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MCDM APPLICATIONS OF NEAR OPTIMAL SOLUTIONS IN DYNAMIC PROGRAMMING

Abstract

One of the methods of scalarization of a multi-criteria problem is the application of a quasi-hierarchy, determined by the decision maker. In discrete problems, to apply this method it is necessary to have an algorithm which generates the optimal solution and the consecutive solutions, contained within the tolerance interval determined by the decision maker. This paper presents algorithms generating the consecutive realizations for a multi-stage deterministic decision-making process as well as an algorithm generating the consecutive strategies for a multi-stage stochastic decision-making process. Algorithms using these solutions in a multi-criteria quasi-hierarchical process are also proposed.

Keywords: multiple objective dynamic programming, quasi-hierarchy, *i*-th process realization, *i*-th strategy, optimality equations.

1 Introduction

In this paper we shall deal with discrete one- and multi-criteria decision-making problems, divided into a finite number of stages. Their characteristic feature is that for each individual stage of the problem, finite sets of feasible states are known, and for each state, the finite set of admissible decisions is also known.

In deterministic processes, the transition from one state to another in the consecutive stages is determined by transition functions, whose arguments are: the state of the process at the beginning of the given stage and the decision made. In stochastic processes we assume that we know the probabilities of the transition, depending on the state of the process at the beginning of the given stage, and of the decision made.

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A realization of the process in the deterministic case consists of a sequence of states and decisions, which transfer the process from an admissible start state to an end state, taking into account the relationships resulting from the transition function. In single-criterion problems we are interested in the optimal realization, that is, a realization maximizing the given multi-stage criterion function. This function is a composition (usually an additive one) of stage criterion functions. In multi-criteria problems we are interested in finding the set of non-dominated process realizations (which is usually very large).

In the stochastic case, a strategy is a function mapping each admissible state to a given decision. In single-criterion problems we are interested in the optimal strategy, that is, a strategy which maximizes the expected value of the multi-stage criterion function. In multi-criteria problems we are interested in finding the set of non-dominated strategies (which, as in the deterministic case, is usually very large).

When solving the problem of finding the optimal strategy of the process, we apply Bellman's optimality principle. Many applications of dynamic programming can be found already in early books in operations research, for instance, in Wager (1975). Multi-criteria decision-making processes were discussed by Trzaskalik, in Trzaskalik (1990, 1998) and in other papers. Extensions and applications for multi-criteria processes can be found, for instance, in Nowak, Trzaskalik (2014, 2013); Trzaskalik, Do Thien Hoa (1999); Trzaskalik, Sitarz (2007, 2009).

While in the single-criterion case usually only one optimal realization of the process exists, in the multi-criteria case the number of non-dominated realizations can be considerable. The search for the set of all efficient realizations can be difficult or even impossible. For that reason, various methods of scalarization of the multi-criteria problem are used.

One of the scalarization methods is the use of a hierarchy of criteria determined by the decision maker. This means that the decision maker is able to formulate a hierarchy of criteria so that the most important criterion is assigned the number 1; the number 2 is reserved for the second-most important criterion, and so on. We assume that all the criteria considered in the problem can be numbered in this way.

We solve the hierarchical problem sequentially. First we find the set of solutions which are optimal with respect to the most important criterion. Out of this set, we select the subset of solutions optimal with respect to the criterion number 2. We continue this procedure until we determine the subset of solutions which are optimal with respect to the least important criterion.

The hierarchical approach has a certain essential shortcoming. It turns out that very often the subset of solutions obtained when an important criterion in the hierarchy is considered has only one element. As a result, the selection of the solution with respect to less important criteria is determined and these criteria do not play an essential role in the process of determining the final solution. For that reason, a quasi-hierarchical approach is often applied. It consists in taking into account, once the (single-criterion) problem has been solved with respect to the most important criterion, not only the best solution, but also those solutions which are close to the optimal solution and contained within the tolerance interval determined in advance by the decision maker. Among the solutions found this way we find the best solution with respect to the second criterion and in the next step we take into account this solution as well as those solutions which are close to the optimal solution and contained within the tolerance interval with respect to the second criterion fixed in advance by the decision maker. This procedure is continued until the least important criterion.

In the application of the quasi-hierarchical procedure the possibility of generating not only the optimal solution, but also near optimal solutions, plays a key role. The solutions considered with respect to the consecutive criteria should be ordered so as to place the optimal solution first, the solution having the second value, second, etc. The ordering of solutions with respect to the first (most important) criterion is of particular importance. The consecutive solutions should be generated as long as they are contained within the tolerance intervals determined by the decision maker.

The problems of generating near optimal solutions in dynamic programming and related fields were taken up already in the past. Elmaghraby (1970) described a solution of the problem of seeking the k-th path between two arbitrary nodes in a graph. The search for the consecutive values in the multi-stage deterministic process was described in Trzaskalik (1990). But the problem of generating the consecutive realizations of a process has not been exhaustively described there. The problem of finding near optimal strategies in a decision tree and an application of the algorithm proposed to the quasi-hierarchical approach have been proposed by Nowak (2014), who has observed that the search for near optimal strategies can begin with a strategy differing from the optimal strategy by the decision in one state only. This approach, as applied to multi-criteria stochastic dynamic programming, was developed in Trzaskalik (2015).

The aim of this paper is to describe a method of finding the consecutive solutions in the stochastic and deterministic cases of single-criterion dynamic programming, and to apply this approach to finding solutions of multi-criteria quasi-hierarchical problems.

This paper consists of an introduction, two main sections, final remarks and two appendices. In the second section, which follows the introduction, we will describe deterministic discrete decision-making processes. We will show how to find the consecutive values of the criterion function and to generate the consecutive realizations of a process, on the basis of optimality equations. The algorithm obtained will be used in the quasi-hierarchical procedure proposed. In the third section we will describe stochastic processes. As in the deterministic case, we will show how to find the consecutive expected values of the criteria function and how to generate the consecutive strategies, on the basis of optimality equations. Next, we will present the quasi-hierarchical procedure for the stochastic case. Final remarks conclude the paper. Because of the importance and the degree of complexity of the algorithm generating the *i*-th realization of a process and the *i*-th strategy, complete solutions of these examples are in the appendices.

2 Deterministic case

2.1 *i*-th optimal value and *i*-th process realization

We will use the following notation (Trzaskalik, 1998, 2015):
T – number of stages of the decision process under consideration,
y_t – state of the process at the beginning of stage t ($t = 1,\ldots,T$),
$\mathbf{Y}_t$ – finite set of process states at stage t,
$\mathbf{Y}_{T+1}$ – finite set of process states at the end of the process,
x_t – feasible decision at stage t,
$\mathbf{X}_t(y_t)$ – finite set of decisions feasible at stage t, when the process was in state $y_t \in \mathbf{Y}_t$ at the beginning of this stage,
d_t – process realization in the stage t; we have:

$$d_t = (y_t, x_t) \tag{1}$$

$\mathbf{D}_t$ – set of process realizations in stage t,
$\mathbf{\Omega}_t(y_t, x_t)$ – transition function; we have:

$$y_{t+1} = \mathbf{\Omega}_t(y_t, x_t) \tag{2}$$

d – process realization; we have:

$$d = ((y_1, x_1), (y_2, x_2), \ldots, (y_T, x_T)) \tag{3}$$

where:

$$y_1 \in \mathbf{Y}_1, x_1 \in \mathbf{X}_1(y_1)$$
$$y_2 = \mathbf{\Omega}_1(y_1, x_1)\ x_2 \in \mathbf{X}_2(y_2)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y_T = \mathbf{\Omega}_{T\text{-}1}(y_{T-1}, x_{T-1})\ x_T \in \mathbf{X}_T(y_T)$$
$$y_{T+1} = \mathbf{\Omega}_t(y_T, x_T)$$

$\mathbf{D}$ – set of all process realizations,
$d_{\overline{t,T}}(y_t)$ – shortened realization, starting from y_t and encompassing stages from t to T; we have:

$$d_{\overline{t,T}}(y_t) = [(y_t, x_t), (y_{t+1}, x_{t+1}), \ldots, (y_T, x_T)] \tag{4}$$

$\mathbf{D}_{\overline{t,T}}(y_t)$ – set of all shortened realizations, starting from y_t and encompassing stages from t to T,

$F_t(d_t)$ – stage criterion function,

$F(d)$ – criterion function evaluating process realization d; we have:

$$F_t(d) = \sum_{t=1}^{T} F_t(d_t) \tag{5}$$

The finite set $\mathbf{D}$ of process realizations can be divided into M classes in such a way that:

$$\mathbf{D} = \mathbf{D}^1 \cup \mathbf{D}^2 \cup \ldots \mathbf{D}^{\mathrm{M}} \tag{6}$$

where:

$$\mathbf{D}^i \cap \mathbf{D}^j \text{ for } i \neq j \tag{7}$$

$$\forall_{i=1,\ldots,M} \forall_{d^j, d^k \in \mathbf{D}^i} F(d^j) = F(d^k) \tag{8}$$

$$\forall_{i<j} \forall_{\{x^i\} \in \{\mathbf{X}^i\}} \forall_{\{x^j\} \in \{\mathbf{X}^j\}} G\{x^i\} > G\{x^j\} \tag{9}$$

Let $d^1 \in \mathbf{D}^1$, $d^2 \in \mathbf{D}^2$, …, $d^M \in \mathbf{D}^M$ and $F(\mathbf{D})=\{F(d^1), \ldots, F(d^M)\}$. i-th process value is defined as G^i. We have:

$$G^i = F(d^i) \tag{10}$$

Each realization from the set $\mathbf{D}^i$ is named i-th process realization. We will use notation:

$$\max_i F(\mathbf{D}) = F^i(d) \tag{11}$$

The way of determining i-th process value and i-th process realization is described below.

Algorithm 1

1. Starting from $i = 1$ for each $y_T \in \mathbf{Y}_T$ we calculate the i-th value:

$$G_T^i(y_T) = \max_i F_T(y_T, x_T) \tag{12}$$

   and find the set of shortened process realizations $\mathbf{D}_{\overline{T,T}}(y_T)$, for which this value is attained.

2. Starting from $i = 1$ for stage t, $t \in \overline{T-1,1}$ and each $y_t \in \mathbf{Y}_t$ we calculate the i-th value:

$$G_t^i(y_t) = \max_{\substack{i \\ x_t \in \mathbf{X}(y_t)}} \{F_t(y_t, x_t) + G_{t+1}^j(\Omega_t(y_t, x_t)) : j = 1, \ldots, i\} \tag{13}$$

   and find the set of shortened process realizations $\mathbf{D}_t(y_t)$, for which this value is attained.

3. The i-th process value is calculated from the formula:

$$G^i = \max_i \{G_1^j(y_1) : j = 1, \ldots, i, y_1 \in \mathbf{Y}_1\} \tag{14}$$

4. The set of all i-th process realizations is calculated from the formula:

$$\mathbf{D}^i = \bigcup_{y_1 \in \mathbf{Y}_1} \{\mathbf{D}^j(y_1) : G^j(y_1) = G^i, j = 1, \ldots, i\} \tag{15}$$

Example 1

We consider a three-stage deterministic decision process. The sets of states for the consecutive stages are as follows:

$$\mathbf{Y}_1 = \{1,2\} \qquad \mathbf{Y}_2 = \{3,4\} \qquad \mathbf{Y}_3 = \{5,6\}$$

We have the following set of final states of the process:

$$\mathbf{Y}_4 = \{7,8\}$$

The sets of feasible decisions are as follows:

$$\mathbf{X}_1(1) = \{A, B\} \qquad \mathbf{X}_2(3) = \{E, F\} \qquad \mathbf{X}_3(5) = \{I, J\}$$
$$\mathbf{X}_1(2) = \{C, D\} \qquad \mathbf{X}_2(4) = \{G, H\} \qquad \mathbf{X}_3(6) = \{K,L\}$$

The graph of the process is given in Figure 1.

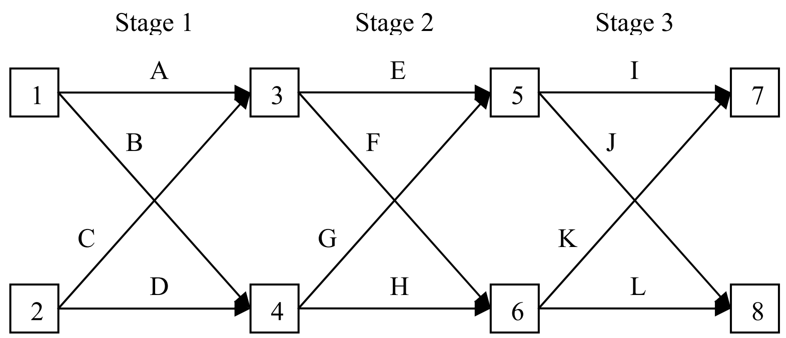

Figure 1. Graph of the process

The values of stage criteria are given in Table 1.

Table 1: Numerical values

Stage	(y_t, x_t)	$F^1(\cdot)$	$F^2(\cdot)$	$F^3(\cdot)$	Stage	(y_t, x_t)	$F_1(\cdot)$	$F_2(\cdot)$	$F_3(\cdot)$
1	(1, A)	6	120	13	2	(4, G)	6	140	16
1	(1, B)	8	110	11	2	(4, H)	4	128	20
1	(2, C)	5	115	14	3	(5, I)	4	102	16
1	(2, D)	9	117	12	3	(5, J)	3	107	15
2	(3, E)	5	132	15	3	(6, K)	5	103	12
2	(3, F)	3	135	14	3	(6, L)	2	101	10

For clarity and due to small size of this illustrative problem, the existing realizations can be written down and numbered from 1 to 16. This numbering is presented in Table 2.

Table 2: List of process realizations

No	Realization	No	Realization	No	Realization	No	Realization
1	(1,A,3,E,5,I)	5	(1,B,4,G,5,I)	9	(2,C,3,E,5,I)	13	(2,D,4,G,5,I)
2	(1,A,3,E,5,J	6	(1,B,4,G,5,I)	10	(2,C,3,E,5,J)	14	(2,D,4,G,5,J)
3	(1,A,3,F,6,K)	7	(1,B,4,H,6,IK	11	(2,C,3,F,6,K)	15	(2,D,4,H,6,K)
4	(1,A,3,F,6,L)	8	(1,B,4,H,6,L)	12	(2,C,3,F,6,L)	16	(2,D,4,H,6,L)

Applying Algorithm 1 for the criterion F^1 we obtain:

$$G^1 = 19,\ \mathbf{D}^1 = \{d^{13}\}$$

$$G^2 = 18,\ \mathbf{D}^2 = \{d^5, d^{14}, d^{15}\}$$

Detailed calculations can be found in Appendix 1.

2.2 MCDM quasi-hierarchical application

We will use the following notation:
K – number of considered criteria,
$F^k(d_t)$ – k-th stage criterion function ($k = 1, \ldots, K$),
$F^k(d)$ − k-th multistage criterion function evaluating process realization d,
ε_k – tolerance limit for k-th multistage criterion function.

We assume that the decision maker in his/her final selection applied the quasi-hierarchical approach. For this reason the criteria have been numbered appropriately, starting with the most important criterion, which is assigned the number 1.

Algorithm 2

1. Using Algorithm 1, find the optimal value $G^1(d)$ for the most important criterion F^1.
2. Ask the decision maker to determine ε^1 for the first criterion.
3. Using Algorithm 1, create the set:
$$\mathbf{D}^{(1)} = \{d \in \mathbf{D}: F^1((d) \geq G^1 - \varepsilon^1\} \quad (16)$$
containing these realizations of the process which are contained within the tolerance interval $[G^1 - \varepsilon^1, G^1]$, determined by the DM for the most important criterion.
4. Set $k = 2$.
5. Determine the optimal realization $d^{(k)}$ in $D^{(k-1)}$, with respect to the k-th criterion:
$$F^k(d^{(k)}) = \max F^k(d): d \in \mathbf{D}^{(k-1)}\} \quad (17)$$
6. Ask the DM to determine ε^k for the k-th criterion.
7. Create the set of realizations $\mathbf{D}^{(k)}$:
$$\mathbf{D}^{(k)} = \{d \in \mathbf{D}^{(k-1)}: F^k(d) \geq F^k(d^{(k)}) - \varepsilon^1\} \quad (18)$$
8. Set $k = k + 1$.
9. If $k \leq K$, go to Step 5.
10. Ask the DM to select the final realization from $\mathbf{D}^{(K)}$.
11. End of procedure.

The algorithm proposed will be illustrated by a numerical example.

Example 2

Now we regard the considered process as a three-criteria hierarchical process, in which the most important is the first criterion, the second-most important is the second criterion, and the least important is the third criterion. Numerical values of stage criteria are given in Table 1.

The determination of the final process realization using the quasi-hierarchical procedure described in **Algorithm 2** is performed as follows:

1. Using Algorithm 1 find the optimal value $G^1 = 19$ for the most important criterion (see Example 1).
2. Ask the DM to determine ε^1 for the first criterion. The DM set $\varepsilon^1 = 2$.
3. Using Algorithm 1, find the set:
$$\mathbf{D}^{(1)} = \{d \in \mathbf{D}: F^1((d) \geq 17\} = \{d^{13}, d^5, d^{14}, d^{15}, d^6, d^7\}$$
4. Set $k = 2$.
5. Determine the optimal realization in $\mathbf{D}^{(1)}$ with respect to the second criterion. To do this, we calculate:
$$F^2(d^{13}) = 359 \; F^2(d^5) = 352 \; F^2(d^{14}) = 364$$
$$F^2(d^{15}) = 348 \quad F^2(d^6) = 357 \; F^2(d^7) = 341$$
From among the values calculated choose the largest one. We have:
$$F^2(d^{(2)}) = F(d^{14}) = 364$$
6. Ask the DM to determine ε^2 for the second criterion. The DM set $\varepsilon^2 = 8$.
7. Create the set $\mathbf{D}^{(2)}$:
$$\mathbf{D}^{(2)} = \{d \in \mathbf{D}^{(1)}: F^2(d) \geq 356\} = \{d^{14}, d^{13}, d^6\}$$
8. Set $k = 3$.
9. Since $k \leq 3$, go to Step 5.
5. Determine the optimal realization in the set $\mathbf{D}^{(2)}$ with respect to the third criterion. To do this, we calculate:
$$F^3(d^{14}) = 43 \quad F^3(d^{13}) = 44 \; F^3(d^6) = 43$$
6. Ask the DM to determine ε^3 for the third criterion. The DM set $\varepsilon^3 = 1$.
7. Create the set $\mathbf{D}^{(3)}$:
$$\mathbf{D}^{(3)} = \{d \in \mathbf{D}^{(2)}: F^2(d) \geq 44\} = \{d^{14}, d^{13}, d^6\}$$
8. Set $k = 4$.
9. Since $k > 3$, go to Step 10.
10. Suggest the selection of the final realization from $\mathbf{D}^3$ to the DM. This selection can be aided by the values of the multi-stage criteria for the following process realizations:
$$F^1(d^{14}) = 18 \quad F^2(d^{14}) = 364 \; F^3(d^{14}) = 43$$
$$F^1(d^{15}) = 19 \quad F^2(d^{15}) = 359 \; F^3(d^{15}) = 44$$
$$F^1(d^6) = 17 \quad F^2(d^6) = 357 \; F^3(d^6) = 43$$
The DM prefers realization d^{14}.
11. End of procedure.

3 Stochastic case

3.1 *i*-th expected value and *i*-th process strategy

We will use additional notation:

$F_t(y_{t+1}| y_t, x_t)$ – value of stage criterion at stage t for the transition from state y_t to state $y_{t+1,}$ when the decision taken was $x_t \in \mathbf{X}_t(y_t)$,

$P_t(y_{t+1}| y_t, x_t)$ – probability of the transition at stage t from state y_t to state y_{t+1}, when the decision taken was $x_t \in \mathbf{X}_t(y_t)$; the following holds:

$$\forall_{t\in\overline{1,T}} \forall_{y_t\in \mathbf{Y}_t} \forall_{x_t\in \mathbf{X}_t(y_y)} \sum_{y_{t+1}\in \mathbf{Y}_{t+1}} P_t(y_{t+1} \mid y_t, x_t) = 1 \tag{19}$$

$\{x(y_1)\}$ – strategy starting from the state y_1 – a function assigning to y_1 and each state $y_t \in \mathbf{Y}_t$ ($t = 2, \ldots, T$) exactly one decision $x_t \in \mathbf{X}_t(y_t)$,

$\{\mathbf{X}(y_1)\}$ − set of all the strategies $\{x(y_1)\}$,

$\{\mathbf{X}\}$ – the set of all strategies of the process under consideration; we have:

$$\{\mathbf{X}\} = \bigcup_{y_1\in \mathbf{Y}_1} \{\mathbf{X}(y_1)\} \tag{20}$$

$\{x\} \in \{\mathbf{X}\}$ – a strategy starting from any state $y_1 \in \mathbf{Y}_1$,

$G\{x\}$ – expected value for strategy $\{x\}$:

$$G\{x^*\} = \max_{\{x\}\in\{\mathbf{X}\}} G\{x\} \tag{21}$$

$\{x_{\overline{t,T}}(y_t)\}$ – shortened strategy, starting from y_t and encompassing stages from t to T,

$\{\mathbf{X}_{\overline{t,T}}(y_t)\}$ – set of all shortened strategies, starting from y_t and encompassing stages from t to T.

Let us consider strategy $\{\overline{x}(y_1)\} \in \{\mathbf{X}(y_1)\}$ starting from any state y_1. The expected value for that strategy is calculated as follows:

Algorithm 3

1. For each state $y_T \in \mathbf{Y}_T$ calculate:

$$G_T(y_T, \{\overline{x}_{\overline{T,T}}\}) = \sum_{y_{T+1}\in \mathbf{Y}_{T+1}} F_T(y_{T+1} \mid y_T, \overline{x}_T) P_T(y_{T+1} \mid y_T, \overline{x}_T) \tag{22}$$

2. For each stage t, $t \in \overline{T-1,1}$ calculate the expected value:

$$G_t(y_t, \{\overline{x}_{\overline{t,T}}\}) = \sum_{y_{t+1}\in \mathbf{Y}_{t+1}} (F_t(y_{t+1} \mid y_t, \overline{x}_t) + G_{t+1}(y_{t+1}, \{\overline{x}_{\overline{t+1,T}}\})) P_t(y_{t+1} \mid y_t, \overline{x}_t) \tag{23}$$

The expected value G of the strategy $\{\overline{x}(y_1)\} \in \{\mathbf{X}(y_1)\}$ is equal to $G_1(y_1, \{x_{\overline{1,T}}(y_1)\})$

The finite set of all strategies $\{\mathbf{X}\}$ can be divided into M classes so, that:

$$\{\mathbf{X}\} = \{\mathbf{X}^1\} \cup \{\mathbf{X}^2\} \cup \ldots \cup \{\mathbf{X}^M\} \tag{24}$$

where:

$$\{\mathbf{X}^i\} \cap \{\mathbf{X}^j\} \quad \text{for } i \neq j \tag{25}$$

$$\forall_{i=1,\ldots,M} \forall_{\{x^k\},\{x^l\} \in \{\mathbf{X}^i\}} G\{x^k\} = G\{x^l\} \tag{26}$$

$$\forall_{i<j} \forall_{\{x^i\} \in \{\mathbf{X}^i\}} \forall_{\{x^j\} \in \{\mathbf{X}^j\}} G\{x^i\} > G\{x^j\} \tag{27}$$

$$G^i = F(d^i)$$

Let $\{x^1\} \in \{\mathbf{X}^1\}$, $\{x^2\} \in \{\mathbf{X}^2\}$, …, $\{x^M\} \in \{\mathbf{X}^M\}$ and $G\{\mathbf{X}\}=\{G\{x^1\}, \ldots, G\{x^M\}\}$. The i-th expected value is defined as G^i. We have:

$$G^i = G\{x^i\} \tag{28}$$

Each strategy from the set $\{\mathbf{X}^i\}$ is called an i-th strategy.

The method of determining the i-th expected value and the i-th optimal strategy is described below.

Algorithm 4

1. Starting from $i = 1$ for each $y_T \in \mathbf{Y}_T$ calculate the i-th expected value:

$$G_T^i(y_T) = \max_{x_T \in \mathbf{X}_T(y_T)}{}_i \sum_{y_{T+1} \in \mathbf{Y}_{T+1}} F_T(y_{T+1} \mid y_T, x_T) \cdot P_T(y_{T+1} \mid y_T, x_T) \tag{29}$$

   and find the set of shortened strategies $\{\mathbf{X}_{\overline{T,T}}(y_T)\}$, for which this value is reached.

2. Starting from $i = 1$ for stage t, $t \in \overline{T-1,1}$ and each $y_t \in \mathbf{Y}_t$ calculate the i-th expected value:

$$G_t^i(y_t) = \max_{x_t \in \mathbf{X}_t(y_t)}{}_i \left\{ \sum_{y_{T+1} \in \mathbf{Y}_{T+1}} \left[F_t(y_{t+1} \mid y_t, x_t) + G_{t+1}^j(y_{t+1})\right] \cdot P_T(y_{T+1} \mid y_T, x_T) : j = 1, \ldots, i \right\} \tag{30}$$

   and find the set of shortened strategies $\{\mathbf{X}_{\overline{t,T}}(y_t)\}$, for which this value is reached.

3. The i-th process value is calculated from the formula:

$$G^i = \max{}_i \left\{ G_1^j(y_1) : j = 1, \ldots, i, y_1 \in \mathbf{Y}_1 \right\} \tag{31}$$

4. The set of all i-th strategies is calculated from the formula:

$$\{\mathbf{X}^i\} = \bigcup_{y_1 \in \mathbf{Y}_1} \left\{ \{\mathbf{X}^j(y_1)\} : G^j(y_1) = G^i, j = 1, \ldots, i \right\} \tag{32}$$

Example 3

We consider a three-stage stochastic decision process. The sets of states for the consecutive stages are as follows:

$$\mathbf{Y}_1 = \{1,2\} \qquad \mathbf{Y}_2 = \{3,4\} \qquad \mathbf{Y}_3 = \{5,6\}$$

We have the following set of final states of the process:

$$\mathbf{Y}_4 = \{7,8\}$$

The sets of feasible decisions are as follows:

$$\mathbf{X}_1(1) = \{A, B\} \quad \mathbf{X}_2(3) = \{E, F\} \quad \mathbf{X}_3(5) = \{I, J\}$$
$$\mathbf{X}_1(2) = \{C, D\} \quad \mathbf{X}_2(4) = \{G, H\} \quad \mathbf{X}_3(6) = \{K,L\}$$

The graph of the process is given in Figure 2. Rectangles denote states of the process in the consecutive stages, circles − random nodes.

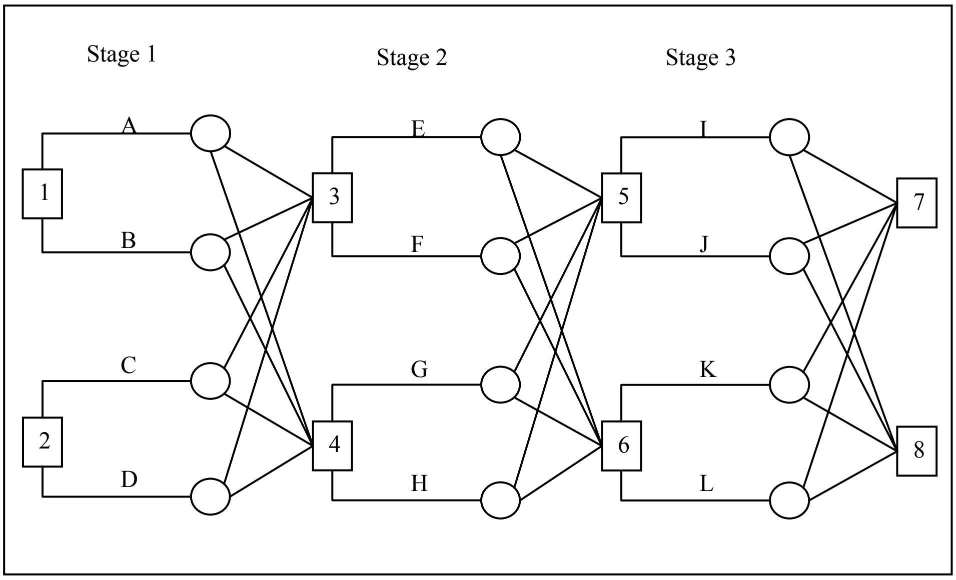

Figure 2. Graph of the process

The possible stage realizations of the process, probabilities of their occurrence, as well as the values of the stage criteria functions are shown in Table 3.

Table 3: Numerical values

Stage	$(y_{t+1}\| y_t, x_t)$	$P(\cdot)$	$F^1(\cdot)$	$F^2(\cdot)$	$F^3(\cdot)$	Stage	$(y_{t+1}\| y_t, x_t)$	$P(\cdot)$	$F^1(\cdot)$	$F^2(\cdot)$	$F^3(\cdot)$
1	(3| 1,A)	0.4	6	15	22	2	(5| 4,G)	0.6	5	15	20
1	(4| 1,A)	0.6	8	17	14	2	(6| 4,G)	0.4	6	18	13
1	(3| 1,B)	0.7	6	15	22	2	(5| 4,H)	0.8	5	15	20
1	(4| 1,B)	0.3	8	17	14	2	(6| 4,H)	0.2	6	18	13
1	(3| 2,C)	0.5	6	15	22	3	(7| 5,I)	0.8	5	30	12
1	(4| 2,C)	0.5	8	17	14	3	(8| 5,I)	0.2	1	12	15
1	(3| 2,D)	0.8	6	15	22	3	(7| 5,J)	0.3	5	30	12
1	(4| 2,D)	0.2	8	17	14	3	(8| 5,J)	0.7	1	12	15
2	(5| 3,E)	0.5	5	15	20	3	(7| 6,K)	0.2	5	30	12
2	(6| 3,E)	0.5	6	18	13	3	(8| 6,K)	0.8	1	12	15
2	(5| 3,F)	0.3	5	15	20	3	(7| 6,L)	0.9	5	30	12
2	(6| 3,F)	0.7	6	18	13	3	(8| 6,L)	0.1	1	12	15

For clarity and due to small size of this illustrative problem, the existing strategies can be written down and numbered from 1 to 64. This numbering is presented in Table 4.

Table 4: List of strategies

No	Decision	No	Decision	No	Decision	No	Decision
1	(A,_,E,G,I,K)	17	(B,_,E,G,I,K)	33	(_,C,E,G,I,K)	49	(_,D,E,G,I,K)
2	(A,_,E,G,I,L)	18	(B,_,E,G,I,L)	34	(_,C,E,G,I,K)	50	(_,D,E,G,I,L)
3	(A,_,E,G,J,K)	19	(B,_,E,G,J,K)	35	(_,C,E,G,J,K)	51	(_,D,E,G,J,K)
4	(A,_,E,G,J,L)	20	(B,_,E,G,J,L)	36	(_,C,E,G,J,L)	52	(_,D,E,G,J,L)
5	(A,_,E,H,I,K)	21	(B,_,E,H,I,K)	37	(_,C,E,H,I,K)	53	(_,D,E,H,I,K)
6	(A,_,E,H,I,L)	22	(B,_,E,H,I,L)	38	(_,C,E,H,I,L)	54	(_,D,E,H,I,L)
7	(A,_,E,H,J,K)	23	(B,_,E,H.J,K)	39	(_,C,E,H,J,K)	55	(_,D,E,H,J,K)
8	(A,_,E,H,J,L)	24	(B,_,E,H,J,L)	40	(_,C,E,H,J,L)	56	(_,D,E,H,J,L)
9	(A,_,F,G,I,K)	25	(B,_,F,G,I,K)	41	(_,C,F,G,I,K)	57	(_,D,F,G,I,K)
10	(A,_,F,G,I,L)	26	(B,_,F,G,I,L)	42	(_,C,F,G,I,L)	58	(_,D,F,G,I,L)
11	(A,_,F,G,J,K)	27	(B,_,F,G,J,K)	43	(_,C,F,G,J,K)	59	(_,D,F,G,J,K)
12	(A,_,F,G,J,L)	28	(B,_,F,G,J,L)	44	(_,C,F,G,J,L)	60	(_,D,F,G,J,L)
13	(A,_,F,H,I,K)	29	(B,_,F,H,I,K)	45	(_,C,F,H,I,K)	61	(_,D,F,H,I,K)
14	(A,_,F,H,I,L)	30	(B,_,F,H,I,L)	46	(_,C,F,H,I,L)	62	(_,D,F,H,I,L)
15	(A,_,F,H,J,K)	31	(A,D,F,H,J,K)	47	(_,C,F,H,J,K)	63	(_,D,F,H,J,K)
16	(A,_,F,H,J,L)	32	(A,D,F,H,J,L)	48	(_,C,F,H,J,L)	64	(_,D,F,H,J,L)

Applying Algorithm 3 for the criterion F^1 we obtain:

$$G^1 = 17.128, \quad \{\mathbf{X}^1\} = \{x^{10}\}$$
$$G^2 = 17.016, \quad \{\mathbf{X}^2\} = \{x^2\}$$

Detailed calculations can be found in Appendix 2.

3.2 MCDM quasi-hierarchical application

We assume again that the decision maker, in his/her final selection, applies the quasi-hierarchical approach. For this reason the criteria have been numbered appropriately, starting with the most important one, which is assigned the number 1.

Algorithm 5

1. Using Algorithm 4 find the expected optimal value G^1 for the most important criterion F^1.
2. Ask the DM to determine ε^1 for the first criterion.
3. Using Algorithm 1, create the set:
$$\{\mathbf{X}^{(1)}\} = \{\{x\} \in \{\mathbf{X}\}: G^1\{x\} \geq G^1 - \varepsilon^1\} \tag{33}$$
which contains, for the most important criterion (number 1) and for each initial state $y_1 \in \mathbf{Y}_1$, the strategies which are contained within the tolerance interval $[G^1 - \varepsilon^1, G^1]$, given by the DM.

4. Set $k = 2$.
5. Determine strategy $\{x^{(k)}\}$ in the set $\{\mathbf{X}^{(k-1)}\}$ which is optimal with respect to the k-th criterion:

$$G^k\{x^{(k)}\} = \max\{G^k\{x^k\}: \{x\} \in \{\mathbf{X}^{(k-1)}\}\} \tag{34}$$

6. Ask the DM to determine ε^k for the k-th criterion.
7. Create the set of strategies $\{\mathbf{X}^{(k)}\}$:

$$\{\mathbf{X}^{(k)}\} = \{\{x\} \in \{\mathbf{X}^{(k-1)}: G^k\{x\} \geq G^k\{x^{(k)}\} - \varepsilon^k\ \} \tag{35}$$

8. Set $k = k + 1$.
9. If $k \leq K$, go to Step 5.
10. Ask the DM to select a strategy from the set $\{\mathbf{X}^{(K)}\}$.
11. End of procedure.

The algorithm proposed will be illustrated by a numerical example.

Example 4

Now we regard the considered process as a three-criteria hierarchical process, in which the most important is the first criterion, the second-most important is the second criterion, and the least important is the third criterion. Numerical values of stage criteria are given in Table 1.

The determination of the final strategy using the quasi-hierarchical procedure described in **Algorithm 5** is performed as follows:

1. Using Algorithm 4 find the expected optimal value $G^1 = 17.128$ (see Example 3) for the most important criterion.
2. Ask the DM to determine ε^1 for the first criterion. The DM set $\varepsilon^1 = 0.342$.
3. Using Algorithm 3, find the set:

$$\{\mathbf{X}^{(1)}\} = \{\{x\} \in \{\mathbf{X}\}: G^1\{x\} \geq 16.585\} =$$
$$= \{\{x^{10}\}, \{x^2\}, \{x^{42}\}, \{x^{14}\}, \{x^6\}, \{x^{34}\}, \{x^{46}\}\}$$

4. Set $k = 2$.
5. Determine the strategy $\{x^{(2)}\}$ in $\{\mathbf{X}^{(1)}\}$ which is optimal with respect to the second criterion. To do this, we calculate:

$$G^2\{x^{10}\} = 48.94\ , \quad G^2\{x^2\} = 48.376, \quad G^2\{x^{42}\} = 55.104, \quad G^2\{x^{14}\} = 49.08$$
$$G^2\{x^6\} = 48.568, \quad G^2\{x^{34}\} = 54.168, \quad G^2\{x^{46}\} = 56.08$$

From among the values found select the largest one. We have:

$$G^2\{x^{(2)}\} = G^2\{x^{46}\} = 56.08$$

6. Ask the DM to determine ε^2 for the second criterion. The DM set $\varepsilon^2 = 5$.
7. Create the set of strategies $\{\mathbf{X}^{(2)}\}$:

$$\{\mathbf{X}^{(2)}\} = \{\{x\} \in \{\mathbf{X}^{(1}\} : G^2\{x\} \geq 51.08\} = \{\{x^{42}\}, \{x^{34}\}, \{x^{46}\}\}$$

8. Set $k = 3$.
9. Since $k \leq 3$, go to Step 5.

5. Determine the strategy $\{x^{(3)}\}$ in $\{\mathbf{X}^{(2)}\}$ which is optimal with respect to the third criterion. To do this, calculate:
$$G^3\{x^{42}\} = 46{,}585,\ \ G^3\{x^{34}\} = 47.315,\ \ G^3\{x^{46}\} = 47.315$$
From among the values found select the largest one. We have:
$$G^3\{x^{(3)}\} = G^2\{x^{34}\} = G^2\{x^{46}\} = 47.315$$
6. Ask the DM to determine ε^3 for the third criterion. The DM set $\varepsilon^3 = 1$.
7. Create the set of strategies $\{\mathbf{X}^{(3)}\}$:
$$\{\mathbf{X}^{(3)}\} = \{\{x\} \in \{\mathbf{X}^{(2)}\} : G^3\{x\} \geq 46{,}315\} = \{\{x^{42}\}, \{x^{34}\}, \{x^{46}\}\}$$
8. Set $k = 4$.
9. Since $k > 3$, go to Step 10.
10. Suggest to the DM the selection of the final strategy from $\{\mathbf{X}^{(3)}\}$. This selection can be aided by the expected values of the multi-stage criteria which are:
$$G^1\{x^{42}\} = 16.97,\quad G^2\{x^{42}\} = 55.104,\ \ G^3\{x^{42}\} = 46{,}585$$
$$G^1\{x^{34}\} = 16.83,\quad G^2\{x^{34}\} = 54.168,\ \ G^3\{x^{34}\} = 47.315$$
$$G^1\{x^{46}\} = 16.83,\quad G^2\{x^{46}\} = 56.08,\quad G^3\{x^{46}\} = 47.315$$
The DM prefers strategy $\{x^{46}\}$.
11. End of procedure.

4 Final remarks

The algorithms presented in this paper, generating the *i*-th realization of a process in the deterministic case and the *i*-th strategy in the stochastic case have both advantages and disadvantages. An advantage of both is that they make it possible to generate the consecutive realizations and strategies, respectively. The decision maker can determine whether the number of the solutions generated is appropriate with regard to the given tolerance interval. If this number is too small or too large, the decision maker can increase or decrease this interval, respectively.

One can also observe certain disadvantages of the quasi-hierarchical approach. The first one is the increasing complexity of the generation of the consecutive solutions and the need for more resource-intensive calculations. The second one is more general and concerns the quasi-hierarchical procedure. An important assumption in all scalarization procedures is that the final solution obtained should be an efficient solution. The quasi-hierarchical procedure does not guarantee this. In the deterministic case it is possible to test the efficiency of the solution obtained and, if this solution is not efficient, to generate efficient solutions better than the solution tested. For the stochastic case, such a procedure has not yet been worked out, which suggest a direction for future research.

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Appendix 1

Stage T = 3

According to formula (12) we obtain:

State y_3 = 6

$G_3^1(6) = \max_1 \{F_3^1(6,K), F_3^1(6,L)\} = \max_1 \{5,2\} = 5$ $\mathbf{D}_{33}^1(6) = (6, K)$

$G_3^2(6) = \max_1 \{F_3^1(6,K), F_3^1(6,L)\} = \max_1 \{5,2\} = 2$ $\mathbf{D}_{33}^2(6) = (6, L)$

State y_3 = 5

$G_3^1(5) = \max_1 \{F_3^1(5,I), F_3^1(5,J)\} = \max_1 (4, 3\} = 4$ $\mathbf{D}_{33}^1(5) = (5, I)$

$G_3^1(5) = \max_2 \{F_3^1(5,I), F_3^1(5,J)\} = \max_2 (4, 3\} = 3$ $\mathbf{D}_{33}^2(5) = (5, J)$

Stage t = 2

According to formula (13) we obtain:

State y_2 = 4

$G_2^1(4) = \max_1\{[F_2^1(4,G) + G_3^1(5)], F_2^1(4,H) + G_3^1(6)] =$

$= \max_1 \{6 + 4, 4 + 5\} = 10$ $\mathbf{D}_{2,3}^1(4) = [(4,G),(5,I)]$

$G_2^2(4) = \max_2 \{F_2^1(4, G) + G_3^j(5), F_2^1(4, H) + G_3^k(6): j, k = 1, 2\} =$

$= \max_2\{[F_2^1(4,G) + G_3^1(5)], [F_2^1(4,G) + G_3^2(5)], [F_2^1(4,H) + G_3^1(6)], F_2^1(4,H) + G_3^2(6)]\} =$

$= \max_2\{6 + 4, 6 + 3, 4 + 5, 4 + 2\} = \max_2\{10, 9, 9, 6\} = 9$

$\mathbf{D}_{2,3}^2(4) = \{[(4,G), (5,J)], [(4,H), (6,K)]\}$

State y_2 = 3

$G_2^1(3) = \max_1\{[F_2^1(3,E) + G_3^1(5)], F_2^1(4,F) + G_3^1(6)] = \max_1 \{5 + 4, 3 + 4\} = \max_1 \{9, 7\} = 9$

$\mathbf{D}_{2,3}^1(3) = [(3, E), (5,I)]$

$G_2^2(3) = \max_2 \{F_2^1(3, E) + G_3^j(5), F_2^1(4, H) + G_3^k(6): j, k = 1, 2\} =$

$= \max_2\{[F_2^1(3,E) + G_3^1(5)], [F_2^1(3,E) + G_3^2(5)], [F_2^1(3,F) + G_3^1(6)], F_2^1(3,F) + G_3^2(6)]\} =$

$= \max_2 \{5 + 4, 5 + 3, 3 + 5, 3 + 2\} = \max_2\{9, 8, 8, 7\}$

$\mathbf{D}_{2,3}^2(3) = \{[(3,E), (5,J)], [(3,F), (6,K)]\}$

Stage 1

According to formula (13) we obtain:

State y_1 = 2

$G_2^1(2) = \max_1\{[F_2^1(2,C) + G_3^1(3)], F_2^1(2,D) + G_3^1(4)] = \max_1 \{5 + 9, 9 + 10\} =$

$= \max_1 \{14, 19\} = 19$

$\mathbf{D}_{1,3}^1(2) = [(2,D),(3, E), (5,I)]$

$G_2^2(2) = \max_2 \{F_2^1(2, C) + G_3^j(3), F_2^1(2, D) + G_3^k(4): j, k = 1, 2\} =$

$= \max_2 \{[F_1^1(2,C) + G_2^1(3)], [F_1^1(2,C) + G_2^2(3)], [F_1^1(2,D) + G_2^1(4)], F_1^1(2,4) + G_2^2(4)]\} =$

$= \max_2 \{5+9, 5+8, 9+10, 9+9\} = \max_2\{14, 13, 19, 18\} = 18$

$\mathbf{D}_{2,3}^2(2) = \{[(2,D), (4,G), (5,J)], [(2,D), (4,H), (6,K)]\}$

State $y_1 = 1$

$G_2^1(1) = \max_1\{[F_1^1(1,A) + G_2^1(3)], F_1^1(1,B) + G_2^1(4)] = \max_1 \{6 + 9, 8 + 10\} =$
$= \max_1 \{15, 18\} = 18$

$\mathbf{D}_{1,3}^1(1) = [(1,B),(4, G), (5,I)]$

$G_2^2(1) = \max_2 \{F_1^1(1, A) + G_2^j(3), F_1^1(1, B) + G_2^k(4): j, k = 1, 2\} =$
$= \max_2 \{[F_1^1(2,C) + G_2^1(3)], [F_1^1(2,C) + G_2^2(3)], [F_1^1(2,D) + G_2^1(4)], F_1^1(2,4) + G_2^2(4)]\} =$
$= \max_2 \{6 + 9, 6 + 8, 8 + 10, 8 + 9\} = \max_2\{15, 14, 18, 17\} = 17$

$\mathbf{D}_{1,3}^2(1) = \{[(1,B), (4,G), (5,J)], [(1,B), (4,H), (6,K)]\}$

1st process value and 1st process realization:
$G_1^1 = \max_1 \{G_1^1(1), G_1^1(2)\} = \max_1\{18, 19\} = 19$
We have: $x_1{*} = 1$ and $d^1 = d_{1,3}^1(2) = [(2,D),(3, E), (5,I)]$.

2nd process value and 2nd process realization:
$G_1^2 = \max_2 \{G_1^1(1), G_1^2(1), G_1^1(2), G_1^2(2)\} = \max_2\{19, 18, 18, 17\} = 18$
$\mathbf{D}^2 = \{\mathbf{D}_{1,3}^1(1), \mathbf{D}_{2,3}^2(2)\} = \{[(2,D), (4,G), (5,J)], [(2,D), (4,H), (6,K)], [(1,B),(4, G), (5,I)]\}$

Appendix 2

Stage T = 3

According to formula (29) we obtain:

State $y_3 = 6$

$G_3^1(6) = \max_1 \{F_3^1(7|6,K)\cdot P_3(7|6,K) + F_3^1(8|6,K)\cdot P_3(8|6,K),$
$F_3^1(7|6,L)\cdot P_3(7|6,L) + F_3^1(8|6,L)\cdot P_3(8|6,L)\} =$
$= \max_1 \cdot\{(5\cdot 0.2 + 1\cdot 0.8), (5\cdot 0.9 + 1\cdot 0.1)\} = \max_1 \{1.8, 4.6\} = 4.6$

$\{X_{3,3}^1(6)\} = \{\_, L\}$

$G_3^2(6) = \max_2 \{F_3^1(7|6,K)\cdot P_3(7|6,L) + F_3^1(8|6,K)\cdot P_3(8|6,K),$
$F_3^1(7|6,L)\cdot P_3(7|6,L) + F_3^1(8|6,L)\cdot P_3(8|6,L)\} =$
$= \max_2 \{(5\cdot 0.2 + 1\cdot 0.8), (5\cdot 0.9 + 1\cdot 0.1)\} = \max_2 \{1.8, 4.6\} = 4.6$

$\{X_{3,3}^2(6)\} = \{\_, K\}$

State $y_3 = 5$

$G_3^1(5) = \max_1 \{F_3^1(7|5,I)\cdot P_3(7|5,I) + F_3^1(8|5,I)\cdot P_3(8|5,I),$
$F_3^1(7|5,J)\cdot P_3(7|5,J) + F_3^1(8|5,J)\cdot P_3(8|5,J)\} =$
$= \max_2 \cdot\{(5\cdot 0.8 + 1\cdot 0.2), (5\cdot 0.3 + 1\cdot 0.7)\} = \max_2 \{4.2, 2.2\} = 4.2$

$\{X_{3,3}^1(5)\} = \{I, \_\}$

$G_3^2(5) = \max_2 \{F_3^1(7|5,I)\cdot P_3(7|5,I) + F_3^1(8|5,I)\cdot P_3(8|5,I),$
$F_3^1(7|5,J)\cdot P_3(7|5,J) + F_3^1(8|5,J)\cdot P_3(8|5,J)\} =$
$= \max_2 \cdot\{(5\cdot 0.8 + 1\cdot 0.2), (5\cdot 0.3 + 1\cdot 0.7)\} = \max_2 \{4.2, 2.2\} = 2.2$

$\{X_{3,3}^2(6)\} = \{J, \_\}$

Stage $t = 2$

According to formula (30) we obtain:

State $y_2 = 4$

$G_2^1(4) = \max_1 \{[F_2^1(5|4,G) + G_3^1(5)]\cdot P_2(5|4,G) + [F_2^1(6|4,G) + G_3^1(6)]\cdot P_2(6|4,G),$
$F_2^1(5|4,H) + G_3^1(5)]\cdot P_2(5|4,H) + [F_2^1(6|4,H) + G_3^1(5)]\cdot P_2(6|4,H)\} =$
$= \max_1 \{(5 + 4.2)\cdot 0.6 + (6 + 4.6)\cdot 0.4, (5 + 4.2)\cdot 0.8 + (6 + 4.6)\cdot 0.2\} = \max_1\{9.76, 9.48\} = 9.76$

$\{X_{2,3}^1(4)\} = \{\_,G, I, L\}$

$G_2^2(4) = \max_2\{[F_2^1(5|4,G) + G_3^1(5)]\cdot P_2(5|4,G) + [F_2^1(6|4,G) + G_3^1(5)]\cdot P_2(5|4,G),$
$[F_2^1(5|4,G) + G_3^2(5)]\cdot P_2(5|4,E) + [F_2^1(6|4,G) + G_3^1(5)]\cdot P_2(5|4,G),$
$[F_2^1(5|4,G) + G_3^1(5)]\cdot P_2(5|4,E) + [F_2^1(6|4,G) + G_3^2(5)]\cdot P_2(5|4,G),$
$[F_2^1(5|4,G) + G_3^2(5)]\cdot P_2(5|4,E) + [F_2^1(6|4,G) + G_3^2(5)]\cdot P_2(5|4,G),$
$[F_2^1(5|4,H) + G_3^2(5)]\cdot P_2(5|4,H) + [F_2^1(6|4,H) + G_3^1(5)]\cdot P_2(5|4,H),$
$[F_2^1(5|4,H) + G_3^1(5)]\cdot P_2(5|4,H) + [F_2^1(6|4,H) + G_3^1(5)]\cdot P_2(5|4,H),$
$[F_2^1(5|4,H) + G_3^2(5)]\cdot P_2(5|4,H) + [F_2^1(6|4,H) + G_3^2(5)]\cdot P_2(5|4,H),$
$[F_2^1(5|4,H) + G_3^1(5)]\cdot P_2(5|4,H) + [F_2^1(6|4,H) + G_3^2(5)]\cdot P_2(5|4,H)\} =$
$= \max_2\{(5 + 4.2)\cdot 0.6 + (6 + 4.6)\cdot 0.4,\ (5 + 2.2)\cdot 0.6 + (6 + 4.6)\cdot 0.4,$
$(5 + 4.2)\cdot 0.6 + (6 + 1.8)\cdot 0.4,\ (5 + 2.2)\cdot 0.6 + (6 + 1.8)\cdot 0.4,$
$(5 + 4.2)\cdot 0.8 + (6 + 4.6)\cdot 0.2,\ (5 + 2.2)\cdot 0.8 + (6 + 4.6)\cdot 0.2,$
$(5 + 4.2)\cdot 0.8 + (6 + 1.8)\cdot 0.2,\ (5 + 2.2)\cdot 0.8 + (6 + 1.8)\cdot 0.2 =$
$= \max_2\{9.76, 8.56, 7.64, 7.44, 9.48, 7.88, 8.76, 7.32\} = 9.48$

$\{X_{2,3}^2(4)\} = \{\_, H, I, L\}$

State $y_2 = 3$

$G_2^1(3) = \max_1\{[F_2^1(5|3,E) + G_3^1(5)]P_2(5|3,E) + [F_2^1(6|3,E) + G_3^1(5)]\cdot P_2(5|3,E),$
$[F_2^1(5|3,F) + G_3^1(5)]\cdot P_2(5|3,F) + [F_2^1(6|3,F) + G_3^1(5)]\cdot P_2(5|3,F)\} =$
$= \max_1\{(5 + 4.2)\cdot 0.5 + (6 + 4.6)\cdot 0.5,\ (5 + 4.2)\cdot 0.3 + (6 + 4.6)\cdot 0.7\} = \max_1\{9.9, 10.18\} = 10.18$

$\{X_{2,3}^1(3)\} = \{F, \_, I, L\}$

$G_3^2(3) = \max_2\{[F_2^1(5|3,E) + G_3^1(5)]\cdot P_2(5|3,E) + [F_2^1(5|4,E) + G_3^1(5)]\cdot P_2(5|4,E),$
$[F_2^1(5|3,E) + G_3^2(5)]\cdot P_2(5|3,E) + [F_2^1(5|4,E) + G_3^1(5)]\cdot P_2(5|4,E),$
$[F_2^1(5|3,E) + G_3^1(5)]\cdot P_2(5|3,E) + [F_2^1(5|4,E) + G_3^2(5)]\cdot P_2(5|4,E),$
$[F_2^1(5|3,E) + G_3^2(5)]\cdot P_2(5|3,E) + [F_2^1(5|4,E) + G_3^2(5)]\cdot P_2(5|4,E),$
$[F_2^1(5|3,F) + G_3^2(5)]\cdot P_2(5|3,F) + [F_2^1(5|4,F) + G_3^1(5)]\cdot P_2(5|4,F),$
$[F_2^1(5|3,F) + G_3^1(5)]\cdot P_2(5|3,F) + [F_2^1(5|4,F) + G_3^1(5)]\cdot P_2(5|4,F),$
$[F_2^1(5|3,F) + G_3^2(5)]\cdot P_2(5|3,F) + [F_2^1(5|4,F) + G_3^2(5)]\cdot P_2(5|4,F),$
$[F_2^1(5|3,F) + G_3^1(5)]\cdot P_2(5|3,F) + [F_2^1(5|4,F) + G_3^2(5)]\cdot P_2(5|4,F)\}=$
$= \max_2\{(5 + 4.2)\cdot 0.5 + (6 + 4.6)\cdot 0.5,\ (5 + 2.2)\cdot 0.5 + (6 + 4.6)\cdot 0.5,$
$(5 + 4.2)\cdot 0.5 + (6 + 1.8)\cdot 0.5,\ (5 + 2.2)\cdot 0.5 + (6 + 1.8)\cdot 0.5,$
$(5 + 4.2)\cdot 0.3 + (6 + 4.6)\cdot 0.7,\ (5 + 2.2)\cdot 0.3 + (6 + 4.6)\cdot 0.7,$
$(5 + 4.2)\cdot 0.3 + (6 + 1.8)\cdot 0.7,\ (5 + 2.2)\cdot 0.3 + (6 + 1.8)\cdot 0.7 =$
$= \max_2\{9.9, 8.93, 8.5, 7.5, 10.18, 9.58, 8.22, 7.62\} = 9.9$

$\{X_{2,3}^2(3)\} = \{E, \_, I, L\}$

Stage 1

According to formula (30) we obtain:

State $y_1 = 2$

$G_2^1(2) = \max_1\{[F_1^1(3|2,C)\cdot P_3(3|2,C) + G_3^1(3)]\cdot P_2(3|2,C) + [F_2^1(4|2,C)\cdot P_3(4|2,C) + G_3^1(4)]\cdot P_2(4|2,C),$
$[F_2^1(5|2,D)\cdot P_3(5|2,D)+G_3^1(5)]\cdot P_2(5|2,D) + [F_2^1(6|2,D)\cdot P_3(5|2,D) + G_3^1(5)]\cdot P_2(5|2,D)\} =$
$= \max_1\{(6 + 10.18)\cdot 0.5 + (8 + 9.76)\cdot 0.5,\ (6 + 10.18)\cdot 0.8 + (8 + 9.76)\cdot 0.2\} =$
$= \max_1\{16.97, 16.496\} = 16.97$

$\{X_{1,3}^1(2)\} = \{\_, C, F, G, I, L\}$

$$\begin{aligned}
G_1^2(2) = \max\nolimits_2\{ & [F_1^1(3|2,C) + G_2^1(3)]\cdot P_1(3|2,C) + [F_1^1(4|2,C) + G_2^1(4)]\cdot P_1(4|4,C),\\
& [F_1^1(3|2,C) + G_2^2(3)]\cdot P_1(3|2,C) + [F_1^1(4|2,C) + G_2^1(4)]\cdot P_1(4|4,C),\\
& [F_1^1(3|2,C) + G_2^1(3)]\cdot P_1(3|2,C) + [F_1^1(4|2,C) + G_2^1(4)]\cdot P_1(4|4,C),\\
& [F_1^1(3|2,C) + G_2^2(3)]\cdot P_1(3|2,C) + [F_1^1(4|2,C) + G_2^1(4)]\cdot P_1(4|4,C),\\
& [F_1^1(3|2,D) + G_2^1(3)]\cdot P_1(3|2,D) + [F_1^1(4|2,D) + G_2^1(4)]\cdot P_1(4|4,D),\\
& [F_1^1(3|2,D) + G_2^2(3)]\cdot P_1(3|2,D) + [F_1^1(4|2,D) + G_2^1(4)]\cdot P_1(4|4,D),\\
& [F_1^1(3|2,D) + G_2^1(3)]\cdot P_1(3|2,D) + [F_1^1(4|2,D) + G_2^1(4)]\cdot P_1(4|4,D),\\
& [F_1^1(3|2,D) + G_2^2(3)]\cdot P_1(3|2,D) + [F_1^1(4|2,D) + G_2^1(4)]\cdot P_1(4|4,D)\} =\\
= \max\nolimits_2\{ & (6 + 10.18)\cdot 0.5 + (8 + 9.76)\cdot 0.5,\ (6 + 9.9)\cdot 0.5 + (8 + 9.76)\cdot 0.5,\\
& (6 + 10.18)\cdot 0.5 + (8 + 9.48)\cdot 0.5,\ (6 + 9.9)\cdot 0.5 + (8 + 9.48)\cdot 0.5,\\
& (6 + 10.18)\cdot 0.8 + (8 + 9.76)\cdot 0.2,\ (6 + 9.9)\cdot 0.8 + (8 + 9.76)\cdot 0.2\\
& (6 + 10.18)\cdot 0.8 + (8 + 9.48)\cdot 0.2,\ (6 + 9.9)\cdot 0.8 + (8 + 9.48)\cdot 0.2\} =\\
= \max\nolimits_2\{ & 16.97, 16.83, 16.83, 16.69, 16.496, 16.152, 16.44, 16.216\} = 16.83
\end{aligned}$$

$$\{X_{1,3}^2(2)\} = \{\{\ \_,C, F, H, I, L\}, \{\ \_,C, E, H, I, L\},\}$$

State $y_1 = 1$

$$\begin{aligned}
G_1^1(1) = \max\nolimits_1\{ & [F_1^1(3|1,A) + G_2^1(3)]\cdot P_1(3|2,C) + [F_1^1(4|1,A) + G_2^1(4)]\cdot P_1(4|1,A),\\
& [F_1^1(3|1,B) + G_3^1(3)]\cdot P_1(3|1,B) + [F_1^1(4|1,B) + G_2^1(4)]\cdot P_1(4|1,B)\} =\\
= \max\nolimits_1\{ & (6 + 10.18)\cdot 0.4 + (8 + 9.76)\cdot 0.6,\ (6 + 10.18)\cdot 0.7 + (8 + 9.76)\cdot 0.3\} =\\
= \max\nolimits_1\{ & 17.128, 16.654\} = 17.128
\end{aligned}$$

$$\{X_{1,3}^1(1)\} = \{A, \_,F, G, I, L\}$$

$$\begin{aligned}
G_1^2(1) = \max\nolimits_2\{ & [F_1^1(3|1,A) + G_2^1(3)]\cdot P_1(3|1,A) + [F_1^1(4|1,A) + G_2^1(4)]\cdot P_1(4|1,A),\\
& [F_1^1(3|1,A) + G_2^2(3)]\cdot P_1(3|1,A) + [F_1^1(4|1,A) + G_2^1(4)]\cdot P_1(4|1,A),\\
& [F_1^1(3|1,A) + G_2^1(3)]\cdot P_1(3|1,A) + [F_1^1(4|1,A) + G_2^2(4)]\cdot P_1(4|1,A),\\
& [F_1^1(3|1,A) + G_2^2(3)]\cdot P_1(3|1,A) + [F_1^1(4|1,A) + G_2^2(4)]\cdot P_1(4|1,A),\\
& [F_1^1(3|1,B) + G_2^1(3)]\cdot P_1(3|1,B) + [F_1^1(4|1,B) + G_2^1(4)]\cdot P_1(4|1,B),\\
& [F_1^1(3|1,B) + G_2^1(3)]\cdot P_1(3|1,B) + [F_1^1(4|1,B) + G_2^1(4)]\cdot P_1(4|1,B),\\
& [F_1^1(3|1,B) + G_2^1(3)]\cdot P_1(3|1,B) + [F_1^1(4|1,B) + G_2^1(4)]\cdot P_1(4|1,B),\\
& [F_1^1(3|1,B) + G_2^1(3)]\cdot P_1(3|1,B) + [F_1^1(4|1,B) + G_2^1(4)]\cdot P_1(4|1,B)\} =\\
= \max\nolimits_2\{ & (6 + 10.18)\cdot 0.4 + (8 + 9.76)\cdot 0.6,\ (6 + 9.9)\cdot 0.4 + (8 + 9.76)\cdot 0.6,\\
& (6 + 10.18)\cdot 0.4 + (8 + 9.48)\cdot 0.6,\ (6 + 9.9)\cdot 0.4 + (8 + 9.48)\cdot 0.6,\\
& (6 + 10.18)\cdot 0.7 + (8 + 9.76)\cdot 0.3,\ (6 + 9.9)\cdot 0.7 + (8 + 9.76)\cdot 0.3,\\
& (6 + 10.18)\cdot 0.7 + (8 + 9.48)\cdot 0.3,\ (6 + 9.9)\cdot 0.7 + (8 + 9.48)\cdot 0.3 =\\
= \max\nolimits_2\{ & 17.128, 17.016, 16.96, 16.848, 16.654, 16.458, 16.546, 16.374\} = 17.016
\end{aligned}$$

$$\{X_{1,3}^2(1)\} = \{A, \_, E, G, I, L\}$$

According to formula (31) we have:

$$G^1 = \max\nolimits_1 \{16.97, 17.128\} = 17.128$$

$$G^2 = \max\nolimits_2 \{16.97, 16.83, 17{,}128, 17.016\} = 17.016$$

According to formula (32) we have:

$$\{X^1\} = \{A, \_,F, G, I, L\}$$

$$\{X^2\} = \{A, \_, E, G, I, L\}$$

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