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ON A SEQUENTIAL DECISION PROCESS WHERE OFFERS ARE DESCRIBED BY TWO TRAITS

Abstract

This article presents a model of searching for some resource, e.g. a job, whose value depends on two quantitative traits. The decision maker observes offers in a random order and must accept precisely one offer. Recall of previously observed offers is not possible. It is assumed that the value of an offer is a linear function of these two traits, which come from a bivariate normal distribution. We consider the following four strategy sets: i) the decision on whether to accept an offer is based purely on the first trait, ii) any decision is only made after observing both traits, iii) after observing the first trait, the decision maker can either immediately accept, immediately reject or observe the second trait and then decide, iv) after observing the first trait, the decision maker can either immediately reject or observe the second trait and then decide. The goal of the decision maker is to maximize his expected reward, where the reward is equal to the value of the offer selected minus the search costs. The optimal strategy from each of these four sets is derived. An example is given.

Keywords: sequential decision process, job search problem, choice based on several traits.

1 Introduction

Anyone who wishes to acquire a particular type of good must i) find offers, ii) assess the value of an offer, iii) decide whether to accept or reject a particular offer. It is assumed that offers appear in a random order. The decision maker must accept one offer and the recall of previously viewed offers is not possible. In the biology literature, this problem is often presented as the mate choice prob-

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lem (in the original version only females are choosy). In the economics literature, this problem often appears as the job search problem (in the original version only job seekers are choosy) or the problem of purchasing a given resource. Stigler (1961) was the first to consider such a model. He assumed that a client is looking for a particular type of good. The goal of the client is to acquire the resource at the lowest possible total cost, where the total cost is assumed to be the price paid for the good plus the search costs. Janetos (1980) presented a similar model within the framework of mate choice.

Classical models assume that decisions are made on the basis of a single trait, which defines the value of an offer. However, Backwell and Passmore (1996) observed that a female of the crab species *Uca annulipes* first observes the size of a male. If a male is sufficiently large, then the female observes the quality of his nest. On the basis of this, she then decides whether to lay eggs or not. Hence, the value of an offer may depend on various traits and a decision maker can collect information on each offer before making a decision. Fawcett and Johnstone (2003); Castellano and Cermelli (2011), as well as Ramsey (2012) presented models of such decision processes. Similar decision processes are also considered in the economics and psychology literature [see Analytis et al., 2014; Baucells et al., 2008; Bearden and Connolly, 2007; Hogarth and Karelaia, 2005, as well as Lim et al., 2006). Ramsey (2012) presents a model of pair formation by mutual acceptance. This model can be interpreted as a job search problem, in which a job seeker first obtains incomplete information about a job (e.g. from an advert). From the point of view of an employer, he obtains incomplete information regarding a job seeker via an application. After receiving these initial signals, if the two parties are still interested in working together, then they can meet for an interview, where both obtain additional information on the value of their prospective partner. This article considers a model in which information is obtained in two steps, but only one side is choosy.

Wiegmann et al. (2010) presented a similar model to the one considered here. They assumed that the order in which traits are observed is fixed. The decision maker incurs general search costs, as well as costs for observing individual traits. They presented the general form of the optimal strategy. In this article, a particular case of such a model, according to which the traits come from a bivariate normal distribution, is considered. The following strategy sets are considered:

i) S_1: the decision on whether to accept an offer is based on the first trait,
ii) S_2: both traits are observed and then a decision is made,
iii) S_3: after observing the first trait, the decision maker can immediately accept, immediately reject or observe the second trait and then decide,
iv) S_4: after observing the first trait, the decision maker can immediately reject or observe the second trait and then decide.

It is useful to consider various strategy sets for two reasons: i) if the gains from observing the second trait or making a decision at a particular moment are small relative to the associated costs, then strategies from the sets S_1 and S_2 can be competitive with strategies from the sets S_3 and S_4, ii) practical aspects of a given problem may mean than some strategies are infeasible. For example, someone wishing to buy a new car may initially collect information (e.g. on reliability, fuel consumption) about various models from the Internet. However, he must visit a dealer before purchasing a car. Hence, strategies from set S_3 are infeasible.

The first goal is to derive the optimal strategy from each set S_i, i =1,2,3,4. The most important results are given in Statements 1-3, which are original results regarding the form of the optimal strategy when the decision maker collects information step by step. The second goal is a description of a numerical procedure for approximating the optimal strategies from sets S_3 and S_4. This method is illustrated using an example. Chapter 2 presents the model. The form of the optimal strategies from sets S_1 and S_2 are derived in Chapter 3. Chapters 4 and 5 consider strategies from the sets S_3 and S_4, respectively. These chapters contain the most important results of this article, namely the statements regarding the form of the optimal strategies from these sets. Chapter 6 presents algorithms which approximate the optimal strategies from sets S_3 and S_4. Chapter 7 presents an example illustrating how these optimal strategies can be approximated and gives numerical results. The summary gives some possible directions for future research.

2 Model

A decision maker observes a sequence of offers whose length is not bounded. He must choose exactly one offer and recall of previously observed offers is not possible. After accepting an offer, the decision maker stops searching. The *i*-th offer appears at moment i and is described by a two-dimensional random variable (X_1, X_2), where X_j denotes the *j*-th trait of the offer. Assume that (X_1, X_2) has a two-dimensional normal distribution with expected value (0, 0) and correlation matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, where $\rho \equiv \rho(X_1, X_2)$ is the coefficient of correlation between these two traits. The value of an offer, V, is given by $V = \alpha X_1 + X_2$, where the parameter α describes the relative weight of the first trait with respect to the second. It is assumed that the first trait must be assessed before the second trait can be assessed.

From these assumptions, V has a normal distribution with mean 0 and variance $\alpha^2 + 1 + 2\rho\alpha$ and $Cov(V, X_1) = \alpha + \rho$. In addition, given that $X_1 = x_1$, the second trait has a normal distribution with mean ρx_1 and variance $1 - \rho^2$, and the value of an offer has a normal distribution with mean $u_1(x_1)$ and variance $1 - \rho^2$, where:

$$u_1(x_1) = E[V|X_1 = x_1] = x_1(\alpha + \rho). \tag{1}$$

Without loss of generality, we may assume that $\alpha > -\rho$, as when $\alpha = -\rho$, then the first trait does not give any information about the value of an offer and thus should not be taken into consideration. When $\alpha < -\rho$, then X_1 is negatively correlated with the value of an offer and thus we can treat $-X_1$ as an indicator of the offer's value.

We consider the four strategy sets $S_1, \dots, S_4$ described in the introduction. It is assumed that the cost of observing trait i is c_i, where $c_i > 0$. The cost of making a decision is d, $d \geq 0$, and the mean cost of finding an offer is $c_0, c_0 > 0$. The payoff of a searcher is equal to the value of the offer chosen minus the sum of the costs incurred. We derive the optimal strategy from each of these four sets. Let u^S denote the expected reward under the optimal strategy from the set S.

It should be noted that under the assumption that the traits come from any two-dimensional normal distribution and the observation and decision costs are linear, then the corresponding search problem can be reduced to the one described above by using the appropriate standardisation procedure.

3 Optimal strategies from the sets S_1 and S_2

Assume that the searcher bases his decision purely on the first trait, X_1, i.e. the strategy belongs to S_1. The observation and decision costs incurred at each moment are $c_0 + c_1 + d$. Since these costs are additive, the optimal strategy is stationary (i.e. the optimal decision of the searcher is independent of the moment). After observing the first trait, the searcher should accept an offer if and only if its expected value is greater than the expected reward from future search. It follows that:

$$u^{S_1} = E[\max\{u^{S_1}, u_1(X_1)\}] - c_0 - c_1 - d. \tag{2}$$

There is no analytic solution to Equation (2), which is of the form $u^{S_1} = h(u^{S_1})$. Differentiating, $0 < h'(u) = 1 - F(u) < 1$, where F is the distribution function of the standard normal distribution. Hence, h is a contraction mapping. It follows that there is exactly one solution to this equation, which can be approximated using the following iterative process: i) $u_1 = 0$, ii) $u_{k+1} = h(u_k)$. Thus $\lim_{k\to\infty} u_k = u^{S_1}$.

Since $u_1(x_1)$ is increasing in x_1, it follows from Equation (2) that the optimal strategy is of the following form: accept an offer as long as its value is at least x_c, where $u_1(x_c) = u^{S_1}$, i.e. $x_c = \frac{u^{S_1}}{\alpha+\rho}$.

Now assume that the searcher assesses both traits before making a decision, i.e. his strategy is from the set S_2. In this case, at each moment the search costs incurred are $c_0 + c_1 + c_2 + d$. The searcher should accept an offer if and only if its value is greater than the expected reward from future search. It follows that:

$$u^{S_2} = E[\max\{u^{S_2}, V\}] - c_0 - c_1 - c_2 - d. \tag{3}$$

Equation (3) can be solved in an analogous way to Equation (2). An offer should be accepted if and only if $V > u^{S_2}$.

4 Optimal strategy from the set S_3

After observing the first trait, the searcher can reject an offer, accept it or observe the second trait. Let $u_3^*(x_1)$ denote the optimal expected reward when the searcher observes the second trait and $X_1 = x_1$. The optimal strategy satisfies the following conditions:

a) After observing the first trait, the searcher should immediately accept an offer if and only if $u_1(x_1) \geq \max\ [u^{S_3}, u_3^*(x_1)]$, i.e. when the expected value of an offer is greater than both the expected reward from search and the optimal expected reward from observing the second trait. Similarly, the searcher should observe the second trait if and only if $u_3^*(x_1) \geq \max\ [u^{S_3}, u_1(x_1)]$, i.e. when the expected reward from observing the second trait is greater than both the expected reward from future search and the expected value of the offer. Otherwise, an offer should be immediately rejected.

b) After observing the second trait, the searcher should accept an offer if and only if $V \geq u^{S_3}$, i.e. when the value of the offer is greater than the expected reward from future search.

The expected reward from observing the second trait when $X_1 = x_1,\ u_3^*(x_1)$, is given by:

$$u_3^*(x_1) = E[\max\{u^{S_3}, V\} | X_1 = x_1] - c_2 - d,$$

$$u_3^*(x_1) = u_1(x_1) + E[\max\{u^{S_3} - u_1(x_1), \sqrt{1-\rho^2}Z\}] - c_2 - d, \tag{4}$$

where *Z* has the standard normal distribution. From Equation (4) and Criterion a), given above, it follows that the searcher should observe the second trait rather than accept an offer immediately after observing the first trait if and only if:

$$E[\max\ \{u^{S_3} - u_1(x_1), \sqrt{1-\rho^2}\, Z\}] - c_2 - d \geq 0.$$

Statement 1 describes the form of the optimal strategy, including a necessary and sufficient condition for the second trait to be observed with a positive probability.

Statement 1: *When:*

$$c_2 + d \geq\ E[\max\ \{0, \sqrt{1-\rho^2}Z\}] = \sqrt{\frac{1-\rho^2}{2\pi}}, \tag{5}$$

then the optimal strategy is of the following form: accept an offer if and only if $X_1 \geq x_c$, where x_c is the threshold used under the optimal strategy from the set S_1. In this case, $u^{S_3} = u^{S_1} = u_1(x_c)$. Otherwise, the optimal strategy is of the following form: there exist three constants $x^{*,1}$, $x^{*,2}$ and u^{S_3}, such that $x^{*,1} < x^{*,2}$ and:

a) when $x_1 < x^{*,1}$, a searcher should immediately reject an offer after observing the first trait,

b) when $x_1 \geq x^{*,2}$, a searcher should immediately accept an offer after observing the first trait,

c) when $x^{*,1} \leq x_1 < x^{*,2}$, a searcher should observe the second trait; after observing the second trait, the searcher should accept the offer if and only if $V \geq u^{S_3}$,

d) the constants $x^{*,1}$, $x^{*,2}$ and u^{S_3} satisfy the following conditions: i) u^{S_3} is the optimal expected reward, ii) when $x_1 = x^{*,1}$, the expected reward from rejecting an offer is equal to the expected reward from observing the second trait, iii) when $x_1 = x^{*,2}$ the expected reward from accepting the offer is equal to the expected reward from observing the second trait.

From these conditions, it follows that $x^{*,1}$, $x^{*,2}$ and u^{S_3} satisfy the following system of equations:

$$u^{S_3} = u^{S_3} F(x^{*,1}) + \int_{x^{*,1}}^{\infty} \max[u_1(x), u_3^*(x)] f(x) dx - c_0 - c_1 - d, \quad (6)$$

$$u^{S_3} = E[\max \ \{u^{S_3}, u_1(x^{1,*}) + \sqrt{1-\rho^2} Z\}] - c_2 - d, \quad (7)$$

$$u_1(x^{*,2}) = E[\max\{u^{S_3}, u_1(x^{2,*}) + \sqrt{1-\rho^2}\, Z\}] - c_2 - d, \quad (8)$$

where f and F denote the density function and the cumulative distribution function, respectively, of the standard normal distribution.

The proof of Statement 1 is given in the Appendix.

The form of the optimal strategy is rather intuitive. When the value of the first trait is particularly low or high, then an offer should be immediately rejected or accepted, as appropriate. However, it is worthwhile observing the second trait when the value of the first trait is neither particularly low nor particularly high, the costs of observing the second trait, c_2, and of making a decision, d, are low and the second trait contains a large amount of information about the value of an offer given the value of the first trait, i.e. $|\rho|$ is small. It should be noted that the condition determining when it is optimal to observe the second trait is independent of c_0, c_1 and α. This results from the properties of the multivariate normal distribution, in particular from the fact that $Var(X_2|X_1 = x_1)$ does not depend on x_1. On the other hand, the qualitative form of the optimal strategy would be similar under more general assumptions regarding the joint distribution of the two traits.

Statement 2, presented below, shows that a simple substitution can transform Equations (7) and (8) into a single equation, which is independent of Equation (6).

Statement 2: *Independently of the value u^{S_3}, the solution to Equations (7) and (8) satisfies $u_1(x^{*,1}) = u^{S_3} - s$ and $u_1(x^{*,2}) = u^{S_3} + s$, where s is the solution of the following equation:*

$$0 = E[\max\{-s, \sqrt{1-\rho^2}\,Z\}] - c_2 - d. \tag{9}$$

From Statement 2, it follows that Equation (6) can be written in the form:

$$U^{S_3} = U^{S_3}F(U^{S_3} - s) + \int_{U^{S_3}-s}^{\infty} \max\ [u_1(x), u_3^*(x)]f(x)dx - c_0 - c_1 - d, \tag{10}$$

where $u_3^*(x)$ is given by Equation (4). Hence, *s* can be derived from Equation (9), and then the only unknown in Equation (10) is U^{S_3}. The proof of Statement 2 is given in the Appendix.

5 The optimal strategy from the set *S₄*

After observing the first trait, the searcher must either reject an offer or observe the second trait. Hence, the optimal strategy must satisfy the following conditions:

i) after observing the first trait, the searcher should observe the second trait if and only if the expected payoff from observing the second trait is greater than the expected reward from future search,
ii) after observing the second trait the searcher should accept an offer if and only if the value of an offer is greater than the expected reward from future search.

Let $u_4^*(x_1)$ denote the expected reward from observing the second trait, when the value of the first trait is x_1. It follows that:

$$u_4^*(x_1) = E[\max\{u^{S_4}, u_1(x_1)\}] - a_2 - d. \tag{11}$$

From the optimality criteria, it follows that the searcher should observe the second trait if and only if:

$$E[\max\{u^{S_4}, u_1(x_1) + \sqrt{1-\rho^2}Z\}] - a_2 - d \geq u^{S_4}.$$

The next statement follows from the equation defining the optimal expected reward when the first trait is being observed, together with the fact that the left hand side of the above equation is increasing in x_1.

Statement 3: *Under the optimal strategy from the set S_4, the searcher observes the second trait if and only if $x \geq x^{3,*}$, where $u_1(x^{*,3}) = u^{S_4} - s$ and s satisfies Equation (9). After observing the second trait, the searcher should accept an offer when its value is at least u^{S_4}. The thresholds $x^{3,*}$ and u^{S_4} satisfy the following pair of equations:*

$$E[max\{u^{S_4}, u_1(x^{3,*}) + \sqrt{1-\rho^2}Z\}] = u^{S_4} + c_2 + d\,, \tag{12}$$

$$u^{S_4} = u^{S_4}F(x^{3,*}) + \int_{x^{3,*}}^{\infty} u_4^*(x)f(x)dx - c_0 - c_1 - d\,. \tag{13}$$

It should be noted that after subtracting u^{S_4} from both sides of Equation (12), we obtain an equation of analogous form to Equation (9). The proof of Statement 3 is analogous to the proof of Statement 1 and is thus omitted.

6 A procedure for determining the optimal strategies from S_3 and S_4

When determining the optimal strategy from either of these sets, we first solve the following equation of the form $s = g(s)$, which is equivalent to Equation (9):

$$s = s + E[\max\{-s, \sqrt{1-\rho^2}Z\}] - c_2 - d.$$

It is possible to solve this equation numerically using the following iterative process:

$$s_1 = 0;\ s_{n+1} = g(s_n).$$

Since $0 < g'(s) = 1 - F(-s) < 1$, this iteration is based on a contraction mapping and there exists exactly one solution of this equation, $s = \lim_{n\to\infty} s_n$.

Now we derive the optimal strategy from the set S_3. Setting $u_1(x^{*,1}) = u^{S_3} - s$, $u_1(x^{*,2}) = u^{S_3} + s$ and $u_1(x_1) = x_1(\alpha+\rho)$, we obtain, $x^{*,1} = \frac{u^{S_3}-s}{\alpha+\rho}$ and $x^{*,2} = \frac{u^{S_3}+s}{\alpha+\rho}$. From Equation (6) it follows that:

$$u^{S_3} = u^{S_3}F(x^{*,1}) + \int_{x^{*,1}}^{x^{*,2}} u_3^*(x)f(x)dx + \int_{x^{*,2}}^{\infty} u_1(x_1)f(x)dx - c_0 - c_1 - d,$$

where $u_3^*(x)$ is defined by Equation (4). This equation is of the form $u^{S_3} = h(u^{S_3})$ and thus can be solved using the iterative procedure: $u_0 = 0$; $u_{n+1} = h(u_n)$. The numerical results obtained by the author suggest that the function h is a contraction mapping, but no proof of this hypothesis could be found.

Now we derive the optimal strategy from the set S_4. Substituting $x^{*,3} = \frac{u^{S_4}-s}{\alpha+\rho}$ into Equation (13), we obtain:

$$u^{S_4} = u^{S_4}F\left(\frac{u^{S_4}-s}{\alpha+\rho}\right) + \int_{\frac{u^{S_4}-s}{\alpha+\rho}}^{\infty} u_4^*(x)f(x)dx - c_0 - c_1 - d,$$

where $u_4^*(x)$ is defined by Equation (11). This equation can also be solved using an iterative numerical procedure. This procedure is illustrated in the following section.

7 Example

We now consider the realization of such a search problem where $\alpha = 1$, i.e. the traits have equal weights, $\rho = 0.5$ (the coefficient of correlation between the traits), $c_0 = 0.1$ (search costs), $c_1 = c_2 = 0.05$ (observation costs), $d = 0$ (costs for making a decision).

7.1 Optimal strategy from the set S_1

From Equation (1), the expected value of an offer when $X_1 = x_1$ is given by $u_1(x_1) =$ $= x_1(\alpha+\rho) = 1.5x_1$. Given the form of the optimal strategy, it follows that the searcher should accept an offer when $x_1 \geq \frac{2u^{S_1}}{3}$. From Equation (2), it follows that:

$$u^{S_1} = E[\max\{u^{S_1}, 1.5Z\}] - c_0 - c_1 - d,$$

$$u^{S_1} = u^{S_1}F\left(\frac{2u^{S_1}}{3}\right) + \frac{1.5}{\sqrt{2\pi}}\exp\left(\frac{-2[u^{S_1}]^2}{9}\right) - 0.15.$$

This equation was solved using an iterative procedure. The optimal expected reward is approximately $u^{S_1} \approx 1.3535$. The searcher should accept an offer if and only if $x_1 \geq \frac{2u^{S_1}}{3} \approx 0.9023$.

7.2 The optimal strategy from set S_2

Now we derive the optimal strategy in the case when both traits are observed automatically. The variance of the value of an offer, σ_V^2, is given by:

$$\sigma_V^2 = \alpha^2 + 1 + 2\rho\alpha = 3.$$

From Equation (3), we obtain:

$$u^{S_2} = u^{S_2}F\left(\frac{u^{S_2}}{\sqrt{3}}\right) + \sqrt{\frac{3}{2\pi}}\exp\left[\frac{-(u^{S_2})^2}{6}\right] - 0.2.$$

Solving this equation numerically, we obtain that the optimal expected reward is approximately $u^{S_2} \approx 1.4250$. Under the optimal strategy, the searcher accepts an offer if and only if the its value satisfies $V \geq u^{S_2} \approx 1.4250$.

7.3 Optimal strategy from the set S_3

From Condition (5), it follows that the optimal strategy is based purely on the first trait when $a_2 + d \geq \sqrt{\frac{1-\rho^2}{2\pi}} \approx 0.3455$. Since this condition is not satisfied, the optimal strategy is thus described by a set of three parameters: $x^{1,*}, x^{2,*}$ and u^{S_3} (see Statement 1). First we derive s, where:

$$s = u_1(x^{*,2}) - u^{S_3} = u^{S_3} - u_1(x^{*,1}).$$

From Equation (9), we obtain:

$$s = s + E[\max\{-s, \sqrt{1-\rho^2}Z\}] - c_2 - d,$$

$$s = sF\left(\frac{s}{\sqrt{0.75}}\right) + \sqrt{\frac{0.75}{2\pi}}\exp\left(\frac{-s^2}{1.5}\right) - 0.05.$$

Solving this equation by an iterative procedure, we obtain $s \approx 1.0271$.

Now we derive the optimal expected reward. We have:

$$u_1(x^{*,1}) = u^{S_3} - s \Rightarrow x^{*,1} = \frac{u^{S_3} - s}{1.5}, \tag{14}$$

$$u_1(x^{*,2}) = u^{S_3} + s \Rightarrow x^{*,2} = \frac{u^{S_3} + s}{1.5}. \tag{15}$$

From Equation (6), we obtain:

$$u^{S_3} = u^{S_3}F(x^{1,*}) + \int_{x^{1,*}}^{x^{2,*}} u_3^*(x)f(x)dx + \int_{x^{2,*}}^{\infty} u_1(x)f(x)dx - 0.15. \qquad (16)$$

Solving this equation by an iterative procedure, we obtain $u^{S_3} \approx 1.5730$. It should be noted that the first integral was approximated using the trapezium rule based on 1000 subintervals of equal length. From Equations (14) and (15), it follows that $x^{1,*} \approx 0.3639, x^{*,2} \approx 1.7334$. Hence, the optimal strategy is of the following form:

a) If $x_1 < 0.3639$, an offer should be immediately rejected.
b) If $x_1 \geq 1.7334$, an offer should be immediately accepted.
c) If $0.3639 \leq x_1 < 1.7334$, the searcher should observe the second trait. After observing the second trait, an offer should be accepted if and only if $x_1 + x_2 \geq 1.5730$.

7.4 Optimal strategy from the set S_4

From Statement 3, we have $u_1(x^{*,3}) = x^{*,3}(1+\rho) = u^{S_4} - s$, where s was derived in Section 7.3. Hence, $x^{3,*} = \frac{u^{S_4}-s}{1+\rho}$. From Equation (13), it follows that:

$$u^{S_4} = u^{S_4}F(x^{*,3}) + \int_{x^{*,3}}^{\infty} u_4^*(x)f(x)dx - c_0 - 0.15,$$

where $u_4^*(x)$ is given by Equation (12). The solution to this equation, $u^{S_4} \approx \approx 1.5641$, was derived using a similar iterative approach to the one used to solve Equation (16). It follows that $x^{*,3} \approx 0.3580$. Hence, the optimal strategy from the set S_4 is of the form:

a) Reject an offer immediately if and only if $x_1 \leq 0.3580$.
b) Otherwise, observe the second trait and based on both traits accept the offer if and only if $x_1 + x_2 \geq 1.5641$.

8 Summary

This article has presented the form of optimal strategies in decision problems where the value of an offer depends on two quantitative traits which come from the bivariate normal distribution. These results can be fairly easily generalized to a larger number of traits, since the form of the conditional distribution of a single trait given the values of the traits that have already been seen is analogous to the conditional distribution of the second trait given the value of the first trait in the model presented above. In particular, the variances of these conditional distributions do not depend on the values of the traits already observed. In this case, it is possible to derive the appropriate threshold (relative to the optimal value) when k traits have yet to be observed by recursion. It is more difficult to derive the optimal strategy when the joint distribution of the traits is not normal.

It was also assumed that the order in which the traits are observed is fixed and offers appear in a random order. In many practical problems of this form, the searcher can choose the order in which objects are seen. For example, when someone wishes to buy a car, then they can choose the order in which models are observed according to the mark of a car. In this case, it is necessary to find the optimal order in which to observe offers. This problem has been considered to some degree in the biology literature (Fawcett and Johnstone, 2003). Hogarth and Karelaia (2005) consider this problem from a psychological point of view. Analytis et al. (2014) consider a problem in which the searcher can choose the order in which offers are observed using *a priori* information about the expected value of each particular offer. Considering an analogous model to the one considered here, but where the assumptions regarding the order in which traits and/or offers are observed and regarding the joint distribution of the traits are relaxed, would seem to be a fruitful area for future research.

Appendix

Proof of Statement 1: Assume that the optimal strategy from the set S_3 is of the following form: accept the first offer such that the value of the first trait is at least x_c, where $u_1(x_c) = u^{S_3}$. It follows from this assumption that $u^{S_3} = u^{S_1}$ and when the value of the first trait is x_c, the searcher prefers to immediately accept this first offer rather that observe the second trait. From Equation (4), we obtain:

$$u_1(x_c) + E[\max\{u^{S_3} - u_1(x_c), \sqrt{1-\rho^2}\,Z\}] - c_2 - d \le u_1(x_c).$$

Hence, $c_2 + d \ge E[\max \{0, \sqrt{1-\rho^2}Z\}]$. In addition:

$$E[\max\{0, \sqrt{1-\rho^2}Z\}] = \sqrt{1-\rho^2}\int_0^{\infty} \frac{xe^{\frac{-x^2}{2}}}{\sqrt{2\pi}}dx = \sqrt{\frac{1-\rho^2}{2\pi}}.$$

It follows that Inequality (5) gives a necessary condition for the optimal strategy to be based on purely one trait. In order to show that it is a sufficient condition, we need to show that when Inequality (5) is satisfied, then i) the searcher prefers to immediately accept an offer rather than observe the second trait whenever $x_1 > x_c$, and ii) the searcher prefers to immediately reject an offer when $x_1 < x_c$. Let:

$$g(x) = E[\max\{x, \sqrt{1-\rho^2}\,Z\}] - c_2 - d.$$

Differentiating, we obtain $0 < g'(x) < 1$. Let $x_1 > x_c$. The searcher prefers to immediately accept an offer rather than immediately reject it. From Condition (4), it follows that the searcher prefers to immediately accept an offer rather than observe the second trait when:

$$E[\max\{u^{S_3} - u_1(x_1), \sqrt{1-p^2}\,Z\}] - c_2 - d \leq 0 \Rightarrow g(k) \leq 0,$$

where $k < 0$, since $u_1(x_1) > u^{S_3}$. This inequality holds since $g'(x) > 0$ and, by assumption, $g(0) \leq 0$. Hence, when $x_1 > x_c$, the searcher prefers to immediately accept an offer rather than observe the second trait.

Now assume $x_1 < x_c$. In this case, the searcher prefers to immediately reject an offer rather than immediately accept it. The expected reward from future search is equal to u^{S_3}. From Equation (4), the expected reward from observing the second trait is equal to:

$$u_3^*(x_1) = E[\max\{u^{S_3}, u_1(x_1) + \sqrt{1-\rho^2}\,Z\}] - c_2 - d.$$

Since u_1 is an increasing function, it follows that u_3^* is also an increasing. By assumption, $u_3^*(x_c) < u^{S_3}$. It follows that for $x_1 < x_c$, $u_3^*(x_1) < u^{S_3}$. Hence, when $x_1 < x_c$, the searcher prefers to immediately reject an offer rather than observe the second trait. Hence, Inequality (5) gives a necessary and sufficient condition for the optimal strategy to be based purely on the first trait.

Now assume that $g(0) > 0$, i.e. Condition (5) is not satisfied. It follows that when $x_1 = x_c$, the searcher prefers to observe the second trait rather than accept an offer at once, i.e. $u^{S_3} > u^{S_1}$. Generally, the searcher prefers to observe the second trait rather than immediately accept the first offer at once when $u_3^*(x_1) >$ $> u_1(x_1)$. It follows that:

$$E\,[\max\{u^{S_3} - u_1(x_1), \sqrt{1-\rho^2}\,Z\}] - c_2 - d > 0. \tag{Z.1}$$

The left hand side of this inequality is increasing in u^{S_3} for fixed x_1 and decreasing in x_1 for fixed u^{S_3}. In addition, when $x_1 \to \infty$, the left hand side of this equation tends to $-c_2 - d$. Hence, for each $u^{S_3} > u_1(x_c)$, there exists exactly one constant $x^{*,2}$, where $x^{*,2} > x_c$ such that:

$$E[\max\{u^{S_3} - u_1(x^{*,2}), \sqrt{1-\rho^2}Z\}] - c_2 - d = 0.$$

Adding $u_1(x^{*,2})$ to both sides, we obtain Equation (8).

It should be noted that when $x_1 = x^{*,2}$, the searcher is indifferent between immediately accepting an offer and observing the second trait. From Inequality (Z.1), it follows that the searcher prefers to immediately accept an offer than to observe the second trait if and only if $x_1 > x^{*,2}$. In addition, since $g(0) > 0$ and $g'(0) > 0$, when $x^1 \geq x^{*,2}$, $u^{S_3} - u_1(x^{*,2}) < 0$, and thus the searcher prefers to immediately accept an offer rather than immediately rejecting it. Hence, the searcher should immediately accept an offer when $x_1 = x^{*,2} > x_c$.

Assume that the searcher is indifferent between observing the second trait and immediately rejecting the offer when $x_1 = x^{*,1}$. We have $x_3^*(x^{*,1}) = u^{S_3}$, i.e. Equation (7) is satisfied. From the form of the function g, it follows that $x^{*,1} < x^{*,2}$. Since u_1 is an increasing function, for $x_1 > x^{*,1}$ we obtain:

$$E[\max\{u^{S_3}, u_1(x_1) + \sqrt{1-\rho^2}\, Z\}] - c_2 - d > u^{S_3}.$$

It follows that for $x_1 > x^{*,1}$, the searcher prefers to observe the second trait rather than immediately reject the offer. Hence, when $x^{*,1} \leq x_1 < x^{*,2}$, the searcher should observe the second trait.

Using an analogous argument, when $x_1 < x^{*,1}$, the searcher prefers to reject an offer at once rather than observe the second trait. It was shown above that when $x_1 < x^{*,2}$, the searcher prefers to observe the second trait rather than immediately accept an offer. Thus when $x_1 < x^{*,1}$, the searcher should immediately reject an offer. It follows that the optimal strategy is of the form described in Statement 1.

In order to derive the optimal strategy, apart from Equations (7) and (8), we require another equation. Since u^{S_3} is the optimal expected reward and the first trait has a standard normal distribution, we obtain:

$$u^{S_3} = \int_{-\infty}^{\infty} \max[\, u^{S_3}, u_1(x), u_3^*(x)] f(x) dx - c_0 - c_1 - d.$$

Using the fact that under the optimal strategy, the searcher should immediately reject an offer when $x_1 \leq x^{*,1}$, we obtain Equation (6).

Proof of Statement 2: It is sufficient to show that Equations (7) and (8) are equivalent to Equation (9). Let $s = u_1(x^{*,2}) - u^{S_3}$. From Equation (8), it follows that:

$$u_1(x^{2,*}) = E[\max\left\{u_1(x^{2,*}) - s, u_1(x^{2,*}) + \sqrt{1-\rho^2}Z\right\}] - c_2 - d,$$

$$0 = E[\max \ \{-s, \sqrt{1-\rho^2}Z\}] - c_2 - d.$$

Let $s = u^{S_3} - u_1(x^{*,1})$. From Equation (7), we obtain:

$$u_1(x^{1,*}) + s = E[\max\{u_1(x^{1,*}) + s, u_1(x^{1,*}) + \sqrt{1-\rho^2}Z\}] - c_2 - d,$$

$$0 = E[\max\{0, -s + \sqrt{1-\rho^2}Z\}] - c_2 - d,$$

$$0 = E[\sqrt{1-\rho^2}Z + \max\{-\sqrt{1-\rho^2}Z, -s\}] - c_2 - d.$$

From the symmetry of the standard normal distribution, it follows that:

$$0 = E[\max\{-s, \sqrt{1-\rho^2}Z\}] - c_2 - d.$$

References

Analytis P.P., Kothiyal A., Katsikopoulos K. (2014), *Multi-attribute Utility Models as Cognitive Search Engines,* Judgment and Decision Making, No. 95, 403-419.

Backwell P.R.Y., Passmore N.I. (1996), *Time Constraints and Multiple Choice Criteria in the Sampling Behaviour and Mate Choice of the Fiddler Crab, Uca Annulipes*, Behavioral Ecology and Sociobiology, No. 38 (6), 407-416.

Baucells M., Carrasco J.A., Hogarth R.M. (2008), *Cumulative Dominance and Heuristic Performance in Binary Multiattribute Choice*, Operations Research, No. 56 (5), 1289-1304.

Bearden J.N., Connolly T. (2007), *Multi-attribute Sequential Search*, Organizational Behavior and Human Decision Processes, No. 103 (1), 147-158.

Castellano S., Cermelli P. (2011), *Sampling and Assessment Accuracy in Mate Choice: A Random-walk Model of Information Processing in Mating Decision*, Journal of Theoretical Biology, No. 274 (1), 161-169.

Fawcett T.W., Johnstone R.A. (2003), *Optimal Assessment of Multiple Cues*, Proceedings of the Royal Society of London B: Biological Sciences, No. 270.1524, 1637-1643.

Hogarth R.M., Karelaia N. (2005), *Simple Models for Multiattribute Choice with Many Alternatives: When It Does and Does Not Pay to Face Trade-offs with Binary Attributes*, Management Science, No. 51 (12), 1860-1872.

Janetos A.C. (1980), *Strategies of Female Mate Choice: A Theoretical Analysis*, Behavioral Ecology and Sociobiology, No. 7 (2), 107-112.

Lim C., Bearden J.N., Smith J.C. (2006), *Sequential Search with Multiattribute Options*, Decision Analysis, No. 3 (1), 3-15.

Ramsey D.M. (2012), *Partnership Formation Based on Multiple Traits*, European Journal of Operational Research, No. 216 (3), 624-637.

Stigler G.J. (1961), *The Economics of Information*, The Journal of Political Economy, No. 69 (3), 213-225.

Wiegmann D.D., Weinersmith K.L., Seubert S.M. (2010), *Multi-attribute Mate Choice Decisions and Uncertainty in the Decision Process: A Generalized Sequential Search Strategy*, Journal of Mathematical Biology, No. 60 (4), 543-572.