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THE NO-SPOILER CONDITION FOR CHOICE CORRESPONDENCES

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Abstract

We show that any choice correspondence which satisfies the weak Pareto criterion and the Majority property must violate the no-spoiler condition. Subsequently we strengthen the weak Pareto criterion. We show that if the number of criteria or individuals or states of nature is odd, then there is no choice correspondence which satisfies this strengthened version of weak Pareto criteria, Majority property and no-loser spoiler condition. However if the number of criteria/individuals/states of nature is even, we need two more properties to ensure the impossibility result. The first of these two properties is top neutrality. The second property is top anonymity.

Keywords: choice correspondence, no-spoiler, no-loser spoiler, majority property.

1 Introduction

In group (or multi-criteria) choice theory, one is concerned with choosing a non-empty set of alternatives from a given set of alternatives for each profile of preferences. Each profile of preferences could either represent the preferences of individual voters (as in group decision theory) or rankings along criteria considered by a single decision maker as in multi-criteria decision making or preferences dependent on the state of nature, once again of a single individual. The multi-criteria decision making interpretation has been nicely motivated by Rubinstein (2012). Similarly, the state dependent preferences interpretation derives its relevance from the stand point of decision making under complete uncertainty (see Lahiri, 2019a, 2019c). In this paper we use the term choice correspondence to describe such procedures concerned with aggregating

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a collection of preferences into a chosen set of outcomes, whether those separate preferences represent the preferences of distinct individuals or rankings along distinct criteria or preferences which depend on the state of nature. Our framework in which profiles of preferences are mapped to chosen outcomes, and which has been discussed for instance in Denicolo (1993) is a generalization of the one normally associated with the work (and the huge literature that has grown out of it) due to Alan Gibbard and Mark Satterthwaite (Gibbard, 1973; Satterthwaite, 1975). In such a situation a choice correspondence has been traditionally referred to as a social choice correspondence.

In an interesting book by Christoph Borgers (2010), a property referred to as no-spoiler condition is introduced, which says the following. Given a preference profile and an alternative x if: (a) x is not the uniquely chosen alternative; and (b) x is removed from the set of available alternatives *then* the new set of chosen alternatives consists of those that were chosen earlier excluding x. A second interesting property introduced subsequently in Borgers (2010) is referred to as the no-loser spoiler condition and says the following. If an un-chosen alternative is removed from the set of available alternatives, then the set of chosen alternatives remains the same as before. It is easy to see from the two definitions that both require the set of available alternatives to be variable – a possibility considered by Kenneth Arrow while presenting his impossibility theorem and discussed in detail in the book by Kelly (1988). In the conventional context in which choice correspondences are discussed, the set of available alternatives is considered to be fixed and we intend to adopt the conventional approach in this paper. So, if we want to use anything like the two properties mentioned above, we will have to adjust the definitions so that they are meaningful in our context. That is precisely what we do here.

Thus, we define the no-spoiler condition as follows. Given two profiles of preferences and an alternative x if: (a) the only difference between the two profiles is that at the second profile every individual/criteria ranks x at the bottom but is otherwise the same as the ranking for the corresponding individual/along the criteria in the first profile; and (b) x is not the uniquely chosen alternative at the first profile, *then* the set of chosen alternatives at the second profile is simply the set of chosen alternatives at the first profile other than x. Similarly we define the no-loser spoiler condition in the following manner. Given two profiles of preferences and an alternative x which is un-chosen at the first profile, if the only difference between the two profiles is that at the second profile every individual/criteria ranks x at the bottom but is otherwise the same as the ranking for the corresponding individual/criteria in the first profile, *then* the set of chosen alternatives at the second profile is the same as the set of chosen alternatives at the first profile.

In this paper we show that any choice correspondence which satisfies the weak Pareto criterion and the Majority property must violate the no-spoiler condition. The Majority property is much weaker than a common assumption in the literature called Condorcet property. The Majority property says that if an alternative is ranked uniquely first by at least half of the total number of individuals/criteria and is ranked first (perhaps not uniquely) by more than half of the total number of individuals/criteria, then the alternative in question is the unique alternative that is chosen. A matter of some concern about the majority property has been voiced by Professor Prasanta Pattanaik in a private communication dated April 27th, 2018, which is being quoted below.

"While this (*interpretation of aggregating multiple attributes into a non--empty set of outcomes*: author) is formally right, the Condorcet property and the Majority property (the appeal of which depends significantly on the underlying intuition of anonymity), are far less persuasive when we deal with a single person's decision making on the basis of multiple criteria: there is no compelling reason why an individual should attach the same weight to each criterion in taking decisions on the basis of multiple criteria. This is so even when we confine ourselves exclusively to orderings in terms of each criterion without taking into account any intensity".

A partial response to Professor Pattanaik's concerns, in the form of empirical evidence in favor of majority rule in individual decision making, can be found in Zhang, Hsee and Xiao (2006).

Subsequently we strengthen the weak Pareto criterion to require that if along all but one criteria (or for all but one individual) x is ranked above y then y is not chosen. We show that if the number of individuals/criteria is odd, then there is no choice correspondence which satisfies this strengthened version of weak Pareto criteria, Majority property and no-loser spoiler condition. However if the number of individuals/criteria is even, we need two more properties to ensure the impossibility result. The first of these two properties is top neutrality. Top neutrality says that if the top two alternatives in all the rankings are x and y, then interchanging the rankings of x and y for every individual or criterion leads to a reversal of the roles of x and y from the perspective of group (or multi-criteria) choice. The second property is top anonymity. Top anonymity says that if the top two alternatives in all the rankings are x and y, then interchanging the names of the individuals/criteria, does not change the outcome of choice for x and y.

The significant use of the majority property in this paper was observed by a mathematician Professor Janez Zerovnik who suggested an alternative conceptualization which assuming his consent we refer to as Point Majority property. What Point Majority property requires is to assign a score of one to the first rank and zero to all other ranks for each ranking of the set of alternatives. If there is a tie for the first rank, then this one unit that is assigned to this rank is shared equally among the alternatives that tie for first rank. The Point Majority property states that if at a profile of rankings there is an alternative whose sum of points acquired over all criteria exceeds half of the total number of criteria, then this alternative is the uniquely chosen alternative. An easy calculation shows that given a profile of rankings, if such an alternative exists, then it must be unique. Further, we are able to show that the Point Majority property implies the Majority property, though on arbitrary domains the converse need not necessarily be true. However, for domains comprising profiles of strict rankings, the two properties − Majority Property and the Point Majority property − are equivalent.

In the rest of the paper instead of referring to "criteria" or "individual/ criteria", we will simply use the word "individual".

2 The model and some assumptions

Let X denote a non-empty finite set of alternatives containing at least three alternatives and $N = \{1,\ldots,n\}$ for some positive integer $n \geq 3$ denote the set of agents. The set of all non-empty subsets of X is denoted by $\Psi(X)$.

In what follows we shall be concerned with binary relations on X.

Given a binary relation R on X and $x,y\in X$, we will denote $(x,y)\in R$ by xRy. Given a binary relation R on X and $A\in\Psi(X)$, let $R|A$ denote the restriction of R to A, i.e. $x(R|A)y$ if and only if $x,y\in A$ and xRy.

Let $W(X)$ denote the set of weak orders on X (i.e. the set of reflexive, complete and transitive binary relations) and $L(X)$ the set of linear orders on X (anti-symmetric weak orders on X).

Given $R\in W(X)$, let $P(R)$ denote its asymmetric part and $I(R)$ denote its symmetric part. When there is no scope for confusion, we may use P instead of $P(R)$ and I instead of $I(R)$.

Further, given $R\in W(X)$ and $A \in\Psi(X)$, let $G(A,R) = \{x\in A|xRy \text{ for all } y\in A\}$. $G(A,R)$ is called the set of **best** or **greatest** elements/alternatives with respect to R in A. It is well known that for an arbitrary $R\in W(X)$, $G(A,R)$ is non-empty.

An element of $[W(X)]^N$ is called a **profile**.

Any non-empty subset Ω of $[W(X)]^N$ is called a **domain**.

Given a domain Ω a **choice correspondence** (CC) on Ω is a function $f:\Omega\rightarrow\Psi(X)$.

An CC f on Ω is said to satisfy **Weak Pareto** if for all $(R_1,\ldots,R_n)\in\Omega$ and $x,y\in X$, $[xP(R_j)y]$ for all $j = 1,\ldots,n$ implies $y\notin f(R_1,\ldots,R_n)$.

The following property which is a strengthened version of weak Pareto can be found in Pattanaik and Lahiri (2017). It was suggested to the author by Professor Prasanta Pattanaik in a private communication dated September 5th, 2016.

A CC f on Ω is said to satisfy **Strengthened Weak Pareto** if for all $(R_1,\ldots,R_n)\in\Omega$ and $x,y\in X,|\{k\in N|xP(R_k)y\}|\geq n-1$ implies $y\notin f(R_1,\ldots,R_n)$.

A CC f with domain Ω is said to satisfy **Condorcet Property** if for all profiles $(R_1,\ldots, R_n)\in\Omega$, the following holds: if there exists $x\in X$ such that $\#\{j\in N| xP_jy\}>\#\{j\in N| yP_jx\}$ for all $y\in X\setminus\{x\}$, then $f(R_1,\ldots,R_n)=\{x\}$.

Condorcet property says that if at a profile the number of individuals which rank an alternative x above an alternative y is greater than the number of individuals which rank y above x where y is any alternative other than x, then x is the unique alternative to be chosen by the CC at the profile.

A far weaker condition than Condorcet Property is the following which to the best of our knowledge has been introduced in a paper by Lahiri (2019b).

A CC f with domain Ω is said to satisfy **Majority Property** if for all profiles $(R_1,\ldots, R_n)\in\Omega$, the following holds: if there exists $x\in X$ such that $\#\{j\in N|G(X,R_j)=\{x\}\}\geq\frac{n}{2}$ and $\#\{j\in N|x\in G(X,R_j)\}>\frac{n}{2}$ then $f(R_1,\ldots,R_n)=\{x\}$.

Majority property says that if at a profile x is ranked uniquely first by at least half of the total number of individuals and is ranked first (perhaps not uniquely) by more than half of the total number of individuals then x is the unique alternative to be chosen by the CC at the profile.

First note if an x exists satisfying the requirements of the majority property then it has got to be unique. For if $y\in X\setminus\{x\}$, then y can be uniquely first less than $(n-\frac{n}{2})=\frac{n}{2}$ times. Hence y cannot satisfy the requirements of the majority property.

It is easy to see that Condorcet property implies the majority property although the converse need not be true.

To see that the Condorcet property implies the Majority Property let $x\in X$ such that $\#\{j\in N|G(X,R_j)=\{x\}\}\geq\frac{n}{2}$ and $\#\{j\in N|x\in G(X,R_j)\}>\frac{n}{2}$. Let $y\in X\setminus\{x\}$. Thus, $\#\{j\in N|xR_jy\}>\frac{n}{2}$ and $\#\{j\in N|yP(R_j)x\}<n-\frac{n}{2}=\frac{n}{2}$ Hence by Condorcet property, $f(R_1,\ldots,R_n)=\{x\}$.

That the converse need not be true is shown in the following example.

Example 1

Let $X=\{x,y,z\}$ and $n=3$. Define the CC f as follows: for all $(R_1,R_2,R_3)\in[W(X)]^N$ and $v\in X$ let $v\in f(R_1,R_2,R_3)$ if and only if for all $w\in X$, $\#\{j|v\in G(X,R_j)\}\geq\#\{j|w\in G(X,R_j)\}$. It is easy to see that f satisfies Majority property. However, f does not satisfy Condorcet property. Let $xP(R_1)yP(R_1)z$, $yP(R_2)xP(R_2)z$ and $zP(R_3)yP(R_3)x$. Thus, $f(R_1,R_3,R_n)=\{x,y,z\}$ although $\#\{j\in N| yP(R_j)w\}>\#\{j\in N| wP(R_j)y\}$ for all $w\in X\setminus\{y\}$, whence according to the Condorcet property $f(R_1,R_n,R_3)$ should have been equal to $\{y\}$.

As we shall see in the next section, there is no CC satisfying strengthened weak Pareto if n is equal to three.

The following two conditions are based on similar properties due to Borges (2010) for group choice functions.

An CC f with domain Ω is said to satisfy the **No-spoiler condition** if for all $(R_1,\ldots, R_n)$, $(T_1,\ldots,T_n)\in\Omega$ and $x\in X$ with $f(R_1,\ldots,R_n) \neq \{x\}$, the following is true: $f(T_1,\ldots,T_n) = f(R_1,\ldots,R_n)\backslash\{x\}$, where for all $k\in N$ and $y\in X\backslash\{x\}$, yP_kx and $T_k|X\backslash\{x\} = R_k|X\backslash\{x\}$.

An CC f with domain Ω is said to satisfy the **No-loser spoiler condition** if for all $(R_1,\ldots, R_n)$, $(T_1,\ldots,T_n)\in\Omega$ and $x\in X$ with $x\notin f(R_1,\ldots,R_n)$, the following is true: $f(T_1,\ldots,T_n) = f(R_1,\ldots,R_n)$, where for all $k\in N$ and $y\in X\backslash\{x\}$, yP_kx and $T_k|X\backslash\{x\} = R_k|X\backslash\{x\}$.

Note: It was pointed out to me by Professor Prasanta Pattanaik, that the no-spoiler condition (in fact the no-loser spoiler condition) was indicated in an interesting discussion immediately after Condition 3 (now better known as Arrow's Independence of Irrelevant Alternatives), in Arrow (1950,1963). The discussion pointed to the intuitive plausibility of the no-loser spoiler condition as defined by Borgers (2010) and provided an example where this condition is convincingly violated by an equally intuitively plausible aggregation rule (i.e. the Borda rule).

To establish the incompatibility of the No-loser spoiler condition with Strengthened Weak Pareto and Majority Property when the number of individuals is even and greater than two in Proposition 5, we will require these two additional properties.

An CC f with domain Ω is said to satisfy **Top Neutrality** if for $(R_1,\ldots,R_n)$, $(S_1,\ldots,S_n)\in\Omega\cap[L(X)]^N$ and $x,y\in X$ with $x \neq y$, (i) implies (ii), where:
(i) for all $j\in N$: either $[xP(R_j)yP(R_j)w$, $yP(S_j)xP(S_j)w$ for all $w\in X\backslash\{x,y\}$ and $R_j|(X\backslash\{x,y\}) = S_j|(X\backslash\{x,y\})]$ **or** $[yP(R_j)xP(R_j)w$, $xP(S_j)yP(S_j)w$ for all $w\in X\backslash\{x,y\}$ and $R_j|(X\backslash\{x,y\}) = S_j|(X\backslash\{x,y\})]$.
(ii) $\{x\} = f(R_1,\ldots,R_n)$ if and only if $\{y\}=f(S_1,\ldots,S_n)$; $\{y\}= f(R_1,\ldots,R_n)$ if and only if $\{x\}= f(S_1,\ldots,S_n)$.

An CC f with domain Ω is said to satisfy **Top Anonymity** if for $(R_1,\ldots,R_n)$, $(S_1,\ldots,S_n)\in\Omega\cap[L(X)]^N$ and $x,y\in X$ with $x \neq y$, (i) implies (ii), where:
(i) there exists a permutation π on N (i.e. one-to-one function from N to N) such that for all $j\in N$, $S_j = T_{\pi(j)}$ and for all $j\in N$:either $[xP(R_j)yP(R_j)w$ for all $w\in X\backslash\{x,y\}]$ **or** $[yP(R_j)xP(R_j)w$ for all $w\in X\backslash\{x,y\}]$.
(ii) $x\in f(R_1,\ldots,R_n)$ if and only if $x\in f(S_1,\ldots,S_n)$; $y\in f(R_1,\ldots,R_n)$ if and only if $y\in f(R_1,\ldots,R_n)$.

3 The main results of this paper

In this section we present the main results of this paper.

Proposition 1
If n is an odd integer and $[L(X)]^N \subset \Omega$, then there does not exist any CC on Ω which satisfies Weak Pareto, Majority property and no-spoiler condition.

Proof
The proof proceeds by showing that if n is an odd integer and if a CC satisfies Weak Pareto and Majority Property, then the CC must violate the no-spoiler condition.

Suppose n is an odd integer. If n is an odd integer greater than three, $\frac{n-3}{2}$ is a positive integer. Let x, y, z be three distinct elements of X and let $(R_1,\ldots,R_n) \in [L(X)]^N$ such that:
(i) $xP(R_1)yP(R_1)zP(R_1)w$ for all $w \in X\backslash\{x,y,z\}$,
(ii) $zP(R_2)xP(R_2)yP(R_2)w$ for all $w \in X\backslash\{x,y,z\}$,
(iii) $yP(R_3)zP(R_3)xP(R_3)w$ for all $w \in X\backslash\{x,y,z\}$,
(iv) $xP(R_k)yP(R_k)zP(R_k)w$ for all $w \in X\backslash\{x,y,z\}$ and $k = 4,\ldots, 3+\frac{n-3}{2}$,
(v) $zP(R_k)yP(R_k)xP(R_k)w$ for all $w \in X\backslash\{x,y,z\}$ and $k = 4+\frac{n-3}{2},\ldots,n$.

By weak Pareto for all $w \in X\backslash\{x,y,z\}$ $w \notin f(R_1,\ldots,R_n)$ and so $f(R_1,\ldots,R_n) \subset \{x, y, z\}$.

Case 1: $f(R_1,\ldots,R_n) = \{x, y, z\}$.

Let $(T_1,\ldots,T_n) \in [L(X)]^N$ such that for all $k \in N$ and $w \in X\backslash\{z\}$, we have $wP(T_k)z$ and $T_k|X\backslash\{z\} = R_k|X\backslash\{z\}$.

Since at $(T_1,\ldots,T_n)$ x is ranked uniquely first by more than half of the total number of individuals, by the Majority Property $f(T_1,\ldots,T_n) = \{x\} \neq f(R_1,\ldots,R_n)\backslash\{z\}$, violating the no-spoiler condition.

Case 2: $f(R_1,\ldots,R_n)$ is a two element subset of $\{x, y, z\}$.
(i) Suppose $f(R_1,\ldots,R_n) = \{x, y\}$.

Let $(S_1,\ldots,S_n) \in [L(X)]^N$ such that for all $k \in N$ and $w \in X\backslash\{z\}$, we have $wP(S_k)z$ and $S_k|X\backslash\{z\} = R_k|X\backslash\{z\}$.

Since at $(S_1,\ldots,S_n)$ x is ranked uniquely first by more than half of the total number of individuals, by the Majority Property $f(S_1,\ldots,S_n) = \{x\} \neq f(R_1,\ldots,R_n)\backslash\{z\}$, violating the no-spoiler condition.
(ii) Suppose $f(R_1,\ldots,R_n) = \{x, z\}$.

Let $(S_1,\ldots,S_n) \in [L(X)]^N$ such that for all $k \in N$ and $w \in X\backslash\{y\}$, we have $wP(S_k)y$ and $S_k|X\backslash\{y\} = R_k|X\backslash\{y\}$.

Since at $(S_1,\ldots,S_n)$ z is ranked uniquely first by more than half of the total number of individuals, by the Majority Property $f(S_1,\ldots,S_n) = \{z\} \neq f(R_1,\ldots,R_n)\backslash\{y\}$, violating the no-spoiler condition.

(iii) Suppose $f(R_1,\ldots,R_n) = \{y, z\}$.

Let $(S_1,\ldots,S_n) \in [L(X)]^N$ such that for all $k \in N$ and $w \in X\backslash\{x\}$, we have $wP(S_k)x$ and $S_k|X\backslash\{x\} = R_k|X\backslash\{x\}$.

Since at $(S_1,\ldots,S_n)$ y is ranked uniquely first by more than half of the total number of individuals, by the Majority Property $f(S_1,\ldots,S_n) = \{y\} \neq f(R_1,\ldots,R_n)\backslash\{x\}$, violating the no-spoiler condition.

Case 3: $f(R_1,\ldots,R_n)$ is a singleton subset of $\{x, y, z\}$.

(i) $f(R_1,\ldots,R_n) = \{x\}$.

Let $(U_1,\ldots,U_n) \in [L(X)]^N$ such that for all $k \in N$ and $w \in X\backslash\{y\}$, we have $wP(U_k)y$ and $U_k|X\backslash\{y\} = R_k|X\backslash\{y\}$.

Since at $(U_1,\ldots,U_n)$ z is ranked uniquely first by more than half of the total number of individuals, by the Majority Property $f(U_1,\ldots,U_n) = \{z\} \neq f(R_1,\ldots,R_n)\backslash\{y\}$, violating the no-spoiler condition.

(ii) $f(R_1,\ldots,R_n) = \{y\}$.

Let $(U_1,\ldots,U_n) \in [L(X)]^N$ such that for all $k \in N$ and $w \in X\backslash\{z\}$, we have $wP(U_k)z$ and $U_k|X\backslash\{z\} = R_k|X\backslash\{z\}$.

Since at $(U_1,\ldots,U_n)$ x is ranked uniquely first by more than half of the total number of individuals, by the Majority Property $f(U_1,\ldots,U_n) = \{x\} \neq f(R_1,\ldots,R_n)\backslash\{z\}$, violating the no-spoiler condition.

(iii) $f(R_1,\ldots,R_n) = \{z\}$.

Let $(U_1,\ldots,U_n) \in [L(X)]^N$ such that for all $k \in N$ and $w \in X\backslash\{x\}$, we have $wP(U_k)x$ and $U_k|X\backslash\{x\} = R_k|X\backslash\{x\}$.

Since at $(U_1,\ldots,U_n)$ y is ranked uniquely first by more than half of the total number of individuals, by the Majority Property $f(U_1,\ldots,U_n) = \{y\} \neq f(R_1,\ldots,R_n)\backslash\{x\}$, violating the no-spoiler condition.

Thus $f(R_1,\ldots,R_n) = \phi$, contrary to the definition of an CC.

This proves the proposition. Q.E.D.

Proposition 2

If n is even and $\Omega = [W(X)]^N$, then there does not exist any CC on Ω which satisfies Weak Pareto, Majority property and no-spoiler condition.

Proof

Suppose n is even. Then $n \geq 4$ and if n is an even integer greater than four, $\frac{n-4}{2}$ is a positive integer. Let x,y,z be three distinct elements of X and let $(R_1,\ldots,R_n) \in [L(X)]^N$ such that:

(i) $xP(R_1)yP(R_1)zP(R_1)w$ for all $w \in X \setminus \{x, y, z\}$,
(ii) $zP(R_2)xP(R_2)yP(R_2)w$ for all $w \in X \setminus \{x, y, z\}$,
(iii) $yP(R_3)zP(R_3)xP(R_3)w$ for all $w \in X \setminus \{x, y, z\}$,
(iv) $xI(R_4)yI(R_4)zP(R_4)w$ for all $w \in X \setminus \{x, y, z\}$,
(v) $xP(R_k)yP(R_k)zP(R_k)w$ for all $w \in X \setminus \{x, y, z\}$ and $k = 5,\ldots, 4+\frac{n-3}{2}$,
(vi) $zP(R_k)yP(R_k)xP(R_k)w$ for all $w \in X \setminus \{x, y, z\}$ and $k = 5+\frac{n-3}{2},\ldots,n$.

By weak Pareto $w \in X \setminus \{x, y, z\}$ implies $w \notin f(R_1,\ldots,R_n)$, so that $f(R_1,\ldots,R_n) \subset \{x, y, z\}$.

The rest of the proof is very similar to the proof of proposition 1. Q.E.D.

Proposition 3

If $[L(X)]^N \subset \Omega$ and $n = 3$, then there does not exist any CC on Ω which satisfies Strengthened Weak Pareto.

Proof

Let x, y, z be three distinct elements of X and let $(R_1,\ldots,R_n) \in [L(X)]^N$ such that:
(i) $xP(R_1)yP(R_1)zP(R_1)w$ for all $w \in X \setminus \{x, y, z\}$,
(ii) $zP(R_2)xP(R_2)yP(R_2)w$ for all $w \in X \setminus \{x, y, z\}$,
(iii) $yP(R_3)zP(R_3)xP(R_3)w$ for all $w \in X \setminus \{x, y, z\}$.
By strengthened weak Pareto $f(R_1,\ldots,R_n) \subset \{x, y, z\}$. Further,
(i) $\#\{j \in N | \, xP(R_j)y\} = 2 = n-1$ implies $y \notin f(R_1,\ldots,R_n)$;
(ii) $\#\{j \in N | yP(R_j)z\} = 2 = n-1$ implies $z \notin f(R_1,\ldots,R_n)$;
(iii) $\#\{j \in N | zP(R_j)x\} = 2 = n-1$ implies $x \notin f(R_1,\ldots,R_n)$.

Thus $f(R_1,\ldots,R_n) = \phi$, which is not possible by the definition of an CC.

This proves the proposition. Q.E.D.

Proposition 4

If n is odd and $[L(X)]^N \subset \Omega$, then there does not exist any CC on Ω which satisfies Strengthened Weak Pareto, Majority property and No-loser spoiler condition.

Proof

Suppose n is odd. By proposition 3, this proposition is definitely satisfied if n is equal to three. Hence suppose n is greater than three. Since n is an odd integer greater than three, $\frac{n-3}{2}$ is a positive integer.

Let x, y, z be three distinct elements of X and let $(R_1,\ldots,R_n) \in [L(X)]^N$ such that:
(i) $xP(R_1)yP(R_1)zP(R_1)w$ for all $w \in X \setminus \{x, y, z\}$,
(ii) $zP(R_2)xP(R_2)yP(R_2)w$ for all $w \in X \setminus \{x, y, z\}$,
(iii) $yP(R_3)zP(R_3)xP(R_3)w$ for all $w \in X \setminus \{x, y, z\}$,
(iv) $xP(R_k)yP(R_k)zP(R_k)w$ for all $w \in X \setminus \{x, y, z\}$ and $k = 4,\ldots, 3+\frac{n-3}{2}$,
(v) $zP(R_k)xP(R_k)yP(R_k)w$ for all $w \in X \setminus \{x, y, z\}$ and $k = 4+\frac{n-3}{2},\ldots,n$.

By strengthened weak Pareto $w \in X \backslash \{x,y,z\}$ implies $w \notin f(R_1,\ldots,R_n)$ so that $f(R_1,\ldots,R_n) \subset \{x, y, z\}$.

By strengthened weak Pareto $\#\{j \in N|\ xP(R_j)y\} = n-1$ implies $y \notin f(R_1,\ldots,R_n)$.

Thus, $f(R_1,\ldots,R_n) \subset \{x, z\}$.

Suppose $x \in f(R_1,\ldots,R_n)$ and let $(T_1,\ldots,T_n) \in [L(X)]^N$ such that for all $k \in N$ and $w \in X \backslash \{y\}$, we have $wP(T_k)y$ and $T_k|X \backslash \{y\} = R_k|X \backslash \{y\}$.

Since at $(T_1,\ldots,T_n)$ z is ranked uniquely first by more than half of the total number of individuals, by the Majority Property $f(T_1,\ldots,T_n) = \{z\} \neq f(R_1,\ldots,R_n) \backslash \{y\}$, violating the no-loser spoiler condition.

Thus $f(R_1,\ldots,R_n) = \{z\}$ and $x \notin f(R_1,\ldots,R_n)$.

Let $(S_1,\ldots,S_n) \in [L(X)]^N$ such that for all $k \in N$ and $w \in X \backslash \{x\}$, we have $wP(S_k)x$ and $S_k|X \backslash \{x\} = R_k|X \backslash \{x\}$.

Since at $(S_1,\ldots,S_n)$ y is ranked uniquely first by more than half of the total number of individuals, by the Majority Property $f(S_1,\ldots,S_n) = \{y\} \neq f(R_1,\ldots,R_n) \backslash \{x\}$, violating the no-loser spoiler condition.

Thus, $f(R_1,\ldots,R_n) \neq \{z\}$ and so $f(R_1,\ldots,R_n) = \phi$, contrary to the definition of an CC.

This proves the proposition. Q.E.D.

The next proposition shows the incompatibility of No-loser spoiler condition with Strengthened Weak Pareto, Majority property, Top Anonymity and Top Neutrality, when the number of individuals is even and greater than two.

Proposition 5

If n is an even natural number greater than two and $\Omega = [W(X)]^N$, then there does not exist any CC on Ω which satisfies Strengthened Weak Pareto, Majority property, Top Anonymity, Top Neutrality and No-loser spoiler condition.

Proof

Suppose n is an even natural number greater than 2. Thus $n \geq 4$ and $\frac{n-4}{2}$ is a non--negative integer.

Let x, y, z be three distinct elements of X and let $(R_1,\ldots,R_n) \in [W(X)]^N$ such that:

(i) $xP(R_1)yP(R_1)zP(R_1)w$ for all $w \in X \backslash \{x, y, z\}$,

(ii) $zP(R_2)xP(R_2)yP(R_2)w$ for all $w \in X \backslash \{x, y, z\}$,

(iii) $yP(R_3)zP(R_3)xP(R_3)w$ for all $w \in X \backslash \{x, y, z\}$,

(iv) $xI(R_4)zP(R_4)yP(R_4)w$ for all $w \in X \backslash \{x, y, z\}$,

and if $n > 4$ then

(v) $xP(R_k)yP(R_k)zP(R_k)w$ for all $w \in X \backslash \{x, y, z\}$ and $k = 5,\ldots, 4+\frac{n-4}{2}$,

(vi) $zP(R_k)xP(R_k)yP(R_k)w$ for all $w \in X \backslash \{x, y, z\}$ and $k = 5+\frac{n-4}{2},\ldots,n$,

(vii) $R_j|X \backslash \{x, y, z\} = R_k|(X \backslash \{x, y, z\})$ for all $j,k \in N$.

By strengthened weak Pareto for all $w \in X \backslash \{x, y, z\}$, $w \notin f(R_1,\ldots,R_n)$, so that $f(R_1,\ldots,R_n) \subset \{x, y, z\}$.

By strengthened weak Pareto we have $y \notin f(R_1,\ldots,R_n)$, so that $f(R_1,\ldots,R_n) \subset \{x,z\}$.

Suppose $x \in f(R_1,\ldots,R_n)$ and let $(T_1,\ldots,T_n) \in [L(X)]^N$ such that for all $k \in N$ and $w \in X \backslash \{y\}$, we have $wP(T_k)y$ and $T_k|X \backslash \{y\} = R_k|X \backslash \{y\}$.

Since at $(T_1,\ldots,T_n)$ z is ranked uniquely first by more than half of the total number of individuals, by the Majority Property $f(T_1,\ldots,T_n) = \{z\} \neq f(R_1,\ldots,R_n) \backslash \{y\}$, violating the no-loser spoiler condition.

Thus $f(R_1,\ldots,R_n) = \{z\}$ i.e. $x \notin f(R_1,\ldots,R_n)$.

Let $(S_1,\ldots,S_n) \in [L(X)]^N$ such that for all $k \in N$ and $w \in X \backslash \{x\}$, we have $wP(S_k)x$ and $S_k|X \backslash \{x\} = R_k|X \backslash \{x\}$.

Thus,

(i) $yP(S_1)zP(S_1)wP(S_1)x$ for all $w \in X \backslash \{x, y, z\}$,

(ii) $zP(S_2)yP(S_2)wP(S_2)x$ for all $w \in X \backslash \{x, y, z\}$,

(iii) $yP(S_3)zP(S_3)wP(S_3)x$ for all $w \in X \backslash \{x, y, z\}$,

(iv) $zP(S_4)yP(S_4)wP(S_4)x$ for all $w \in X \backslash \{x, y, z\}$,

and if $n > 4$ then

(v) $yP(S_k)zP(S_k)wP(S_k)x$ for all $w \in X \backslash \{x, y, z\}$ and $k = 5,\ldots, 4+\frac{n-4}{2}$,

(vi) $zP(S_k)yP(S_k)wP(S_k)x$ for all $w \in X \backslash \{x, y, z\}$ and $k = 5+\frac{n-4}{2},\ldots,n$.

By no-loser spoiler condition, $f(S_1,\ldots,S_n) = \{z\}$.

Let $(U_1,\ldots,U_n) \in \Omega$ be such that $U_1 = S_2$, $U_2 = S_1$, $U_3 = S_4$, $U_4 = S_3$, for $k = 5,\ldots, 4+\frac{n-4}{2}$, let $U_k = S_{k+\frac{n-4}{2}}$ and for $k = 5+\frac{n-4}{2},\ldots,n$, let $U_k = S_{k-\frac{n-4}{2}}$. Thus, let π be the permutation on N such that $\pi(1) = 2$, $\pi(2) = 1$, $\pi(3) = 4$, $\pi(4) = 3$, $\pi(k) = k + \frac{n-4}{2}$ for $k = 5,\ldots, 4+\frac{n-4}{2}$ and $\pi(k) = k - \frac{n-4}{2}$ for $k = 5+\frac{n-4}{2},\ldots,n$.

By Strengthened Weak Pareto, $f(U_1,\ldots,U_n) \subset \{y, z\}$.

By Top Anonymity $y \in f(R_1,\ldots,R_n)$ if and only if $y \in f(U_1,\ldots,U_n)$ and $z \in f(R_1,\ldots,R_n)$ if and only if $z \in f(U_1,\ldots,U_n)$. Thus, $f(U_1,\ldots,U_n) = \{z\}$.

However, for all $j \in N$: either $[yP(S_j)zP(S_j)w,\ zP(U_j)yP(U_j)w$ for all $w \in X \backslash \{y, z\}$ and $S_j|(X \backslash \{y, z\}) = U_j|(X \backslash \{y, z\})]$ **or** $[zP(S_j)yP(S_j)w,\ yP(U_j)zP(U_j)w$ for all $w \in X \backslash \{y, z\}$ and $S_j|(X \backslash \{y, z\}) = U_j|(X \backslash \{y, z\})]$.

Thus by Top Neutrality, $f(S_1,\ldots,S_n) = \{z\}$ implies $f(U_1,\ldots,U_n) = \{y\} \neq \{z\}$ leading to a contradiction.

This proves the proposition. Q.E.D.

4 Point Majority property and its implications

This section emerged from a very stimulating discussion with Professor Janez Zerovnik (here after referred to as JZ).

Given $R\in W(X)$ and $x\in X$, let first $(x,R) = 1$ if x is ranked first at R and first $(x,R) = 0$, otherwise. It is easy to see that for all $R\in W(X)$, $\sum_{y\in X}\text{first}\,(y,R)$ is a positive integer and if $R\in L(X)$, then $\sum_{y\in X}\text{first}\,(y,R) = 1$. For $R\in W(X)$ and $x\in X$, let point $(x,R) = \frac{\text{first}\,(x,R)}{\sum_{y\in X}\text{first}\,(y,R)}$.

If for $R\in W(X)$ and $x\in X$ it is the case that x is the only alternative to be ranked first at R, then point $(x,R) = 1$ and point $(y,R) = 0$ for all $y\in X\backslash\{x\}$. In particular if for $R\in L(X)$, x is ranked first at R, then point $(x,R) = 1$, and is equal to 0, otherwise. In general (i.e. $R\in W(X)$), point $(x,R) = \frac{1}{\text{cardinality of the set of alternatives ranked first at }R}$ if x is ranked first at R and point $(x,R) = 0$, otherwise. Further, $\sum_{x\in X}\text{point}\,(x,R) = 1$ for all $R\in W(X)$.

Given $(R_1,\ldots, R_n)\in [W(X)]^N$ if $\{x\in X|\sum_{j=1}^{n}\text{point}(x,R_j) > \frac{n}{2}\} \neq \phi$, then this set must be a singleton. This follows from the fact that if for $x, z\in X$, with $x \neq z$, it is the case that $\sum_{j=1}^{n}\text{point}(x,R_j) > \frac{n}{2}$ and $\sum_{j=1}^{n}\text{point}(z,R_j) > \frac{n}{2}$, then $n = \frac{n}{2} + \frac{n}{2} < \sum_{j=1}^{n}\text{point}(x,R_j) + \sum_{j=1}^{n}\text{point}(z,R_j) = \sum_{j=1}^{n}[\text{point}(x,R_j) + \text{point}(z,R_j)] = \sum_{j=1}^{n}[\frac{\text{first}\,(x,R_j)+\text{first}\,(z,R_j)}{\sum_{y\in X}\text{first}\,(y,R_j)}] \leq n$, leading to a contradiction.

The following concept was orally suggested to me by JZ.

A CC f with domain Ω is said to satisfy **Point Majority Property** if for all profiles $(R_1,\ldots, R_n)\in\Omega$, the following holds: if there exists $x\in X$ such that $\sum_{j=1}^{n}\text{point}(x,R_j) > \frac{n}{2}$ then $f(R_1,\ldots,R_n) = \{x\}$.

Proposition 6

(i) If a CC f on a domain Ω satisfies Point Majority Property, then it satisfies Majority Property. If $\Omega \cap [W(X)]^N \neq \phi$, then the converse need not be true.

(ii) A CC f on $[L(X)]^N$ satisfies Point Majority Property if and only if it satisfies Majority Property.

Proof

(i) Suppose f on a domain Ω satisfies Point Majority Property. Let $(R_1,\ldots,R_n)\in\Omega$ such that for some $x\in X$, $\#\{j\in N|G(X,R_j) = \{x\}\} \geq \frac{n}{2}$ and $\#\{j\in N|x\in G(X,R_j)\} > \frac{n}{2}$. Thus, we must have either (a) $\#\{j\in N|G(X,R_j) = \{x\}\} > \frac{n}{2}$, or (b) $\#\{j\in N|G(X,R_j) = \{x\}\} = \frac{n}{2}$ and $\#\{j\in N|\{x\}\subset\subset G(X,R_j)\} - \#\{j\in N|G(X,R_j) = \{x\}\} > 0$.

Then since for all $j\in N$ with $G(X,R_j) = \{x\}$ we must have point (x,R_j) and for all $j\in N$, with $\{x\}\subset\subset G(X,R_j)$, we have $0 <$ point $(x,R_j) < 1$, in either case (i.e. in both cases (a) and (b)) we have $\sum_{j=1}^{n} \text{point}(x, R_j) > \frac{n}{2}$.

Hence by Point Majority property, we get $f(R_1,\dots,R_n) = \{x\}$ and so f satisfies Majority Property.

In order to show that if $\Omega \cap [W(X)]^N \neq \phi$, then the converse need not be true, let $X = \{x,y,z\}$ with $n = 3$. Suppose that f is a CC on Ω such that for all $(R_1, R_2, R_3)\in\Omega$, if for some $w\in X$ it is the case that $\#\{j\in N|G(X,R_j) = \{w\}\} \geq 2$, then $f(R_1,R_2,R_3) = \{w\}$; otherwise $f(R_1, R_2, R_3) = G(X,R_1)$. Clearly f satisfies the Majority property.

Let $(R_1, R_2, R_3)\in\Omega$ with $G(X, R_1) = \{x,y\}$, $G(X,R_2) = \{x,z\}$ and $G(X,R_3) = \{x\}$. Then by the definition of f, we get $f(R_1, R_2, R_3) = \{x,y\}$, although point (x, R_1) + point (x, R_2) + point $(x,R_3) = \frac{1}{2} + \frac{1}{2} +1 = 2 > \frac{3}{2}$, leading to a violation of the Point Majority property.

This counterexample can be generalized to arbitrary n and arbitrary non--empty finite X containing at least two alternatives. Let $x\in X$ and suppose there exists a preference profile $(S_1,\dots,S_n)\in\Omega$, such that $G(X,S_j) = \{x,w_j\}$ for $j = 1,\dots,n-1$ and $G(X,S_n) = \{x\}$, where $w_1,\dots,w_n\in X\setminus\{x\}$, not necessarily distinct. Let f be the CC on Ω such that if there exists $y\in X$ satisfying $\#\{j\in N|G(X,R_j) = \{y\}\} \geq \frac{n}{2}$ and $\#\{j\in N|y\in G(X,R_j)\}> \frac{n}{2}$ then $f(R_1,\dots,R_n) = \{y\}$. Clearly, f satisfies Majority property and $f(S_1,\dots,S_n) = \{x,w_1\} \neq \{x\}$. However $\sum_{j=1}^{n} \text{point}(x, S_j) > \frac{n}{2}$ and thus $f(S_1,\dots,S_n) \neq \{x\}$ implies a violation of the point majority property.

(ii) In order to show that for $\Omega = [L(X)]^N$, the two properties are equivalent, in view of (i) we need to show that for $\Omega = [L(X)]^N$, Majority property implies the Point Majority property. Thus let f be a CC on $[L(X)]^N$ satisfying the Majority Property and suppose that for some $(R_1, R_2,\dots,R_n)\in[L(X)]^N$ it is the case that $\sum_{j=1}^{n} \text{point}(x, R_j) > \frac{n}{2}$. Since $(R_1, R_2,\dots,R_n)\in[L(X)]^N$ for all $y\in X$ and $j\in N$, point $(y,R_j) \in\{0,1\}$. Hence it must be the case that $\#\{j\in N|G(X,R_j) = \{x\}\} > \frac{n}{2}$. By Majority property we get $f(R_1, R_2,\dots,R_n) = \{x\}$ and so f satisfies the Point Majority property.

This proves the above proposition. Q.E.D.

In view of the above proposition, we have the following corollaries to propositions 1, 2, 4 and 5, whose proofs being very simple are being stated without proofs.

Corollary of proposition 1

If n is an odd integer and $[L(X)]^N \subset \Omega$, then there does not exist any CC on Ω which satisfies Weak Pareto, Point Majority property and No-spoiler condition.

Corollary of proposition 2

If n is even and $\Omega = [W(X)]^N$, then there does not exist any CC on Ω which satisfies Weak Pareto, Point Majority property and No-spoiler condition.

Corollary of proposition 4

If n is odd and $[L(X)]^N \subset \Omega$, then there does not exist any CC on Ω which satisfies Strengthened Weak Pareto, Point Majority property and No-loser spoiler condition.

Corollary of proposition 5

If n is an even natural number greater than two and $\Omega = [W(X)]^N$, then there does not exist any CC on Ω which satisfies Strengthened Weak Pareto, Point Majority property, Top Anonymity, Top Neutrality and No-loser spoiler condition.

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