02, b03) $\succ_{Torque}$ (b04, b06).

Applying the proposed method to the above preference information, and using the linear model of preferences, we got distances to the ideal solution (d^+) and to the anti-ideal solution (d^-) in the value space, presented in Table 2.

Table 2
Distances of 12 buses to the ideal and the anti-ideal solutions in the value space

BusId	d^+	d^-
b01	2.0968	0.3116
b02	2.0968	0.3116
b03	2.2227	0.0302
b04	2.0518	0.4654
b05	2.0239	0.6197
b06	2.0519	0.4843
b07	2.0469	0.5035
b08	2.0652	0.4454
b09	2.0601	0.4646
b10	2.0900	0.3678
b11	1.9893	0.9483
b12	2.0394	0.5806

Table 3
Final ranking of 12 buses according to relative closeness scores in the value space

Rank	c^*	BusId
1	0.3296	b11
2	0.2406	b05
3	0.2223	b12
4	0.2027	b01
4	0.2027	b02
5	0.2025	b07
6	0.1945	b06
7	0.1912	b04
8	0.1876	b09
9	0.1830	b08
10	0.1669	b03
11	0.1527	b10

Based on the above distance measures we calculate the relative closeness score (c^*) of each alternative to the ideal solution in the value space. Then, we rank all alternatives according to the descending order of this relative closeness. The final ranking including relative closeness scores is presented in Table 3.

6 Conclusion

We presented a new MCDA ranking method that combines the advantages of the robust ordinal regression approach and the ranking score based on the aggregate distance measure function coming from the $TOPSIS$ method. In this method, the preference model is composed of a set of additive value functions compatible with the preference information given by the DM. Based on this set of compatible value functions, we introduce an aggregate function representing relative closeness to the reference point (ideal solution) in the value space. The ranking score determined by this distance measure function is then used to rank all alternatives.

The main advantages of the proposed method are:

1. It models the user's preferences in terms of additive value functions, that can be composed of linear, piecewise-linear or very general monotonic marginal value functions.

2. It takes into account the preference information expressed by the user in a very simple and intuitive way i.e. in the form of comparisons of some reference alternatives and/or in the form of intensities of preference between some pairs of reference alternatives. Moreover, intensities of preference can be specified comprehensively, on all criteria, or partly, on specific criteria.

3. It permits to detect inconsistent preference information with respect to an assumed form of the preference model. When the ordinal regression fails to find any compatible value function, the DM can remove this impossibility in one of two ways: the preference model can be changed to a more complex one (i.e. from piecewise-linear to general additive), or the inconsistent subset of pairwise comparisons can be detected and removed.

4. Calculating distance in the value space permits to avoid the normalization procedure that is used in $TOPSIS$ to transform original evaluations on different scales into a common unit. This normalization makes $TOPSIS$, and other methods based on the distance measure function, non-meaningful.

5. It can deal with ordinal scales of criteria while in the original $TOPSIS$ method the calculation of distance in the original ordinal criteria space does not make sense. Here, the ordinal criteria scales are translated

into marginal value scales which are interval scales. These scales give a correct interpretation to the distance in the value space.

6. It provides robust conclusions. The proposed relative closeness score used to rank the alternatives is based on all compatible value functions, rather than on only one or a few among the many possible functions, as it is usual in MCDA.

In brief, the robust ordinal regression applied to $TOPSIS$ not only facilitates the preference elicitation but also solves the problem of non-meaningfulness of $TOPSIS$.

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References

Abo-Sinna M.A., Amer A.H. (2005), *Extensions of TOPSIS for Multiobjective Large-scale Nonlinear Programming Problems*, Applied Mathematics and Computation, 162, p. 243–256.

Braglia M., Frosolini M., Montanari R. (2003), *Fuzzy TOPSIS Approach for Failure Mode, Effects and Criticality Analysis*, Quality and Reliability Engineering International, 19, p. 425–443.

Chen C.T. (2000), *Extension of the TOPSIS for Group Decision-making under Fuzzy Environment*, Fuzzy Sets and Systems, 114, p. 1–9.

Chen M.F., Tzeng G.H. (2004), *Combining Grey Relation and TOPSIS Concepts for Selecting an Expatriate Host Country*, Mathematical and Computer Modelling, 40, p. 1473–1490.

Chen S., Hwang C. (1992), in: *Fuzzy Multiple Attribute Decision Making:Methods and Applications*, *Lecture Notes in Economics and Mathematical Systems*, Vol. 375, Springer-Verlag.

Chu T.C. (2002), *Facility Location Selection Using Fuzzy TOPSIS under Group Decisions*, Fuzziness and Knowledge-Based Systems, 10, p. 687–701.

Deng H., Yeh C.H., Willis R.J. (2000), *Inter-company Comparison Using Modified TOPSIS with Objective Weights*, Computers and Operations Research, 27, p. 963–973.

Figueira J., Greco S., Ehrgott M. (2005), *Multiple Criteria Decision Analysis: State of the Art Surveys*, Springer, New York.

Figueira J., Greco S., Słowiński R. (2009), *Building a Set of Additive Value Functions Representing a Reference Preorder and Intensities of Preference: GRIP Method*, European Journal of Operational Research, 195(2), p. 460–486.

Greco S., Kadziński M., Słowiński R. (2011), *Selection of a Representative Value Function in Robust Multiple Criteria Sorting*, Computers & Operations Research, 38, p. 1620–1637.

Greco S., Matarazzo B., Słowiński R. (2001), *Rough Sets Theory for Multicriteria Decision Analysis*, European Journal of Operational Research, 129, p. 1–47.

Greco S., Mousseau V., Słowiński R. (2008), *Ordinal Regression Revisited: Multiple Criteria Ranking with a Set of Additive Value Functions*, European Journal of Operational Research, 191, p. 415–435.

Greco S., Słowiński R., Figueira J., Mousseau V. (2010), *Robust ordinal regression*, in: *New Advances in Multiple Criteria Decision Analysis*, Ed. M. Ehrgott, et al., Springer, p. 273–320.

Hwang C., Yoon K. (1981), in: *Multiple Attribute Decision Making Methods and Applications*, Springer, Berlin Heidelberg.

Jacquet-Lagrèze E., Siskos Y. (1982), *Assessing a Set of Additive Utility Functions for Multicriteria Decision-making, the UTA Method*, European Journal of Operational Research, 10, p. 151–164.

Kadziński M., Greco S., Słowiński R. (2012), *Selection of a Representative Value Function in Robust Multiple Criteria Ranking and Choice*, European Journal of Operational Research, 217(3), p. 541–553.

Kiss L., Martel J., Nadeau R. (1994), *ELECCALC - an Interactive Software for Modelling the Decision Maker's Preferences*, Decision Support Systems, 12, p. 757–777.

Lai Y.J., Liu T.Y., Hwang C.L. (1994), *TOPSIS for MODM*, European Journal of Operational Research, 76, p. 486–500.

Liao H.C. (2003), *Using PCR-TOPSIS to Optimize Taguchis Multi-response Problem*, The International Journal of Advanced Manufacturing Technology, 22, p. 649–655.

March J. (1978), *Bounded Rationality, Ambiguity and the Engineering of Choice*, Bell Journal of Economics, 9, p. 587–608.

Martel J., Roy B. (2006), *Analyse de la signifiance de diverses procédures d'agrégation multicritere*, INFOR, 44, p. 191–214.

Mousseau V., Słowiński R. (1998), *Inferring an ELECTRE TRI Model from Assignment Examples*, Journal of Global Optimization, 12, p. 157–174.

Mousseau V., Słowiński R., Zielniewicz P. (2000), *A User-oriented Implementation of the ELECTRE-TRI Method Integrating Preference Elicitation Support*, Computers & Operations Research, 27, p. 757–777.

Olson D.L. (2004), *Comparison of Weights in TOPSIS Models*, Mathematical and Computer Modelling, 40, p. 721–727.

Opricovic S., Tzeng G.H. (2004), *Compromise Solution by MCDM Methods: A Comparative Analysis of VIKOR and TOPSIS*, European Journal of Operational Research, 156, p. 445–455.

Pekelman D., Sen S. (1974), *Mathematical Programming Models for the Determination of Attribute Weights*, Management Science, 20, p. 1217–1229.

Roy B., Bouyssou D. (1993), *Aide Multicritère à la Décision : Méthodes et Cas*, Economica.

Srinivasan V., Shocker A. (1973), *Estimating the Weights for Multiple Attributes in a Composite Criterion Using Pairwise Judgments*, Psychometrika, 38, p. 473–493.