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TRADE-OFF GUIDED SEARCH FOR APPROXIMATE PARETO OPTIMAL PORTFOLIOS

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Abstract

In this paper, we attempt to represent the Pareto Front in the Markowitz mean-variance model by two-sided discrete approximations. We discuss the possibility of using such approximations for portfolio selection. The potential of the approach is illustrated by the results of preliminary numerical experiments.

Keywords: portfolio optimization, Pareto front approximation, Pareto front navigation.

1 Introduction

The standard approach to solving the Markowitz mean-variance model Markowitz (1952), Markowitz (1991), Gondzio et al. (2007), Elton et al. (2014) is to solve a number of optimization problems with a quadratic objective function, representing portfolio variance, and one linear function, representing portfolio mean return, constrained to be equal to a specific value. By this, a number of efficient (in the sense of Pareto) portfolios and the corresponding pairs of

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mean return and variance values (elements of the Pareto Front) can be derived. We refer to the mean-variance model, because the vast majority of research this subject is focused on this particular problem, rather than on the mean-standard deviation model. The mean-variance model has been a subject of extensive investigations which focused mostly on additional constraints included in the model such as skewness Briec et al. (2013), liquidity Lo et al. (2003) or portfolio size Chiam et al. (2008). Examples of large-scale portfolio optimization problems were discussed in Steuer et al. (2011) and details related to the large-scale optimization with multiple objectives are discussed in Steuer et al. (2006). To solve large-scale portfolio optimization problems, customized evolutionary algorithms were proposed, e.g., in Chen et al. (2011) and Deb et al. (2011).

This standard approach, however completely overshadows the inherent multiple criteria (here: bi-criteria) nature of the problem, where an investor looking for a portfolio he/she would prefer the most, trades risk (captured as variance) for gains (captured as mean returns) and vice versa.

Here we aim to assist investors which invest in portfolios on any scale, with a specific focus on investing in a large number of assets. We start with the observation that investors, with a possible exception of complete novices in the business, have some, maybe vague, idea about their individual risk profiles, for instance: being risk-prone, risk-neutral or risk-averse. This roughly narrows their potential investments to specific segments of the Pareto Front (PF). Hence, this makes the derivation of the whole PF superflous. To enable investors to locate such investor risk-specific segments, we propose a framework to search for the preferable combination of mean return and variance.

The framework consists in capturing the investors' risk profiles as they are willing to make on the unachievable ideal portfolio of zero risk and the maximal mean return to get an efficient portfolio. In the mean-variance model, the maximal mean return is yielded by investing the whole capital into the asset of the highest return.

To avoid having to solve optimization problems at the early stage of the decision making process, instead of the element of the PF of the required risk and mean return, investors can be provided with lower and upper bounds on each of these values. Bounds, if sufficiently tight, are used to decide if portfolios with risk and mean return within those bounds are satisfactory. If so, an element of the PF (and the corresponding portfolio) is derived by solving one quadratic optimization problem.

The outline of this paper is as follows. In Section 2, we recall the Markowitz model and provide necessary preliminaries. In Section 3, we show how the investor can be supported in his/her portfolio selection decisions by approximate valuations of elements of the PF in regions of that set of his/her temporary interest. In Section 4 and Section 5, we show how to populate two specific sets necessary to provide approximate valuations. Section 6 provides some illustrative numerical examples, whereas Section 7 concludes the paper.

2 Preliminaries

The problem of portfolio investments is formulated as follows. Given a number of risky assets, find a portfolio with the most preferred risk and return characteristic.

The underlying model for that problem is the Markowitz mean-variance (MV) model:

$$\begin{aligned} &\min\ f_1(x) = x^T Q x \quad \text{(minimize variance)} \\ &\max\ f_2(x) = e^T x \qquad \text{(maximize mean)} \end{aligned} \tag{1}$$

$$\text{subject to } x \in X_0 = \left\{ x \;\middle|\; \begin{array}{l} u^T x = 1, \ \text{(all capital to be consumed)}, \\ x \geq 0, \end{array} \right\},$$

where x is the vector of fractions of the capital, spent on buying individual assets, Q is the covariance matrix, e is the vector of means, u is the all-ones vector, x, u, $e \in R^n$, Q is an $n \times n$ matrix, and n is the number of assets.

Below we use the standard definition of solution (here: portfolio) efficiency in the sense of Pareto, and we refer to the set of valuations of efficient portfolios as the Pareto Front.

3 Selecting an investment portfolio

The method of preference-driven navigation over the Pareto front (PF), as proposed in Kaliszewski (2006), relies on the notion of *the vector of concessions*, serving as *the preference carrier*.

In what follows, we make use of an element y^* of R^2, $y_1^* = 0$, $y_2^* = \max_{x \in X_0} e^T x + \varepsilon$, $\varepsilon > 0$. Element y^* clearly does not represent any portfolio composed of risky assets.

Since y^* is unattainable, to get a feasible portfolio represented on the PF, the investor has to compromise on y^*, and he or she can do this by selecting a vector of concessions (see Kaliszewski et al., 2016) τ, $\tau_1 < 0, \tau_2 > 0$.

Vector τ defines proportions in which the investor agrees to sacrifice unattainable values of risk and mean return represented by y^* in a quest for an element of the PF.

Vector τ can be defined by the investor explicitly, in the *atomistic* (or *parametric*) *manner*, by indicating its components τ_1, τ_2 (in absolute numbers, e.g. $\tau = (-2, 7)$ or relative quantities, e.g. $\tau = (-\frac{2}{2+4}, \frac{7}{2+4})$), or implicitly, in the *holistic manner*, by indicating a *base point* y, i.e. a variant from the set $\{y \mid y_1 \in y_1^* + R_+, y_2 \in y_2^* - R_+\}$, R_+ – the set of real positive numbers, which defines τ as $\tau_1 = y_1 - y_1^*$, $\tau_2 = y_2^* - y_2$. The latter manner is a special version of the reference point paradigm (cf. e.g. Kaliszewski, 2006; Ehrgott, 2005; Miettinen, 1999; Wierzbicki, 1999).

If the investor expresses his or her preferences in the form of vectors τ, then for any such a vector, he or she can be provided with bounds:

$$L_l(S_L, \tau) \leq f_l(x^\tau) \leq U_l(VS_U, \tau), \; l = 1, 2, \tag{2}$$

where x^τ would be a solution to:

$$\min_{x \in X_0} \max\{\frac{1}{\tau_1}(f_1(x) - y_1^*), \frac{1}{\tau_2}(y_2^* - f_2(x))\} \tag{3}$$

if this problem were solved to optimality, S_L is a set of feasible portfolios, VS_U is a set of elements of R^2 located "above" the PF (for definitions of S_L and VS_U and formulas for $L_l(S_L, \tau)$ and $U_l(VS_U, \tau)$ see Miroforidis (2010), Kaliszewski et al. (2009), Kaliszewski et al. (2010), Kaliszewski et al. (2012a)). Figure 1 illustrates the derivation of bounds for given τ and unknown x^τ.

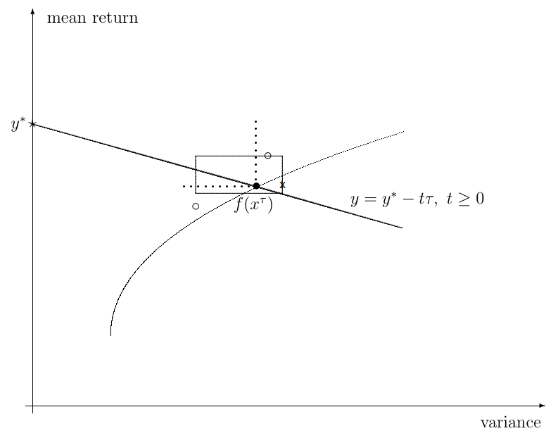

Figure 1. Derivation of bounds for given τ and unknown x^τ (bullet) with VS_U consisting of two elements (circles) and with S_L consisting of one portfolio (marked with an X). The North-West corner of the rectangle determined with these three elements is a lower bound for $f_1(x^\tau)$ and an upper bound for $f_2(x^\tau)$, whereas the South-East corner is an upper bound for $f_1(x^\tau)$ and a lower bound for $f_2(x^\tau)$

4 Derivation of S_L

To ensure effectiveness of lower bound calculations, sets S_L (termed *lower shells* in Miroforidis (2010), Kaliszewski et al. (2009), Kaliszewski et al. (2010), Kaliszewski et al. (2012), Kaliszewski et al. (2012a)) should be composed of elements which are not dominated by any other element in this set.

To populate S_L, we propose the following procedure:

Procedure 1: Derivation of the lower approximation S_L

```
begin
1   Initialize A with assets
2   for a number of pairs of portfolios in A do
3       For each pair derive a number of their convex combinations
4   S_L := A;
5   Delete the dominated portfolios from S_L; go to Step 2.
```

If the lower bounds obtained with this population procedure are not satisfactory, i.e. the differences between the lower and upper bounds are not as small as required, the procedure can be extended to combinations of more than two portfolios.

5 Derivation of VS_U

Similarly to set S_L, to ensure effectiveness of upper bound calculations, sets VS_U (termed *virtual upper shells* in Kaliszewski et al. (2012)) should be composed of elements which do not dominate any other element in this set.

Let us consider the following modification of model (1):

$$\begin{aligned} &\min\ f_1(x) = x^T Q x \\ &\max\ f_2'(x) = \tfrac{1}{\alpha}\, e^T x \end{aligned} \tag{4}$$

$$\text{subject to } x \in X_0\,.$$

Figure 2 represents PFs of this model for different α. It is clearly seen that for $\alpha = 1$ the PF to model (4) coincides with the PF of model (1) and for $\alpha < 1$ the PF satisfies the requirements for set VS_U for model (1).

Assume for the moment that some algorithm applied to model (1) has produced the PF and hence to model (4) with $\alpha = 1$. Then a simple rescaling of that PF in the model (4) with $\alpha < 1$ produces the PF to model (1) and hence VS_U (Figure 2).

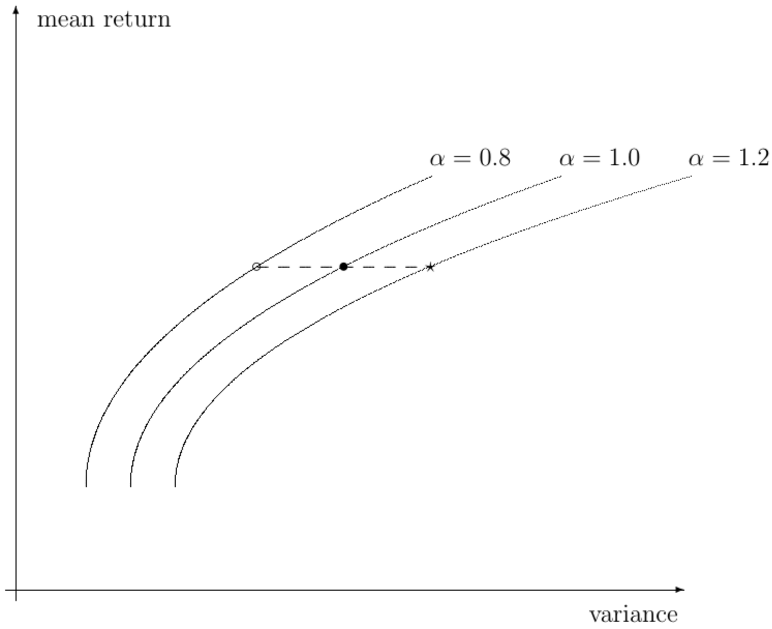

Figure 2. Pareto Fronts for model (4) with different α

This observation is of no practical value, because the PF for model (1) is the very object sought, not a given one. However, below we make use of the idea of objects shifting in the mean-variance space along the variance axis to populate VS_U.

Suppose that an algorithm is able produce S_L such that $\max_{x \in S_L}(f_1(x) - f_1(x')) \leq \beta$, where $f(x')$ are elements of the PF, such that for each $x \in S_L$, $f_1(x) = f_1(x')$.

Then, the set $\{y \mid y_1 = f_1(x) - \beta, y_2 = f_2(x),\ x \in S_L\}$ is clearly a valid VS_U.

At the moment we are not in the position to propose any exact method to derive β except in-sample testing. In the next section, we present results of a few such tests.

6 An illustrative example

We illustrate the idea on an example from the Beasley OR Library (1991), namely the problem *port4.txt* with 98 assets. For that problem the library provides the Pareto Front with 2000 elements uniformly covering the range of accessible returns (Figure 3).

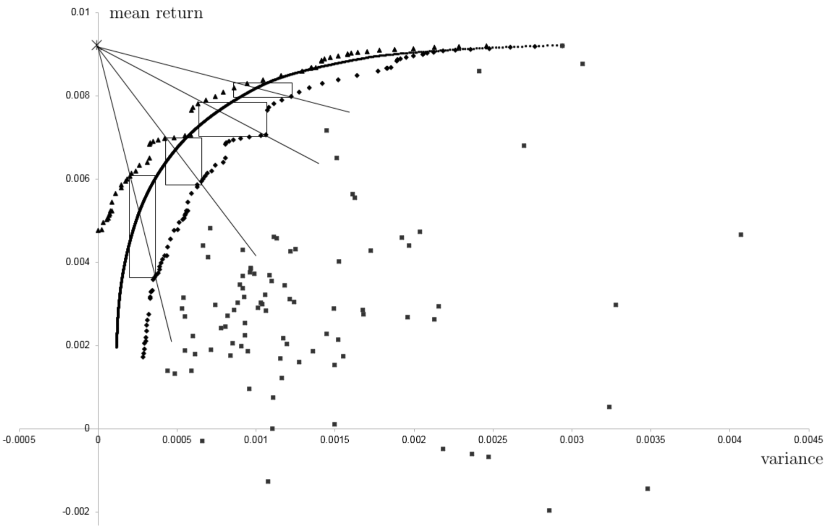

Figure 3. The first iteration – VS_U (▲), $f(S_L)$ (⧫) and images of assets (■) in the example problem. For each τ and the corresponding compromise half line, the North-West corners of the rectangles represent the lower bound for variance and the upper bound for mean, and the South-East corners represent the upper bound for variance and the lower bound for mean

For this problem we have derived lower shell S_L by forming portfolios from pairs of assets (Figure 3). Next, for each portfolio from S_L we have calculated the difference in variance between this portfolio and an element of the Pareto Front with the same return. If there has been no element of the Pareto Front with the return equal to that of the portfolio, we have taken the element of the closest return. With 2000 elements of the Pareto Front, we miss the correct value of return by at most $0.1815 \cdot 10^{-5}$. Next, we shifted S_L along the horizontal axis by the value of the maximal difference and we have obtained a valid VS_U. With S_L and VS_U in place, we have calculated bounds on $f_l(x^\tau),\ l = 1, 2$, as in Table 1.

Table 1: Lower and upper bounds on components of $f(x^\tau)$ for selected τ, first iteration

τ	$L_1(S_L, \tau)$	$L_2(S_L, \tau)$	$U_1(VS_U, \tau)$	$U_2(VS_U, \tau)$
(1,1)	0.00123	0.00798	0.00086	0.00828
(0.5,1)	0.00107	0.00706	0.00063	0.00788
(0.2,1)	0.00065	0.00593	0.00042	0.00699
(0.0667,1)	0.00037	0.00367	0.00020	0.00613

In the second iteration, we have derived S_L by forming portfolios from pairs of portfolios in S_L of the first iteration and calculated bounds again. The bounds are given in Table 2.

Table 2: Lower and upper bounds on components of $f(x^\tau)$ for selected τ, second iteration

τ	$L_1(S_L,\tau)$	$L_2(S_L,\tau)$	$U_1(VS_U,\tau)$	$U_2(VS_U,\tau)$
(1,1)	0.00090	0.00811	0.00100	0.00821
(0.5,1)	0.00074	0.00752	0.00080	0.00773
(0.2,1)	0.00047	0.00641	0.00054	0.00669
(0.0667,1)	0.00025	0.00452	0.00031	0.00537

Table 3 presents maximal errors which occur when taking $L(S_L,\tau)$ to represent $f(x^\tau)$, defined as:

$$err_l = 100 \cdot \frac{U_l(VS_U,\tau) - L_l(S_L,\tau)}{L_l(S_L,\tau)}, \ l = 1,2$$

or:

$$\overline{err}_l = 100 \cdot \frac{f(x^\tau) - L_l(S_L,\tau)}{L_l(S_L,\tau)}, \ l = 1,2,$$

with $f(x^\tau)$ approximated by solving problem (3) over the discrete approximation of the Pareto Front, as provided in Beasley OR Library for that problem. The errors have been calculated for the first and second iteration.

Table 3: Maximal relative errors when taking $L(S_L,\tau)$ to represent $f(x^\tau)$

	First iteration				**Second iteration**			
τ	err_1	err_2	$\overline{err}_1$	$\overline{err}_2$	err_1	err_2	$\overline{err}_1$	$\overline{err}_2$
	%				%			
(1,1)	43.01	3.83	18.33	2.69	10.09	1.16	8.00	0.27
(0.5,1)	69.81	11.64	23.02	8.35	11.63	2.13	5.83	1.10
(0.2,1)	54.51	17.78	17.97	13.07	11.68	4.65	0.96	4.16
(0.0667,1)	83.27	66.82	32.33	41.92	22.69	18.36	5.00	14.89

The numbers in Table 3 illustrate the phenomenon of fast improvement of approximations of the Pareto Front by lower shells. As in the first iteration, the relative errors are absolutely unacceptable, in the second iteration they drop to the level which, if still unacceptable, makes sense to proceed to the third and possibly successive iterations. And it should be stressed that this is only by taking pairwise combinations of portfolios[1].

[1] A similar behavior was observed for the other portfolio selection problems from the Beasley OR Library; due to limited space we confined ourselves to presenting numerical results for one problem only.

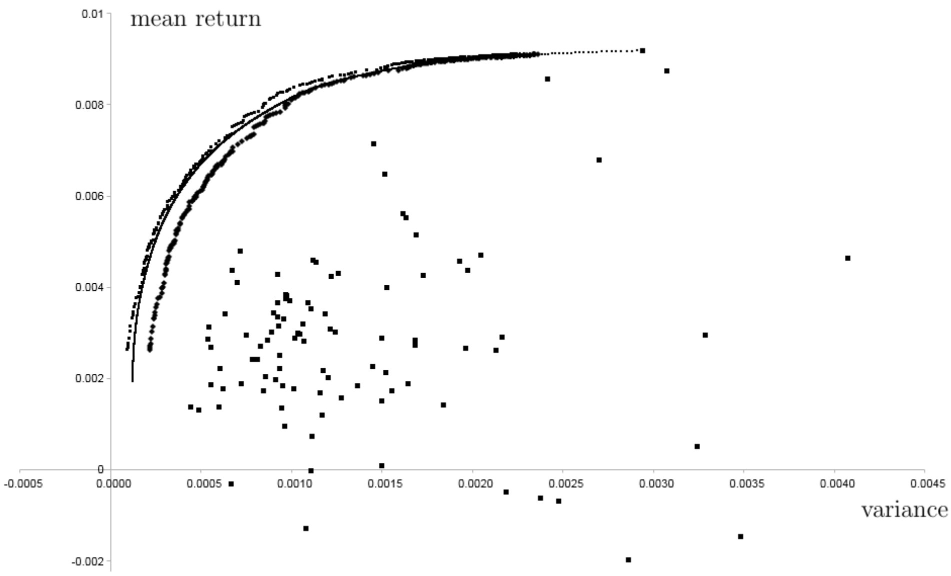

Figure 4. The second iteration – VS_U (▲), $f(S_L)$ (♦) and images of assets (■) in the example problem

Thus far we have attempted to approximate the entire Pareto Front, which is of limited use in practical applications. As we now know where to improve approximations locally (i.e. in the regions pointed to by the DM's preferences represented by the vectors of concessions and the corresponding compromise half line), we can limit computations solely to those regions.

7 Conclusions

In this paper, we have attempted to apply the general bounding methodology to the classical Markowitz portfolio selection mean-variance model. The methodology allows investors to express their preferences in a natural manner with the help of *vectors of concessions* and thus limit the search for Pareto optimal (efficient) portfolios directly to the regions of investors' interests. The existence of two-sided bounds on Pareto suboptimal portfolios allows to control the extent of Pareto suboptimality of feasible portfolios when they are considered for the most preferred portfolio.

We perceive inexact approaches to the portfolio selection problems to be a valid alternative to exact methods when the number of assets available for a portfolio exceeds a thousand. We raise three issues to support our view. First, inexact methods can provide feasible portfolios relatively quickly.

The only problem is their accuracy, but we have solved that problem by providing lower and upper bounds on portfolio variance and mean. Second, inexact methods are generally much simpler to code than exact methods, so they can be often coded in-house. This eliminates the need to acquire (often on the basis of a costly license) an exact solver. Third, in the case of problems admitting more constraints (e.g., cardinality constraints), exact methods become inefficient as the size and complexity of portfolio selection problems grows.

We have shown that in the case of the mean-variance portfolio selection problem, portfolios with a limited number of assets can provide reasonable approximations of the Pareto optimal portfolios. This observation needs to be further verified on various large-scale test problems. Applications of this observation to more complex portfolio selection problems will be the subject of the authors' further investigations.

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